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Analysis of the Pareto equilibrium in multi-objective games using semi-tensor product*

Fanyueyang ZHANG, Jun'e FENG‡

School of Mathematics, Shandong University, Jinan 250100, China

E-mail: zhangfanyueyang@mail.sdu.edu.cn; fengjune@sdu.edu.cn

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Abstract: Multi-objective games (MOGs) have received much attention in recent years as a class of games with vector payoffs. Based on the semi-tensor product (STP), this paper discusses the MOG, including the existence, finite-step reachability, and finite-step controllability of Pareto equilibrium of this model, from both static and dynamic perspectives. First, the MOG concept is presented using multi-layer graphs, and STP is used to convert the payoff function into its algebraic form. Then, from the static perspective, two necessary and sufficient conditions are proposed to verify whether all players can meet their expectations and whether the strategy profile is a Pareto equilibrium, separately. Furthermore, from the dynamic perspective, a strategy updating rule is designed to investigate the finite-step reachability of the evolutionary MOG. Finally, the finite-step controllability of the evolutionary MOG is analyzed by adding pseudo-players, and a backward search algorithm is provided to find the shortest evolutionary process and control sequence.

Key words: Multi-objective game; Pareto equilibrium; Semi-tensor product; Finite-step reachability; Finite-step controllability

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1 Introduction

As one of the important tools in researching interaction mechanisms and behavior among individuals, game theory has a wide range of applications in areas such as cybernetics, biology, and economics. It has also become a popular interdisciplinary field.

The payoff function is one of the important components of finite games and can be categorized into scalar and vector forms. Most of the existing references focus on the cases where payoffs are scalar. However, games with vector payoffs, which are commonly known as multi-objective games (MOGs) or multi-criteria games, are ubiquitous in reality. Games with vector payoffs offer a noteworthy study topic when internal motivation is considered (Puerto and Perea, 2018). For instance, considering tobacco consumers as players, as discussed by Ismaili (2018), every player's payoff vector contains three components: the pleasure of smoking, the cost of cigarettes, and the impact on life expectancy. Consequently, the payoff can be modeled in the form of a vector, the numerical components of which demonstrate commodities (Hamel and Löhne, 2018), and the ultimate payoff depends on each component of the payoff vector. For example, Rădulescu et al. (2020) applied this type of game to MOG multi-agent systems, and every component of the payoff vector represented the performance on a different objective.

Equilibrium analysis is an important issue in game theory. For games with scalar payoffs, the existence of equilibria has been investigated, such as pure Nash equilibrium (Shah-Mansouri and Wong, 2018),

‡ Corresponding author

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ORCID: Fanyueyang ZHANG, https://orcid.org/0009-0000-0188-087X; Jun'e FENG, https://orcid.org/0000-0002-2192-6481

Bayesian Nash equilibrium (He et al., 2024), fuzzy strong Nash equilibrium (Huang and Liu, 2022), and stationary Nash equilibrium (Jaśkiewicz and Nowak, 2023). Therefore, it is reasonable and practical to explore the equilibrium of MOGs. Because there is more than one objective for every player in an MOG, an accepted equilibrium is the Pareto equilibrium (Qu et al., 2015). Pareto equilibrium is a strategy profile in which no player can be better without making the others worse (Smith et al., 2014; Yan and Hayakawa, 2024). Based on methods such as Ky Fan's minimax inequality and fixed-point theorem (Wang SY, 1993; Lu, 2008), the stochastic maximum principle with Markov jumps and Poisson jumps and the Lagrangian multiplier technique (Zhang et al., 2018; Hu et al., 2024), and the theorem of quasi-equilibrium existence (Ding, 2000a, 2000b), the existence of Pareto equilibrium has been investigated. To date, there has been no reference to finite-step reachability or finite-step controllability of Pareto equilibria in MOGs.

Specifically, for the Pareto equilibrium problem in MOGs, it is necessary to discuss the individual expectations of players, because cooperation and competition complement each other in the game process. In particular, for MOGs, if we examine only Pareto equilibrium which is cooperation-oriented, then the equilibrium points may be altruistic but not egoistic, which is not true in most real scenarios. Equally important, if we examine only Nash equilibrium, then such strictly competitive games do not reach an equilibrium point (Choobineh and Mohagheghi, 2019; Peng et al., 2022), because it is nearly impossible for all objectives to simultaneously achieve the best equilibrium. Consequently, by setting the payoff vector and ultimate payoff expectations for every player in an MOG and then researching the Pareto equilibrium, we can achieve more flexible simulation of various problems, where the ultimate payoff can be expressed using the weighted sum, weighted average, polynomial function, and so on. Capătă (2021) provided sufficient conditions for the existence of Pareto equilibrium over set constraints for an MOG with two players. Additionally, several references have explained payoff vector constraints based on reality. For example, Fournier et al. (2021) took investment companies and scholarships of students as examples to explain the practical significance of payoff vector limitations. The payoff vector of an investment company may include the value of the portfolio, the Sharpe ratio, and the conditional value at risk. The company usually requires the value of its portfolio to be higher than the value of other investment companies, requires the Sharpe ratio to be above a specific level, and requires the risk conditionality to be lower than a specific level. Additionally, a student's grade is influenced by several factors, like study progress, extracurricular classes, and time spent with friends and family. To keep receiving the fellowship, the student is expected to meet specific standards to maintain his/her average grade above a certain level. MOGs have also been used in dynamic networks, where the supplier tried to meet a multi-dimensional demand (Bertsekas and Rhodes, 1971; Hart, 2005; Bauso et al., 2015). Accordingly, it is meaningful to research MOGs with restricted payoff vectors (Bauso et al., 2008; Fournier et al., 2021).

In recent years, as one of the important mathematical tools, the semi-tensor product (STP) of matrices has been widely used in Boolean control networks (Liu et al., 2023), k-valued logical control networks (Wang YF et al., 2023), game theoretical control (Kong et al., 2020; Li L et al., 2023; Jia et al., 2024), and equilibrium analysis (Cheng and Ji, 2022; Wu et al., 2022). In particular, STP is an effective tool for solving equilibrium problems with scalar payoffs (Guo and Wang, 2016; Li HT et al., 2018; Le et al., 2020). Wang JH et al. (2022) took a class of congestion games with player-specific costs and resource failures as the research target, and provided a necessary and sufficient condition to guarantee the global convergence of the target Nash equilibrium via STP. Wang YH and Cheng (2016) used STP to obtain a sufficient condition that can ensure the stability of a class of delayed evolutionary potential games at a pure Nash equilibrium. Yu et al. (2024) discussed the existence and convergence of weighted Nash equilibria in games with multiple payoffs via STP. To our knowledge, STP has not been used to study the Pareto equilibrium of MOGs, including its existence, finite-step reachability, and finite-step controllability.

In this study, the existence, finite-step reachability, and finite-step controllability of the Pareto equilibrium of MOGs are considered using the STP technique. The main contributions of this paper are as follows:

1. The algebraic expressions of MOGs are

obtained using multi-layer graphs and STP.

2. Two necessary and sufficient conditions are proposed to verify whether all players can meet their expectations and to verify the existence of Pareto equilibrium, separately.

3. A new strategy updating rule is designed, and evolutionary MOGs are considered in algebraic form, based on which some results concerning finite-step reachability and finite-step controllability are presented.

4. A backward search algorithm is presented to determine the shortest evolutionary process and control sequence.

2 Preliminaries

The important symbols used in this paper are as follows: Let $\mathbf{0}_k$ be $(\underbrace{0,0,\ldots,0}_{k})^{\mathrm{T}}$, $\mathbf{0}$ be a zero matrix with appropriate dimensions, $\mathbf{1}_k$ be $(\underbrace{1,1,\ldots,1}_{k})^{\mathrm{T}}$, $\mathcal{M}_{m\times n}$ be the set of $m\times n$ real matrices, $\mathcal{M}^{\mathrm{B}}_{m\times n}$ be the set of $m\times n$ matrices which consist of 0 and 1, $\mathbb{N}_+$ be the set of positive integers, $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^k$ be the set of k-dimensional real vectors, and $\mathbb{R}^k_+$ be the set $\{(b_1,b_2,\ldots,b_k)\in\mathbb{R}^k|b_j\geq 0,\ j=1,2,\ldots,k\}$. $\boldsymbol{\delta}^i_k$ denotes the i^{th} column of the identity matrix $\boldsymbol{I}_k$, and the set Δ_k consists of $\boldsymbol{\delta}^i_k,\ i=1,2,\ldots,k$. $|B|$ represents the number of elements in set B. For matrix $\boldsymbol{A}\in\mathcal{M}_{m\times n}$, $\mathrm{Col}_j(\boldsymbol{A})$ is the j^{th} column of $\boldsymbol{A}$ and $A_{i,j}$ is the element in row i and column j, where $i\in\{1,2,\ldots,m\}$ and $j\in\{1,2,\ldots,n\}$. $\mathrm{Col}(\boldsymbol{A}):=\{\mathrm{Col}_j(\boldsymbol{A})|j=1,2,\ldots,n\}$. For vectors $\boldsymbol{\alpha}=[\alpha_1,\alpha_2,\ldots,\alpha_m]$, $\boldsymbol{\beta}=[\beta_1,\beta_2,\ldots,\beta_m]\in\mathbb{R}^m$, α_i is defined as the i^{th} element of $\boldsymbol{\alpha}$ where $i\in\{1,2,\ldots,m\}$, and $\boldsymbol{\alpha}<_{\exists}\boldsymbol{\beta}$ means that there exists at least one $j\in\{1,2,\ldots,m\}$ such that $\alpha_j<\beta_j$ holds. For m-dimensional Boolean vectors $\boldsymbol{p}=[p_1,p_2,\ldots,p_m]$ and $\boldsymbol{q}=[q_1,q_2,\ldots,q_m]$, denote $\boldsymbol{p}\vee\boldsymbol{q}=[p_1\vee q_1,p_2\vee q_2,\ldots,p_m\vee q_m]$. In detail, for every $j=1,2,\ldots,m$,

$$p_j\vee q_j=\begin{cases}0, & p_j=q_j=0,\\ 1, & \text{otherwise.}\end{cases}$$

2.1 STP

Definition 1 (Cheng et al., 2011) Given $\boldsymbol{M}\in\mathcal{M}_{m\times n}$ and $\boldsymbol{N}\in\mathcal{M}_{p\times q}$, $\boldsymbol{M}\ltimes\boldsymbol{N}$ is the STP of $\boldsymbol{M}$ and $\boldsymbol{N}$, and can be expressed as

$$\boldsymbol{M}\ltimes\boldsymbol{N}:=(\boldsymbol{M}\otimes\boldsymbol{I}_{t/n})(\boldsymbol{N}\otimes\boldsymbol{I}_{t/p})\in\mathcal{M}_{(mt/n)\times(qt/p)},$$

where $t=\mathrm{lcm}\{n,p\}$ is the least common multiple of n and p, and "$\otimes$" is the Kronecker product.

For brevity, this paper omits the symbol "$\ltimes$" during the calculation.

Lemma 1 (Cheng, 2014) For any function v: $\Delta_{k^n}\to\mathbb{R}$, the row vector $\boldsymbol{V}\in\mathbb{R}^{k^n}$, which satisfies the following equation:

$$v(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n)=\boldsymbol{V}\ltimes_{i=1}^{n}\boldsymbol{x}_i,$$

is called the structure vector of v, where $\ltimes_{i=1}^{n}\boldsymbol{x}_i=\boldsymbol{x}_1\ltimes\boldsymbol{x}_2\ltimes\ldots\ltimes\boldsymbol{x}_n$ and $\boldsymbol{V}$ is unique for v.

Lemma 2 (Cheng, 2014) Denote

$$\boldsymbol{D}^{[p,q]}_{\mathrm{f}}=\mathbf{1}^{\mathrm{T}}_p\otimes\boldsymbol{I}_q\in\mathcal{M}^{\mathrm{B}}_{q\times(pq)},$$
$$\boldsymbol{D}^{[p,q]}_{\mathrm{r}}=\boldsymbol{I}_p\otimes\mathbf{1}^{\mathrm{T}}_q\in\mathcal{M}^{\mathrm{B}}_{p\times(pq)}.$$

Then, for $\boldsymbol{X}\in\Delta_p$ and $\boldsymbol{Y}\in\Delta_q$, the following equations hold:

$$\boldsymbol{D}^{[p,q]}_{\mathrm{f}}\boldsymbol{X}\boldsymbol{Y}=\boldsymbol{Y},\quad \boldsymbol{D}^{[p,q]}_{\mathrm{r}}\boldsymbol{X}\boldsymbol{Y}=\boldsymbol{X}.$$

Lemma 3 (Cheng and Liu, 2017) For any two column vectors $\boldsymbol{X}\in\mathbb{R}^m$ and $\boldsymbol{Y}\in\mathbb{R}^n$, we have

$$\boldsymbol{W}_{[m,n]}\ltimes\boldsymbol{X}\ltimes\boldsymbol{Y}=\boldsymbol{Y}\ltimes\boldsymbol{X},$$

where $\boldsymbol{W}_{[m,n]}=(\boldsymbol{\delta}^1_n\ltimes\boldsymbol{\delta}^1_m,\ldots,\boldsymbol{\delta}^n_n\ltimes\boldsymbol{\delta}^1_m,\ldots,\boldsymbol{\delta}^1_n\ltimes\boldsymbol{\delta}^m_m,\ldots,\boldsymbol{\delta}^n_n\ltimes\boldsymbol{\delta}^m_m)\in\mathcal{M}_{(mn)\times(mn)}$ is the swap matrix.

2.2 Game model

The MOG was defined by Wang SY (1993). On that basis, a set of weights is added in the following definition. Particularly, the weight of every objective should be equivalent if all players have no preference for any of the objectives.

Definition 2 An MOG $=(N,S,\boldsymbol{a},\boldsymbol{K},\boldsymbol{C})$ consists of the following factors:

1. $N=\{1,2,\ldots,n\}$ is the set of players.

2. $S=S_1\times S_2\times\ldots\times S_n$ is the set of strategy profiles, where S_i is the strategy set of player i and "$\times$" is the Cartesian product. $\boldsymbol{s}=(s_1,s_2,\ldots,s_n)\in S$ is a strategy profile where player i chooses strategy s_i, $i=1,2,\ldots,n$, $s_i\in S_i$.

3. $\boldsymbol{a}=(a_1,a_2,\ldots,a_m)$, where a_j represents the weight of the j^{th} layer, $j=1,2,\ldots,m$. Different layers represent the different objectives, and $m\in\mathbb{N}_+$ is the number of objectives in this game.

4. $\boldsymbol{K} = (K_1, K_2, \ldots, K_n)$, where $K_i = \{k_1^i, k_2^i, \ldots, k_{m_i}^i\}$ is the set of layers where player i resides, meaning that K_i consists of the objectives with which player i is concerned, $m_i \in \mathbb{N}_+$, $m \geq m_i \geq 1$ and $i = 1, 2, \ldots, n$.

5. $\boldsymbol{C}(\boldsymbol{s}) = (\boldsymbol{c}_1(\boldsymbol{s}), \boldsymbol{c}_2(\boldsymbol{s}), \ldots, \boldsymbol{c}_n(\boldsymbol{s}))$, where $\boldsymbol{c}_i^{\mathrm{T}}(\boldsymbol{s}) = \left[c_i^{k_1^i}(\boldsymbol{s}), c_i^{k_2^i}(\boldsymbol{s}), \ldots, c_i^{k_{m_i}^i}(\boldsymbol{s})\right]$ is the payoff vector of player i under the strategy profile $\boldsymbol{s} \in S$. Specifically, $c_i^j(\boldsymbol{s}) = \sum\limits_{l \in U_j(i)} c_{i,l}^j(s_i, s_l)$ is the payoff of the j^{th} objective for player i, and $c_{i,l}^j(s_i, s_l)$ is the payoff of player i in a game with player l in terms of the j^{th} objective, where $U_j(i)$ is the set of players who have games with player i at the j^{th} layer, $i \in N$, $j \in K_i$.

Remark 1 It is worth noting that the dimensions of vectors $\boldsymbol{c}_i(\boldsymbol{s})$ and $\boldsymbol{c}_j(\boldsymbol{s})$ may be different because the size of K_i may be unequal to the size of K_j, $i, j \in N$, $i \neq j$, and $\boldsymbol{s} \in S$.

To describe the game model more intuitively, the following example is provided.

Example 1 Consider the scholarship selection policy for students. In terms of bonus points for scientific research results, the value is usually adjusted according to the level of the published article. There are five students, and they are numbered 1 through 5. Students 2 and 4 have published first-level articles, students 1, 2, 3, and 4 have published second-level articles, and students 3, 4, and 5 have published third-level articles.

This situation can be modeled as in Fig. 1: MOG$= (N, S, \boldsymbol{a}, \boldsymbol{K}, \boldsymbol{C})$. Here, $N = \{1, 2, \ldots, 5\}$, $\boldsymbol{K} = (K_1, K_2, \ldots, K_5)$, $K_1 = \{2\}$, $K_2 = \{1, 2\}$, $K_3 = \{2, 3\}$, $K_4 = \{1, 2, 3\}$, $K_5 = \{3\}$, $\boldsymbol{C}(\boldsymbol{s}) = (c_1(\boldsymbol{s}), \boldsymbol{c}_2(\boldsymbol{s}), \boldsymbol{c}_3(\boldsymbol{s}), \boldsymbol{c}_4(\boldsymbol{s}), c_5(\boldsymbol{s}))$, $c_1(\boldsymbol{s}) = c_1^2(\boldsymbol{s})$, $\boldsymbol{c}_2(\boldsymbol{s}) = [c_2^1(\boldsymbol{s}), c_2^2(\boldsymbol{s})]^{\mathrm{T}}$, $\boldsymbol{c}_3(\boldsymbol{s}) = [c_3^2(\boldsymbol{s}), c_3^3(\boldsymbol{s})]^{\mathrm{T}}$, $\boldsymbol{c}_4(\boldsymbol{s}) = [c_4^1(\boldsymbol{s}), c_4^2(\boldsymbol{s}), c_4^3(\boldsymbol{s})]^{\mathrm{T}}$, and $c_5(\boldsymbol{s}) = c_5^3(\boldsymbol{s})$. $\boldsymbol{a} = (a_1, a_2, a_3)$ consists of the weights of articles in different levels. In this example, the following represents the ultimate bonus points for student i: $\sum\limits_{j \in K_i} a_j c_i^j(\boldsymbol{s})$, $\boldsymbol{s} \in S$.

Definition 3 (Wang SY, 1993) A strategy $\bar{s}_i \in S_i$ of player i is called a Pareto-efficient strategy against $\bar{\boldsymbol{s}}$ if there is no strategy $s_i \in S_i$ that satisfies

$$c_i(s_i, \bar{s}_{-i}) - c_i(\bar{s}_i, \bar{s}_{-i}) \in \mathbb{R}_+^{k_i} \backslash \{\boldsymbol{0}_{k_i}\}.$$

Definition 4 (Wang SY, 1993) A strategy profile $\bar{\boldsymbol{s}} = (\bar{s}_i, \bar{s}_{-i}) \in S$ is called a Pareto equilibrium if, for each player i, $\bar{s}_i \in S_i$ is a Pareto-efficient strategy against $\bar{\boldsymbol{s}}$.

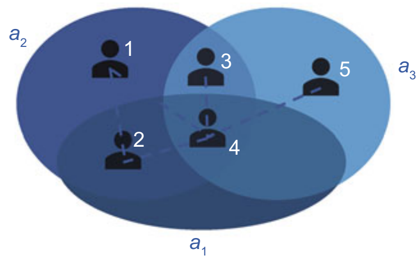

Fig. 1 Three-layer graph representation of the MOG in Example 1

3 Main results

This section describes the static and dynamic analyses. For convenience of representation, the MOG mentioned in this section satisfies $N = \{1, 2, \ldots, n\}$, $S_i = S_0 = \{1, 2, \ldots, k\}$, and $\boldsymbol{a} = (a_1, a_2, \ldots, a_m)$. Meanwhile, $\boldsymbol{\delta}_k^j \in \Delta_k$ stands for the algebraic form of strategy $j \in S_0$, and the algebraic form of S_0 is Δ_k. In this way, we can express the strategy profile $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ as $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$, where $\boldsymbol{x}_i$ $(i = 1, 2, \ldots, n)$ is the algebraic form of s_i, $s_i \in S_0$, $\boldsymbol{x}_i \in \Delta_k$.

Remark 2 This paper also applies to situations where the number of strategies is different among players because these strategy sets in algebraic form can be unified by Lemma 2.

3.1 Static analysis

In an MOG, the payoff vector of player i is related to the objectives with which player i is concerned and the players who have games with player i. Therefore, assume that $\mathbb{C}_i(\boldsymbol{s})$ is the ultimate payoff for player i under the strategy profile $\boldsymbol{s} \in S$, which can be expressed as

$$\mathbb{C}_i(\boldsymbol{s}) = \sum_{j \in K_i} a_j c_i^j(\boldsymbol{s}), \quad i = 1, 2, \ldots, n.$$

Assume that player i has $d_{i,j}$ neighbors in the j^{th} layer. Then, it is noticed that

$$\begin{aligned} c_i^j(\boldsymbol{s}) &= \sum_{l \in U_j(i)} c_{i,l}^j(s_i, s_l) \\ &= \boldsymbol{V}_{c_i^1}^j \boldsymbol{x}_i \boldsymbol{x}_{i_1} + \boldsymbol{V}_{c_i^2}^j \boldsymbol{x}_i \boldsymbol{x}_{i_2} + \ldots + \boldsymbol{V}_{c_i^{d_{i,j}}}^j \boldsymbol{x}_i \boldsymbol{x}_{i_{d_{i,j}}}, \end{aligned} \tag{1}$$

where $\boldsymbol{V}_{c_i^l}^j$ is the structure vector of player i when he/she plays with the l^{th} neighbor in the j^{th} layer, $i \in N,\ j \in K_i,\ l \in \{1, 2, \ldots, d_{i,j}\}$.

Next, we analyze the payoff of player i in the j^{th} layer, $j \in K_i$. Define the $k^2 \times k^n$ matrix:

$$\boldsymbol{F}_{i,j} = \begin{cases} \mathbf{1}_{k^{i-1}}^{\mathrm{T}} \otimes \boldsymbol{I}_k \otimes \mathbf{1}_{k^{j-i-1}}^{\mathrm{T}} \otimes \boldsymbol{I}_k \otimes \mathbf{1}_{k^{n-j}}^{\mathrm{T}}, & i < j, \\ \mathbf{0}, & i = j, \\ \mathbf{1}_{k^{j-1}}^{\mathrm{T}} \otimes \boldsymbol{I}_k \otimes \mathbf{1}_{k^{i-j-1}}^{\mathrm{T}} \otimes \boldsymbol{I}_k \otimes \mathbf{1}_{k^{n-i}}^{\mathrm{T}}, & i > j. \end{cases}$$

Subsequently, according to Lemma 2, Eq. (1) can be transformed into

$$c_i^j(\boldsymbol{s}) = \boldsymbol{\mathcal{F}}_i^j \ltimes_{p=1}^n \boldsymbol{x}_p,$$

where $\boldsymbol{\mathcal{F}}_i^j = \sum\limits_{l \in U_j(i)} \boldsymbol{V}_{c_i^l}^j \boldsymbol{F}_{i,l} \in \mathbb{R}^{k^n}$ is a row vector.

Thus, the ultimate payoff of player i in the MOG can be rewritten as

$$\mathbb{C}_i(\boldsymbol{s}) = \boldsymbol{\mathcal{G}}_i \ltimes_{p=1}^n \boldsymbol{x}_p, \tag{2}$$

where

$$\boldsymbol{\mathcal{G}}_i = \sum_{j \in K_i} a_j \boldsymbol{\mathcal{F}}_i^j. \tag{3}$$

Without loss of generality, we assume that the expectation of player i is r_i, where r_i is a known constant. If the ultimate payoff of player i is in line with r_i, then the following algebraic inequality holds:

$$\mathbb{C}_i(\boldsymbol{s}) \geq r_i, \quad i \in N.$$

Therefore, combining Eq. (2), we obtain

$$\begin{bmatrix} \boldsymbol{\mathcal{G}}_1 \ltimes_{p=1}^n \boldsymbol{x}_p \\ \boldsymbol{\mathcal{G}}_2 \ltimes_{p=1}^n \boldsymbol{x}_p \\ \vdots \\ \boldsymbol{\mathcal{G}}_n \ltimes_{p=1}^n \boldsymbol{x}_p \end{bmatrix} \geq \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix};$$

that is,

$$\boldsymbol{\mathcal{G}} \ltimes_{p=1}^n \boldsymbol{x}_p \geq \mathbf{0}_n,$$

where

$$\boldsymbol{\mathcal{G}} = \begin{bmatrix} \boldsymbol{\mathcal{G}}_1 - r_1 \mathbf{1}_{k^n}^{\mathrm{T}} \\ \boldsymbol{\mathcal{G}}_2 - r_2 \mathbf{1}_{k^n}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{\mathcal{G}}_n - r_n \mathbf{1}_{k^n}^{\mathrm{T}} \end{bmatrix}. \tag{4}$$

In this way, we can obtain the following theorem:

Theorem 1 The individual expectations $(r_1, r_2, \ldots, r_n)$ can be achieved in the MOG $= (N, S, \boldsymbol{a}, \boldsymbol{K}, \boldsymbol{C})$ if and only if there exists a positive integer j satisfying

$$\mathrm{Col}_j(\boldsymbol{\mathcal{G}}) \geq \mathbf{0}_n,$$

where $\boldsymbol{\mathcal{G}}$ is shown in Eq. (4).

Remark 3 The ultimate payoffs in other forms can also be discussed in a similar way.

Example 2 Consider an MOG $= (N, S, \boldsymbol{a}, \boldsymbol{K}, \boldsymbol{C})$; the three-layer graph is shown in Fig. 2. Here, we set $N = \{1, 2, 3, 4\}$, $S_1 = S_2 = S_3 = S_4 = S_0 = \{1, 2\}$, $\boldsymbol{a} = (3, 2, 2)$, $\boldsymbol{K} = (\{1\}, \{1, 3\}, \{1, 2, 3\}, \{2\})$. $\boldsymbol{C}(\boldsymbol{s}) = (c_1(\boldsymbol{s}), \boldsymbol{c}_2(\boldsymbol{s}), \boldsymbol{c}_3(\boldsymbol{s}), c_4(\boldsymbol{s}))$ is shown in Table 1, and the payoff expectations of players 1–4 are $r_1 = 21$, $r_2 = 28$, $r_3 = 36$, and $r_4 = 12$. Then, from Table 1, we can observe

$$\begin{aligned} \boldsymbol{V}_1^{\mathrm{c}} &= [7\ 7\ 1\ 9\ 8\ 8\ 2\ 4\ 3\ 9\ 7\ 6\ 5\ 12\ 8\ 3], \\ \boldsymbol{V}_2^{\mathrm{c}} &= \begin{bmatrix} 5 & 3 & 7 & 10 & 3 & 6 & 9 & 9 & 5 & 6 & 3 & 5 & 1 & 8 & 5 & 3 \\ 4 & 2 & 4 & 4 & 9 & 1 & 2 & 6 & 3 & 2 & 5 & 2 & 3 & 4 & 7 & 6 \end{bmatrix}, \\ \boldsymbol{V}_3^{\mathrm{c}} &= \begin{bmatrix} 6 & 3 & 6 & 7 & 3 & 0 & 5 & 5 & 6 & 3 & 7 & 3 & 4 & 8 & 5 & 3 \\ 1 & 10 & 1 & 4 & 7 & 1 & 2 & 4 & 1 & 10 & 7 & 1 & 6 & 5 & 6 & 3 \\ 2 & 2 & 1 & 5 & 7 & 3 & 3 & 2 & 9 & 4 & 2 & 5 & 3 & 8 & 2 & 4 \end{bmatrix}, \\ \boldsymbol{V}_4^{\mathrm{c}} &= [9\ 2\ 5\ 8\ 10\ 7\ 5\ 5\ 2\ 3\ 7\ 3\ 7\ 12\ 6\ 4]. \end{aligned} \tag{5}$$

Combining Eqs. (2), (3), and (5), we can obtain

$$\begin{aligned} \boldsymbol{\mathcal{G}}_1 &= [21\ 21\ 3\ 27\ 24\ 24\ 6\ 12\ 9\ 27\ 21\ 18\ 15\ 36\ 24\ 9], \\ \boldsymbol{\mathcal{G}}_2 &= [23\ 13\ 29\ 38\ 27\ 20\ 31\ 39\ 21\ 22\ 19\ 19\ 9\ 32\ 29\ 21], \\ \boldsymbol{\mathcal{G}}_3 &= [24\ 33\ 22\ 39\ 37\ 8\ 25\ 27\ 38\ 37\ 39\ 21\ 30\ 50\ 31\ 23], \\ \boldsymbol{\mathcal{G}}_4 &= [18\ 4\ 10\ 16\ 20\ 14\ 10\ 10\ 4\ 6\ 14\ 6\ 14\ 24\ 12\ 8]. \end{aligned}$$

Furthermore, based on Eq. (4), $\boldsymbol{\mathcal{G}}$ can be calculated as Eq. (6) (at the top of the next page). Thus, it can be verified that $j = 4$ and $j = 14$ satisfy Theorem 1; that is, payoff vectors of all players are in line with their expectations under the strategy profiles $\boldsymbol{\delta}_{16}^4 \sim (1, 1, 2, 2)$ and $\boldsymbol{\delta}_{16}^{14} \sim (2, 2, 1, 2)$.

Next, we discuss the existence of Pareto equilibria in MOGs. Denote

$$\boldsymbol{V}_i^{\mathrm{c}} = \begin{bmatrix} \boldsymbol{\mathcal{F}}_i^{k_1} \\ \boldsymbol{\mathcal{F}}_i^{k_2} \\ \vdots \\ \boldsymbol{\mathcal{F}}_i^{k_{m_i}} \end{bmatrix}.$$

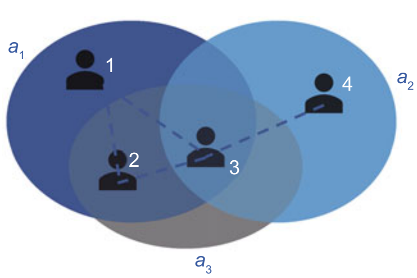

Fig. 2 Three-layer graph representation of the MOG in Example 2

$$\boldsymbol{\mathcal{G}}=\begin{bmatrix} 14 & 14 & -4 & 20 & 17 & 17 & -1 & 5 & 2 & 20 & 14 & 11 & 8 & 29 & 17 & 2\\ -5 & -15 & 1 & 10 & -1 & -8 & 3 & 11 & -7 & -6 & -9 & -9 & -19 & 4 & 1 & -7\\ -12 & -3 & -14 & 3 & 1 & -28 & -11 & -9 & 2 & 1 & 3 & -15 & -6 & 14 & -5 & -13\\ 12 & -2 & 4 & 10 & 14 & 8 & 4 & 4 & -2 & 0 & 8 & 0 & 8 & 18 & 6 & 2 \end{bmatrix}. \quad (6)$$

Table 1 Payoffs of Example 2

	$\boldsymbol{c}_i(\boldsymbol{s})$							
	$\boldsymbol{s}_1$=(1, 1, 1, 1)	$\boldsymbol{s}_2$=(1, 1, 1, 2)	$\boldsymbol{s}_3$=(1, 1, 2, 1)	$\boldsymbol{s}_4$=(1, 1, 2, 2)	$\boldsymbol{s}_5$=(1, 2, 1, 1)	$\boldsymbol{s}_6$=(1, 2, 1, 2)	$\boldsymbol{s}_7$=(1, 2, 2, 1)	$\boldsymbol{s}_8$=(1, 2, 2, 2)
c_1	7	7	1	9	8	8	2	4
$\boldsymbol{c}_2$	$[5,4]^{\mathrm{T}}$	$[3,2]^{\mathrm{T}}$	$[7,4]^{\mathrm{T}}$	$[10,4]^{\mathrm{T}}$	$[3,9]^{\mathrm{T}}$	$[6,1]^{\mathrm{T}}$	$[9,2]^{\mathrm{T}}$	$[9,6]^{\mathrm{T}}$
$\boldsymbol{c}_3$	$[6,1,2]^{\mathrm{T}}$	$[3,10,2]^{\mathrm{T}}$	$[6,1,1]^{\mathrm{T}}$	$[7,4,5]^{\mathrm{T}}$	$[3,7,7]^{\mathrm{T}}$	$[0,1,3]^{\mathrm{T}}$	$[5,2,3]^{\mathrm{T}}$	$[5,4,2]^{\mathrm{T}}$
c_4	9	2	5	8	10	7	5	5

	$\boldsymbol{c}_i(\boldsymbol{s})$							
	$\boldsymbol{s}_9$=(2, 1, 1, 1)	$\boldsymbol{s}_{10}$=(2, 1, 1, 2)	$\boldsymbol{s}_{11}$=(2, 1, 2, 1)	$\boldsymbol{s}_{12}$=(2, 1, 2, 2)	$\boldsymbol{s}_{13}$=(2, 2, 1, 1)	$\boldsymbol{s}_{14}$=(2, 2, 1, 2)	$\boldsymbol{s}_{15}$=(2, 2, 2, 1)	$\boldsymbol{s}_{16}$=(2, 2, 2, 2)
c_1	3	9	7	6	5	12	8	3
$\boldsymbol{c}_2$	$[5,3]^{\mathrm{T}}$	$[6,2]^{\mathrm{T}}$	$[3,5]^{\mathrm{T}}$	$[5,2]^{\mathrm{T}}$	$[1,3]^{\mathrm{T}}$	$[8,4]^{\mathrm{T}}$	$[5,7]^{\mathrm{T}}$	$[3,6]^{\mathrm{T}}$
$\boldsymbol{c}_3$	$[6,1,9]^{\mathrm{T}}$	$[3,10,4]^{\mathrm{T}}$	$[7,7,2]^{\mathrm{T}}$	$[3,1,5]^{\mathrm{T}}$	$[4,6,3]^{\mathrm{T}}$	$[8,5,8]^{\mathrm{T}}$	$[5,6,2]^{\mathrm{T}}$	$[3,3,4]^{\mathrm{T}}$
c_4	2	3	7	3	7	12	6	4

The four values of $\boldsymbol{s}_i$ in the brackets represent the corresponding selected strategies of the four players

Based on Lemma 3, it can be implied that

$$\boldsymbol{c}_i(\boldsymbol{s})=\boldsymbol{L}_i\boldsymbol{x}_i\ltimes_{p\neq i}^{n}\boldsymbol{x}_p, \quad (7)$$

where $\boldsymbol{L}_i=\boldsymbol{V}_i^{\mathrm{c}}\boldsymbol{W}_{[k,k^{i-1}]}\in\mathcal{M}_{m_i\times k^n}$. Then, divide $\boldsymbol{L}_i$ into k parts:

$$\boldsymbol{L}_i=[\boldsymbol{L}_i^1,\boldsymbol{L}_i^2,\ldots,\boldsymbol{L}_i^k], \quad (8)$$

where $\boldsymbol{L}_i^r\in\mathcal{M}_{m_i\times k^{n-1}}$, $r=1,2,\ldots,k$. Subsequently, arrange the k parts in columns to construct a new matrix, and let $\boldsymbol{H}_i^l$ be the l^{th} column of this matrix, that is,

$$\boldsymbol{H}_i^l=\begin{bmatrix}H_i^{l,1}\\ H_i^{l,2}\\ \vdots\\ H_i^{l,k}\end{bmatrix}=\mathrm{Col}_l\left(\begin{bmatrix}\boldsymbol{L}_i^1\\ \boldsymbol{L}_i^2\\ \vdots\\ \boldsymbol{L}_i^k\end{bmatrix}\right)\in\mathbb{R}^{km_i}. \quad (9)$$

Based on the above, the following theorem holds:

Theorem 2 For an MOG $=(N,S,\boldsymbol{a},\boldsymbol{K},\boldsymbol{C})$, there exists a Pareto equilibrium $\boldsymbol{s}^*=(s_1^*,s_2^*,\ldots,s_n^*)$ if and only if one of the following conclusions holds:

(i) For every $i\in N$, there exists a vector $\boldsymbol{\alpha}\in\mathbb{R}^{m_i}$ such that $\boldsymbol{H}_i^{-i}=\boldsymbol{1}_k\otimes\boldsymbol{\alpha}$ holds;

(ii) For every $i\in N$, there exists a constant $r\in S_0\backslash\{s_i^*\}$ such that $\boldsymbol{H}_i^{-i,r}<_{\exists}\boldsymbol{H}_i^{-i,s_i^*}$ holds, where $-i$ satisfies $\ltimes_{p\neq i}^{n}\boldsymbol{x}_p^*=\boldsymbol{\delta}_{k^{n-1}}^{-i}$, $i=1,2,\ldots,n$.

Proof As can be obtained from Eqs. (7) and (8), $\mathrm{Col}_l(\boldsymbol{L}_i^q)$ is the payoff vector of player i as i chooses strategy $q\in S_0$ and l satisfies $\ltimes_{p\neq i}^{n}\boldsymbol{x}_p=\boldsymbol{\delta}_{k^{n-1}}^{l}$. Hence, it can be derived that $\boldsymbol{H}_i^l$ as shown in Eq. (9) consists of all possible payoff vectors of player i, where $i\in N$ and $\ltimes_{p\neq i}^{n}\boldsymbol{x}_p=\boldsymbol{\delta}_{k^{n-1}}^{l}$.

(Necessity) According to Definitions 3 and 4, it can be deduced that the payoff vector of player i $\boldsymbol{c}_i(s_i,s_{-i}^*)$ satisfies either condition (10) or condition (11):

$$\boldsymbol{c}_i(s_i,s_{-i}^*)=\boldsymbol{c}_i(s_i^*,s_{-i}^*), \quad (10)$$

$$\boldsymbol{c}_i(s_i,s_{-i}^*)<_{\exists}\boldsymbol{c}_i(s_i^*,s_{-i}^*), \quad (11)$$

where (s_i^*,s_{-i}^*) is the Pareto equilibrium, $s_i\in S_0$. Setting $\ltimes_{p\neq i}^{n}\boldsymbol{x}_p^*=\boldsymbol{\delta}_{k^{n-1}}^{-i}$ and combining Eq. (9), we have

$$\boldsymbol{c}_i(s_i,s_{-i}^*)=\boldsymbol{L}_i\boldsymbol{x}_i\ltimes_{p\neq i}^{n}\boldsymbol{x}_p^*=\boldsymbol{H}_i^{-i,s_i},$$

$$\boldsymbol{c}_i(s_i^*,s_{-i}^*)=\boldsymbol{L}_i\boldsymbol{x}_i^*\ltimes_{p\neq i}^{n}\boldsymbol{x}_p^*=\boldsymbol{H}_i^{-i,s_i^*}.$$

Furthermore, based on the arbitrariness of s_i, Eq. (10) is equivalent to

$$\boldsymbol{H}_i^{-i,1}=\boldsymbol{H}_i^{-i,2}=\cdots=\boldsymbol{H}_i^{-i,k}\in\mathbb{R}^{m_i}.$$

Therefore, there exists a vector $\boldsymbol{\alpha}$ such that $\boldsymbol{H}_i^{-i}=\boldsymbol{1}_k\otimes\boldsymbol{\alpha}$ holds.

Similarly, inequality (11) is equivalent to

$$\boldsymbol{H}_i^{-i,s_i} <_{\exists} \boldsymbol{H}_i^{-i,s_i^*}.$$

(Sufficiency) From Eq. (9) it can be inferred that $\boldsymbol{H}_i^{-i} = \boldsymbol{1}_k \otimes \boldsymbol{\alpha}$ means

$$\boldsymbol{c}_i(s_i=1, s_{-i}^*) = \boldsymbol{c}_i(s_i=2, s_{-i}^*) = \ldots = \boldsymbol{c}_i(s_i=k, s_{-i}^*)$$
$$\Longleftrightarrow \quad \boldsymbol{c}_i(s_i, s_{-i}^*) = \boldsymbol{c}_i(s_i^*, s_{-i}^*), \quad \forall s_i \in S_i. \quad (12)$$

Similarly, $\boldsymbol{H}_i^{-i,s_i} <_{\exists} \boldsymbol{H}_i^{-i,s_i^*}$ means

$$\boldsymbol{c}_i(r, s_{-i}^*) <_{\exists} \boldsymbol{c}_i(s_i^*, s_{-i}^*), \quad r \in S_0 \backslash \{s_i^*\}. \quad (13)$$

Hence, if expression (12) or (13) holds, when player i changes strategy from s_i^* to s_i and other players do not change strategies, at least one component of the payoff vector of player i decreases, that is,

$$\boldsymbol{c}_i(s_i, s_{-i}^*) - \boldsymbol{c}_i(s_i^*, s_{-i}^*) \notin \mathbb{R}_+^{k_i} \backslash \{\boldsymbol{0}_{k_i}\}, \quad s_i \in S_0 \backslash \{s_i^*\}.$$

Example 3 Consider the didactic toy example in Ocean Shores city. As shown in Ismaili (2018), there are five players, each player has two strategies, and each player focuses on two objectives. Thus, this example can be described by MOG $= (N, S, \boldsymbol{a}, \boldsymbol{K}, \boldsymbol{C})$, and the two-layer graph is shown in Fig. 3. Here, $N = \{1, 2, \ldots, 5\}$, $\boldsymbol{s} \in S$ ($S = S_1 \times S_2 \times \ldots \times S_5$), $S_i = S_0 = \{1, 2\}$, $\boldsymbol{a} = (1, 1)$, $\boldsymbol{K} = (K_1, K_2, \ldots, K_5)$, and $K_i = K_0 = \{1, 2\}$ ($i = 1, 2, \ldots, 5$). The payoff of every player is shown in Fig. 1 in Ismaili (2018). Then, $\boldsymbol{V}_i^{\mathrm{c}}$ can be deduced as Eq. (14) (at the top of the next page), $i = 1, 2, \ldots, 5$. For $j = 1, 2, 3, 4$,

$$\boldsymbol{W}_{[2,2^j]} = \boldsymbol{\delta}_{2^{j+1}}[1, 3, \ldots, 2^{j+1}-1, 2, 4, \ldots, 2^{j+1}].$$

Then, it can be calculated that

$$\boldsymbol{c}_i(s_i, s_{-i}) = \boldsymbol{L}_i \boldsymbol{x}_i \ltimes_{p \neq i}^5 \boldsymbol{x}_p, \quad i = 1, 2, \ldots, 5,$$

where $\boldsymbol{L}_1$–$\boldsymbol{L}_5$ are shown in Eq. (15) (on the next page).

For instance, it can be verified that when $\boldsymbol{s} = (2, 1, 1, 1, 1)$, $\ltimes_{j \neq 1}^5 \boldsymbol{x}_j = \boldsymbol{\delta}_{16}^1$ and $\ltimes_{j \neq i}^5 \boldsymbol{x}_j = \boldsymbol{\delta}_{16}^9$ hold, $i = 2, 3, 4, 5$. Furthermore, according to Eq. (9) and Theorem 2, it can be checked that

$$\boldsymbol{H}_1^{1,2} <_{\exists} \boldsymbol{H}_1^{1,1}, \quad \boldsymbol{H}_2^{9,1} <_{\exists} \boldsymbol{H}_2^{9,2},$$
$$\boldsymbol{H}_3^{9,1} <_{\exists} \boldsymbol{H}_3^{9,2}, \quad \boldsymbol{H}_4^{9,1} <_{\exists} \boldsymbol{H}_4^{9,2},$$
$$\boldsymbol{H}_5^{9,1} <_{\exists} \boldsymbol{H}_5^{9,2},$$

which reveals that strategy profile (2,1,1,1,1) is a Pareto equilibrium. In a similar way, it can be validated that strategy profiles (2,1,1,1,2), (2,1,1,2,1), (2,1,1,2,2), (2,1,2,1,1), (2,1,2,1,2), (2,2,1,1,1), (2,2,1,1,2), (2,2,1,2,1), (2,2,1,2,2), (2,2,2,2,1), and (2,2,2,2,2) are also Pareto equilibria, which is consistent with the result of Fig. 1 in Ismaili (2018).

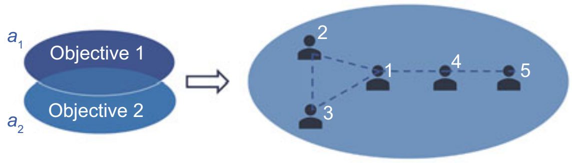

Fig. 3 Two-layer graph representation of the MOG in Example 3

Example 4 Recall Example 2. Next, we explore the existence of its Pareto equilibrium. Combining Lemma 3 with Eqs. (5) and (7), we can obtain

$$\boldsymbol{c}_i(s_i, s_{-i}) = \boldsymbol{L}_i \boldsymbol{x}_i \ltimes_{p \neq i}^4 \boldsymbol{x}_p, \quad i = 1, 2, 3, 4,$$

where

$$\boldsymbol{L}_1 = \begin{bmatrix} 7\ 7\ 1\ 9\ 8\ 8\ 2\ 4\ 3\ 9\ 7\ 6\ 5\ 12\ 8\ 3 \end{bmatrix},$$
$$\boldsymbol{L}_2 = \begin{bmatrix} 5\ 3\ 7\ 10\ 5\ 6\ 3\ 5\ 3\ 6\ 9\ 9\ 1\ 8\ 5\ 3 \\ 4\ 2\ 4\ 4\ 3\ 2\ 5\ 2\ 9\ 1\ 2\ 6\ 3\ 4\ 7\ 6 \end{bmatrix},$$
$$\boldsymbol{L}_3 = \begin{bmatrix} 6\ 3\ 3\ 0\ 6\ 3\ 4\ 8\ 6\ 7\ 5\ 5\ 7\ 3\ 5\ 3 \\ 1\ 10\ 7\ 1\ 1\ 10\ 6\ 5\ 1\ 4\ 2\ 4\ 7\ 1\ 6\ 3 \\ 2\ 2\ 7\ 3\ 9\ 4\ 3\ 8\ 1\ 5\ 3\ 2\ 2\ 5\ 2\ 4 \end{bmatrix},$$
$$\boldsymbol{L}_4 = \begin{bmatrix} 9\ 5\ 10\ 5\ 2\ 7\ 7\ 6\ 2\ 8\ 7\ 5\ 3\ 3\ 12\ 4 \end{bmatrix}.$$

Moreover, based on Eq. (9), we have the following results, where $l \in \{1, 2, \ldots, 8\}$:

$$\boldsymbol{H}_1^l = \mathrm{Col}_l\left(\begin{bmatrix} 7 & 7 & 1 & 9 & 8 & 8 & 2 & 4 \\ 3 & 9 & 7 & 6 & 5 & 12 & 8 & 3 \end{bmatrix}\right),$$

$$\boldsymbol{H}_2^l = \mathrm{Col}_l\left(\begin{bmatrix} 5 & 3 & 7 & 10 & 5 & 6 & 3 & 5 \\ 4 & 2 & 4 & 4 & 3 & 2 & 5 & 2 \\ 3 & 6 & 9 & 9 & 1 & 8 & 5 & 3 \\ 9 & 1 & 2 & 6 & 3 & 4 & 7 & 6 \end{bmatrix}\right),$$

$$\boldsymbol{H}_3^l = \mathrm{Col}_l\left(\begin{bmatrix} 6 & 3 & 3 & 0 & 6 & 3 & 4 & 8 \\ 1 & 10 & 7 & 1 & 1 & 10 & 6 & 5 \\ 2 & 2 & 7 & 3 & 9 & 4 & 3 & 8 \\ 6 & 7 & 5 & 5 & 7 & 3 & 5 & 3 \\ 1 & 4 & 2 & 4 & 7 & 1 & 6 & 3 \\ 1 & 5 & 3 & 2 & 2 & 5 & 2 & 4 \end{bmatrix}\right),$$

$$\boldsymbol{H}_4^l = \mathrm{Col}_l\left(\begin{bmatrix} 9 & 5 & 10 & 5 & 2 & 7 & 7 & 6 \\ 2 & 8 & 7 & 5 & 3 & 3 & 12 & 4 \end{bmatrix}\right).$$

$$
\begin{aligned}
\boldsymbol{V}_1^{\mathrm{c}} &= \begin{bmatrix} 14 & 14 & 3 & 3 & 4 & 4 & 10 & 10 & 5 & 5 & 1 & 1 & 11 & 11 & 12 & 12 & 7 & 7 & 10 & 10 & 6 & 6 & 1 & 1 & 6 & 6 & 3 & 3 & 1 & 1 & 15 & 15 \\ 4 & 4 & 15 & 15 & 10 & 10 & 12 & 12 & 10 & 10 & 3 & 3 & 13 & 13 & 15 & 15 & 14 & 14 & 6 & 6 & 13 & 13 & 8 & 8 & 15 & 15 & 11 & 11 & 5 & 5 & 14 & 14 \end{bmatrix}, \\
\boldsymbol{V}_2^{\mathrm{c}} &= \begin{bmatrix} 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 \end{bmatrix}, \\
\boldsymbol{V}_3^{\mathrm{c}} &= \begin{bmatrix} 14 & 14 & 14 & 14 & 1 & 1 & 1 & 1 & 14 & 14 & 14 & 14 & 1 & 1 & 1 & 1 & 5 & 5 & 5 & 5 & 7 & 7 & 7 & 7 & 5 & 5 & 5 & 5 & 7 & 7 & 7 & 7 \\ 18 & 18 & 18 & 18 & 8 & 8 & 8 & 8 & 18 & 18 & 18 & 18 & 8 & 8 & 8 & 8 & 12 & 12 & 12 & 12 & 11 & 11 & 11 & 11 & 12 & 12 & 12 & 12 & 11 & 11 & 11 & 11 \end{bmatrix}, \\
\boldsymbol{V}_4^{\mathrm{c}} &= \begin{bmatrix} 9 & 16 & 12 & 6 & 9 & 16 & 12 & 6 & 9 & 16 & 12 & 6 & 9 & 16 & 12 & 6 & 15 & 11 & 13 & 12 & 15 & 11 & 13 & 12 & 15 & 11 & 13 & 12 & 15 & 11 & 13 & 12 \\ 16 & 14 & 16 & 1 & 16 & 14 & 16 & 1 & 16 & 14 & 16 & 1 & 16 & 14 & 16 & 1 & 8 & 15 & 10 & 5 & 8 & 15 & 10 & 5 & 8 & 15 & 10 & 5 & 8 & 15 & 10 & 5 \end{bmatrix}, \\
\boldsymbol{V}_5^{\mathrm{c}} &= \begin{bmatrix} 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 & 10 & 16 & 12 & 2 \\ 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 & 8 & 4 & 7 & 10 \end{bmatrix}.
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\boldsymbol{L}_1 &= \begin{bmatrix} 14 & 14 & 3 & 3 & 4 & 4 & 10 & 10 & 5 & 5 & 1 & 1 & 11 & 11 & 12 & 12 & 7 & 7 & 10 & 10 & 6 & 6 & 1 & 1 & 6 & 6 & 3 & 3 & 1 & 1 & 15 & 15 \\ 4 & 4 & 15 & 15 & 10 & 10 & 12 & 12 & 10 & 10 & 3 & 3 & 13 & 13 & 15 & 15 & 14 & 14 & 6 & 6 & 13 & 13 & 8 & 8 & 15 & 15 & 11 & 11 & 5 & 5 & 14 & 14 \end{bmatrix}, \\
\boldsymbol{L}_2 &= \begin{bmatrix} 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 \end{bmatrix}, \\
\boldsymbol{L}_3 &= \begin{bmatrix} 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\ 18 & 18 & 18 & 18 & 18 & 18 & 18 & 18 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 \end{bmatrix}, \\
\boldsymbol{L}_4 &= \begin{bmatrix} 9 & 16 & 9 & 16 & 9 & 16 & 9 & 16 & 15 & 11 & 15 & 11 & 15 & 11 & 15 & 11 & 12 & 6 & 12 & 6 & 12 & 6 & 12 & 6 & 13 & 12 & 13 & 12 & 13 & 12 & 13 & 12 \\ 16 & 14 & 16 & 14 & 16 & 14 & 16 & 14 & 8 & 15 & 8 & 15 & 8 & 15 & 8 & 15 & 16 & 1 & 16 & 1 & 16 & 1 & 16 & 1 & 10 & 5 & 10 & 5 & 10 & 5 & 10 & 5 \end{bmatrix}, \\
\boldsymbol{L}_5 &= \begin{bmatrix} 10 & 12 & 10 & 12 & 10 & 12 & 10 & 12 & 10 & 12 & 10 & 12 & 10 & 12 & 10 & 12 & 16 & 2 & 16 & 2 & 16 & 2 & 16 & 2 & 16 & 2 & 16 & 2 & 16 & 2 & 16 & 2 \\ 8 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 4 & 10 & 4 & 10 & 4 & 10 & 4 & 10 & 4 & 10 & 4 & 10 & 4 & 10 & 4 & 10 \end{bmatrix}.
\end{aligned}
\tag{15}
$$

Therefore, it can be checked that Theorem 2 holds for every $i \in N$ as $l = 1, 4, 5, 8, 14$, and 15, meaning that $\boldsymbol{\delta}_{16}^1$, $\boldsymbol{\delta}_{16}^4$, $\boldsymbol{\delta}_{16}^5$, $\boldsymbol{\delta}_{16}^8$, $\boldsymbol{\delta}_{16}^{14}$, and $\boldsymbol{\delta}_{16}^{15}$ are Pareto equilibria. Thus, combining with the result in Example 2, we find that strategy profiles $\boldsymbol{\delta}_{16}^4 \sim (1, 1, 2, 2)$ and $\boldsymbol{\delta}_{16}^{14} \sim (2, 2, 1, 2)$ are Pareto equilibria, and all players can meet their expectations under these strategy profiles.

3.2 Dynamic evolutionary analysis

In this subsection, a strategy updating rule is proposed and then the finite-step reachability and finite-step controllability of Pareto equilibrium for evolutionary MOGs (EMOGs) are investigated.

Denote $(s_1(t), s_2(t), \ldots, s_n(t))$ as the strategy profile at time t; its algebraic form is described as $(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t))$. Let $\boldsymbol{x}_{-i}(t) = (\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_{i-1}(t), \boldsymbol{x}_{i+1}(t), \ldots, \boldsymbol{x}_n(t))$. Given $\boldsymbol{x}_i \in \Delta_k \backslash \{\boldsymbol{x}_i(t)\}$ and $\boldsymbol{X}_{-i}(t) = \ltimes_{j \neq i}^{n} \boldsymbol{x}_j(t) \in \Delta_{k^{n-1}}$, set $f_i(t, \boldsymbol{x}_i) = \boldsymbol{c}_i(\boldsymbol{x}_i, \boldsymbol{X}_{-i}(t)) - \boldsymbol{c}_i(\boldsymbol{x}_i(t), \boldsymbol{X}_{-i}(t))$, $P_i(t) = \{\boldsymbol{x}_i | f_i(t, \boldsymbol{x}_i) \in \mathbb{R}_+^{k_i} \backslash \{\boldsymbol{0}_{k_i}\}\}$, and $Q_i(t) = \{\boldsymbol{x}_i | -f_i(t, \boldsymbol{x}_i) \in \mathbb{R}_+^{k_i} \backslash \{\boldsymbol{0}_{k_i}\}\}$. Thus, the strategy updating rule is designed as

$$
O_i(t) = \begin{cases} \{\boldsymbol{x}_i(t)\}, & |Q_i(t)| = k - 1, \\ P_i(t), & |Q_i(t)| + |P_i(t)| = k - 1 \\ & \quad \text{and } |Q_i(t)| < k - 1, \\ \Delta_k \backslash Q_i(t), & \text{otherwise}. \end{cases} \tag{16}
$$

Here, $O_i(t)$ is the set of strategies that can be chosen at time $t + 1$ for player i. Particularly, $O_i(t)$ with $\boldsymbol{X}_{-i}(t) = \boldsymbol{\delta}_{k^{n-1}}^l$ is designated as O_i^l, and it is assumed that player i chooses the strategies in O_i^l with equal probability, $i = 1, 2, \ldots, n$, $l = 1, 2, \ldots, k^{n-1}$.

Remark 4 The determination of strategy updating rules is an important prerequisite for the dynamic analysis of evolutionary games. Several common strategy updating rules, such as myopic best response adjustment, unconditional imitation, and simplified Fermi rule, are all dependent on the player's own payoff at the present moment as a comparison, whereas these rules assume that the strategies chosen by other players remain unchanged at the next moment.

Let $\bar{\boldsymbol{x}}_i(t+1)$ represent the feasible strategy of player i at time $t+1$; based on Eq. (16), it can be expressed as

$$\bar{\boldsymbol{x}}_i(t+1) = \vee_{\boldsymbol{x}_i \in O_i(t)} \boldsymbol{x}_i.$$

Using O_i^l, define $\boldsymbol{R}_i = [\vee_{\boldsymbol{x}_i \in O_i^1} \boldsymbol{x}_i, \vee_{\boldsymbol{x}_i \in O_i^2} \boldsymbol{x}_i, \ldots, \vee_{\boldsymbol{x}_i \in O_i^{k^{n-1}}} \boldsymbol{x}_i]$. Thereby,

$$\bar{\boldsymbol{x}}_i(t+1) = \boldsymbol{R}_i \boldsymbol{X}_{-i}(t), \quad \boldsymbol{X}_{-i}(t) \in \Delta_{k^{n-1}}. \tag{17}$$

Combined with Lemma 2, Eq. (17) is equivalent to

$$\bar{\boldsymbol{x}}_i(t+1) = \boldsymbol{J}_i \boldsymbol{X}(t), \tag{18}$$

where $\boldsymbol{J}_i = \boldsymbol{R}_i \left(\boldsymbol{I}_{k^{i-1}} \otimes \mathbf{1}_k \otimes \boldsymbol{I}_{k^{n-i}}\right)$, $\boldsymbol{X}(t) \in \Delta_{k^n}$.

Based on the above, the one-step evolutionary equation can be expressed as $\bar{\boldsymbol{X}}(t+1)$, satisfying

$$\bar{\boldsymbol{X}}(t+1) = \boldsymbol{J}\boldsymbol{X}(t), \tag{19}$$

where $\mathrm{Col}_j(\boldsymbol{J}) = \mathrm{Col}_j(\boldsymbol{J}_1) \ltimes \mathrm{Col}_j(\boldsymbol{J}_2) \ltimes \ldots \ltimes \mathrm{Col}_j(\boldsymbol{J}_n)$, $\boldsymbol{X}(t) \in \Delta_{k^n}$, $j = 1, 2, \ldots, k^n$. It is worth noting that $J_{p,q} = 1$ and $\mathrm{Col}_q(\boldsymbol{J}) \neq \boldsymbol{\delta}_{k^n}^p$ mean that the strategy profile $\boldsymbol{X}(t) = \boldsymbol{\delta}_{k^n}^q$ can evolve to the strategy profile $\boldsymbol{\delta}_{k^n}^p$ with a positive probability.

Theorem 3 Consider the EMOG with strategy updating rule (16) and all players simultaneously update their strategies. The number of Pareto equilibria n_{p} satisfies

$$n_{\mathrm{p}} = \mathrm{trace}(\boldsymbol{J}).$$

Proof First, from Eq. (16) it can be verified that $O_i(t)$ contains all Pareto-efficient strategies for player i, $i \in N$. The details are as follows:

Case 1: $|Q_i(t)| = k-1$. The elements in $Q_i(t)$ satisfy Definition 3. Consequently, $\boldsymbol{x}_i(t)$ is the Pareto-efficient strategy for player i.

Case 2: $|Q_i(t)| + |P_i(t)| = k-1$, $|Q_i(t)| < k-1$. Here, $\Delta_k \backslash \{\boldsymbol{x}_i(t)\} = Q_i(t) \cup P_i(t)$ and $P_i(t) \neq \varnothing$. From the definition of $P_i(t)$, it is clear that $P_i(t)$ contains all Pareto-efficient strategies against the strategy profile at time t.

If the two cases above are not satisfied, then $\Delta_k \backslash \{\boldsymbol{x}_i(t)\} = Q_i(t) \cup P_i(t) \cup P_i'(t)$, where $P_i'(t) = \{\boldsymbol{x}_i | f_i(t, \boldsymbol{x}_i) = \mathbf{0}_{k_i}$, or $f_i(t, \boldsymbol{x}_i)$ has both positive and negative elements$\}$. According to Definition 3, it can be derived that the strategy $\boldsymbol{x}_i \in P_i'(t)$ is a Pareto-efficient strategy against $(\boldsymbol{x}_i, \boldsymbol{x}_{-i}(t))$, and the strategy $\boldsymbol{x}_i(t)$ is also a Pareto-efficient strategy against $(\boldsymbol{x}_i(t), \boldsymbol{x}_{-i}(t))$. Thus, combining with case 2, we find that the set $\Delta_k \backslash Q_i(t) = P_i(t) \cup P_i(t) \cup \{\boldsymbol{x}_i(t)\}$ consists of all Pareto-efficient strategies in this case.

In addition, for any $r \in \{1, 2, \ldots, k^n\}$, $J_{r,r} = 1$ demonstrates that if $\boldsymbol{X}(t) = \boldsymbol{\delta}_{k^n}^r$, then one of the feasible strategy profiles at time $t+1$ is $\boldsymbol{\delta}_{k^n}^r$ based on Eq. (19). Meanwhile, Eq. (19) is obtained by equivalent transformation of Eq. (17) based on Lemma 2, and Eq. (17) relies on strategy updating rule (16). Thus, combining Definitions 3 and 4, we find that the strategy profile $\boldsymbol{\delta}_{k^n}^r$, where every player takes the Pareto-efficient strategy, is a Pareto equilibrium, which is equivalent to $J_{r,r} = 1$, $r \in \{1, 2, \ldots, k^n\}$.

Theorem 4 Consider the EMOG with the one-step evolutionary equation (19). There are $J_{p_*,p_\circ}^s$ possible processes to reach the strategy profile $\boldsymbol{\delta}_{k^n}^{p_*}$ from $\boldsymbol{\delta}_{k^n}^{p_\circ}$ at step s with a positive probability, $s \in \mathbb{N}_+$.

Proof According to Eq. (19) and $\boldsymbol{J} \in \mathcal{M}_{k^n \times k^n}^{\mathrm{B}}$, we have that $J_{p,q} = 1$ holds if and only if the strategy profile $\boldsymbol{\delta}_{k^n}^q$ can evolve to the strategy profile $\boldsymbol{\delta}_{k^n}^p$ at step 1 with a positive probability. Note that the k^n strategy profiles can be regarded as k^n nodes. In this way, the matrix $\boldsymbol{J}^{\mathrm{T}}$ represents the adjacency matrix of the directed graph consisting of these k^n nodes. Next, mathematical induction is used to prove this theorem.

(i) Theorem 4 is obviously true for $s = 1$.

(ii) $s = 2$. $J_{p_*,v} J_{v,p_\circ} = 1$ means that the two-step evolutionary process $\boldsymbol{\delta}_{k^n}^{p_\circ} \rightarrow \boldsymbol{\delta}_{k^n}^v \rightarrow \boldsymbol{\delta}_{k^n}^{p_*}$ exists with a positive probability because $J_{p_*,v} J_{v,p_\circ} = 1$ holds if and only if $J_{v,p_\circ} = J_{p_*,v} = 1$ holds, $v \in \{1, 2, \ldots, k^n\}$. As a result, $J_{p_*,p_\circ}^2 = \sum_{v=1}^{k^n} J_{p_*,v} J_{v,p_\circ}$ is the number of the evolutionary processes from $\boldsymbol{\delta}_{k^n}^{p_\circ}$ to $\boldsymbol{\delta}_{k^n}^{p_*}$ at step 2 with a positive probability.

(iii) Suppose that Theorem 4 is true for $s = t-1 > 2$. Moreover, when $s = t$, the following equation holds:

$$J_{p_*,p_\circ}^t = \left(\boldsymbol{J}^{t-1}\boldsymbol{J}\right)_{p_*,p_\circ} = \sum_{v=1}^{k^n} J_{p_*,v}^{t-1} J_{v,p_\circ},$$

where $J_{v,p_\circ}$ represents the number of evolutionary processes from $\boldsymbol{\delta}_{k^n}^{p_\circ}$ to $\boldsymbol{\delta}_{k^n}^v$ at step 1 with a positive probability and $J_{p_*,v}^{t-1}$ represents the number of evolutionary processes from $\boldsymbol{\delta}_{k^n}^v$ to $\boldsymbol{\delta}_{k^n}^{p_*}$ at step $t-1$ with a positive probability. Thus, $J^{t-1}_{p_*,v}J_{v,p_\mathrm{o}}$ is the number of paths starting from $\boldsymbol{\delta}^{p_\mathrm{o}}_{k^n}$, passing through $\boldsymbol{\delta}^{v}_{k^n}$, and finally reaching $\boldsymbol{\delta}^{p_*}_{k^n}$, $v \in \{1,2,\ldots,k^n\}$. Therefore, $J^{t}_{p_*,p_\mathrm{o}}$ is the number of paths from $\boldsymbol{\delta}^{p_\mathrm{o}}_{k^n}$ to $\boldsymbol{\delta}^{p_*}_{k^n}$ at step t with a positive probability.

Next, we discuss the fact that the EMOG with the one-step evolutionary equation (19) cannot naturally evolve to the target Pareto equilibrium at step s with a positive probability, where $s \in \mathbb{N}_+$ is stated. In this case, motivated by the control theory, we introduce m pseudo-players as external controls, who participate in the EMOG and update their strategies without being constrained by Eq. (16). Specifically, $N^\mathrm{u} = \{1,2,\ldots,m\}$ represents the set of pseudo-players, and S^u_i ($S^\mathrm{u}_i = S_0$) represents the set of strategies for every $i \in N^\mathrm{u}$. $\boldsymbol{u}_j(t) \in \Delta_k$ is the strategy in algebraic form chosen by pseudo-player j at time t, $j \in N^\mathrm{u}$. Furthermore, $\boldsymbol{u}(t) = \ltimes^{m}_{j=1}\boldsymbol{u}_j(t) \in \Delta_{k^m}$. Combined with Lemma 2 and Eqs. (16)–(19), the controlled one-step evolutionary equation can be written as

$$\bar{\boldsymbol{X}}_\mathrm{u}(t+1) = \boldsymbol{J}_\mathrm{u}\boldsymbol{u}(t)\boldsymbol{X}_\mathrm{u}(t), \quad \boldsymbol{X}_\mathrm{u}(t) \in \Delta_{k^n}, \tag{20}$$

where $\boldsymbol{J}_\mathrm{u} \in \mathcal{M}^\mathrm{B}_{k^n \times k^{m+n}}$ and $\bar{\boldsymbol{X}}_\mathrm{u}(t+1)$ is the feasible strategy profile at time $t+1$ after adding m pseudo-players.

For ease of representation, divide $\boldsymbol{J}_\mathrm{u}$ into k^m parts:

$$\boldsymbol{J}_\mathrm{u} = [\boldsymbol{J}_{\mathrm{u},1}, \boldsymbol{J}_{\mathrm{u},2}, \ldots, \boldsymbol{J}_{\mathrm{u},k^m}], \tag{21}$$

where $\boldsymbol{J}_{\mathrm{u},r} \in \mathcal{M}^\mathrm{B}_{k^n \times k^n}$, $r = 1,2,\ldots,k^m$. Similar to Eq. (19), if there exists $j \in N^\mathrm{u}$ such that $(\boldsymbol{J}_{\mathrm{u},j})_{p,q} = 1$ and $\mathrm{Col}_q(\boldsymbol{J}_{\mathrm{u},j}) \neq \boldsymbol{\delta}^{p}_{k^n}$ hold, then strategy profile $\boldsymbol{\delta}^{q}_{k^n}$ can be controlled by $\boldsymbol{\delta}^{j}_{k^m}$ to $\boldsymbol{\delta}^{p}_{k^n}$ at step 1 with a positive probability.

Furthermore, define

$$\bar{\boldsymbol{J}}_\mathrm{u} = [\bar{\boldsymbol{J}}_{\mathrm{u},1}, \bar{\boldsymbol{J}}_{\mathrm{u},2}, \ldots, \bar{\boldsymbol{J}}_{\mathrm{u},k^n}] \in \mathcal{M}^\mathrm{B}_{k^n \times k^n}, \tag{22}$$

where $\bar{\boldsymbol{J}}_{\mathrm{u},l} = \vee^{k^m}_{r=1}\mathrm{Col}_l(\boldsymbol{J}_{\mathrm{u},r})$, $l = 1,2,\ldots,k^n$. Then, the following result holds:

Theorem 5 Consider the EMOG with the controlled one-step evolutionary equation (20). The initial strategy profile $\boldsymbol{X}(0) = \boldsymbol{\delta}^{p_\mathrm{o}}_{k^n}$ can be controlled to the target strategy profile $\boldsymbol{X}(s) = \boldsymbol{\delta}^{p_*}_{k^n}$ at step s with a positive probability if and only if $(\bar{\boldsymbol{J}}^{s}_\mathrm{u})_{p_*,p_\mathrm{o}} > 0$.

Proof In accordance with Eqs. (21) and (22), $(\bar{\boldsymbol{J}}_\mathrm{u})_{p,q} = 1$ indicates that there is at least one control sequence such that the strategy profile can be controlled from $\boldsymbol{\delta}^{q}_{k^n}$ to $\boldsymbol{\delta}^{p}_{k^n}$ at step 1 with a positive probability. Hence, matrix $\bar{\boldsymbol{J}}_\mathrm{u}$ contains all possible evolutionary processes at step 1. Combined with the proof of Theorem 4 and setting $\boldsymbol{J} = \bar{\boldsymbol{J}}_\mathrm{u}$, Theorem 5 holds.

However, although the initial strategy profile can be controlled at step s with a positive probability, it sometimes does not meet the requirements of the problem. To further demonstrate the impacts of pseudo-players, the shortest evolutionary process from the initial strategy profile to the target strategy profile is studied. Algorithm 1 is designed for this purpose and its effectiveness has been proven.

For ease of representation, Algorithm 1 can be summarized in the following four steps:

Step 1: Identify the initial strategy profile $\boldsymbol{\delta}^{p_\mathrm{o}}_{k^n}$ and the target strategy profile $\boldsymbol{\delta}^{p_*}_{k^n}$, initialize the length of the shortest evolutionary process $t = 0$, and

Algorithm 1 Backward search method (to find the shortest evolutionary process and control sequence)

Input: $p_\mathrm{o},\ p_*,\ \boldsymbol{J}_{\mathrm{u},i},\ i = 1,2,\ldots,k^m$
Output: t, $\boldsymbol{P}$, $\boldsymbol{Q}$

Initialize $t = 0,\ \boldsymbol{q} = \boldsymbol{0}^\mathrm{T}_{k^n}$;
while $q_{p_\mathrm{o}} \neq 1$ **do**
  $t = t + 1$;
  for $j = 1 : k^m$ **do**
    for $r = 1 : k^n$ **do**
      $\boldsymbol{\alpha} = \boldsymbol{0}^\mathrm{T}_{k^n}$;
      if $t == 1$ and $(\boldsymbol{J}_{\mathrm{u},j})_{p_*,r} == 1$ **then**
        $\alpha_r = 1$;
      else if $t \neq 1$ and $\{l|(\boldsymbol{J}_{\mathrm{u},j})_{l,r} = 1\} \subset \{l|Q_{t-1,l} = 1\}$ **then**
        $\alpha_r = 1$;
      end if
    end for
    if $j == 1$ **then**
      $\boldsymbol{p} = \boldsymbol{\alpha}$;
    else
      $\boldsymbol{p} = [\boldsymbol{p}; \boldsymbol{\alpha}]$;
    end if
  end for
  for $r = 1 : k^n$ **do**
    $q_r = p_{1,r} \vee p_{2,r} \vee \ldots \vee p_{k^m,r}$;
  end for
  if $t == 1$ **then**
    $\boldsymbol{P} = \boldsymbol{p}$;
    $\boldsymbol{Q} = \boldsymbol{q}$;
  else
    $\boldsymbol{P} = [\boldsymbol{P};\ \boldsymbol{p}]$;
    $\boldsymbol{Q} = [\boldsymbol{Q};\ \boldsymbol{q}]$;
  end if
end while

define $\mathit{\Gamma}(\boldsymbol{x}) = \{r|x_r \neq 0, \boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n, r \in \{1, 2, \ldots, n\}\}$.

Step 2: $t \leftarrow t+1$. Find the strategy profiles that can be controlled by $\boldsymbol{\delta}_{k^m}^{j}$ to $\boldsymbol{\delta}_{k^n}^{p_*}$ at step 1 with probability 1 and denote these strategy profiles by set $Z_j(t)$, that is, $Z_j(t) = \{\boldsymbol{\delta}_{k^n}^{r}|(\boldsymbol{J}_{\mathrm{u},j})_{p_*,r} = 1, r \in \{1, 2, \ldots, k^n\}\}$, $j = 1, 2, \ldots, k^m$. Put all $Z_j(t)$ together as $Z(t)$, that is, $Z(t) = \cup_{j=1}^{k^m} Z_j(t)$.

Step 3: Determine whether $\boldsymbol{\delta}_{k^n}^{p_\mathrm{o}}$ can be controlled to $\boldsymbol{\delta}_{k^n}^{p_*}$ at step t with probability 1. If $\boldsymbol{\delta}_{k^n}^{p_\mathrm{o}} \in Z(t)$ holds, then output t; otherwise, set $t \leftarrow t+1$.

Step 4: Find the strategy profiles that can be controlled to the strategy profiles that are in $Z(t)$ at step 1 with probability 1, and then use these strategy profiles to obtain $Z_j(t)$, that is, $Z_j(t) = \{\boldsymbol{\delta}_{k^n}^{r}|\mathit{\Gamma}(\mathrm{Col}_r(\boldsymbol{J}_{\mathrm{u},j})) \subset Z(t-1)\}$. Return to step 3.

Theorem 6 Consider the EMOG with the controlled one-step evolutionary equation (20). If the game can be controlled from $\boldsymbol{X}(0) = \boldsymbol{\delta}_{k^n}^{p_\mathrm{o}}$ to the target strategy profile $\boldsymbol{\delta}_{k^n}^{p_*}$ within a finite number of steps with probability 1, then Algorithm 1 can be used to find the shortest evolutionary process and control sequence.

Proof Assume that $\boldsymbol{X}(0) = \boldsymbol{\delta}_{k^n}^{p_\mathrm{o}}$ can be controlled to the strategy profile $\boldsymbol{\delta}_{k^n}^{p_*}$ within a finite number of steps with probability 1. In addition, t acts as the output of Algorithm 1, and assume that $(\boldsymbol{X}(0), \boldsymbol{X}(1), \ldots, \boldsymbol{X}(t-1), \boldsymbol{X}(t))$ is the shortest evolutionary process and $(\boldsymbol{u}(0), \boldsymbol{u}(1), \ldots, \boldsymbol{u}(t-1))$ is the control sequence. It is clear that $\boldsymbol{X}(t-1)$ must be able to be controlled to $\boldsymbol{X}(t) = \boldsymbol{\delta}_{k^n}^{p_*}$ at step 1 with probability 1. Hence, as described in step 2, we have

$$\boldsymbol{X}(t-1) \in Z(1) \xrightarrow[\boldsymbol{u}(t-1) \in \{\boldsymbol{\delta}_{k^m}^{j}|Z_j(t) \neq \varnothing\}]{\text{probability 1}} \boldsymbol{X}(t) = \boldsymbol{\delta}_{k^n}^{p_*}.$$

Case 1: $\boldsymbol{\delta}_{k^n}^{p_\mathrm{o}} \in Z(1)$. It is known that $\boldsymbol{\delta}_{k^n}^{p_\mathrm{o}}$ can be controlled to $\boldsymbol{\delta}_{k^n}^{p_*}$ at step 1 with probability 1. In this case, $t = 1$ and $\boldsymbol{u}(0) \in \{\boldsymbol{\delta}_{k^m}^{j}|\mathrm{Col}_{p_\mathrm{o}}(\boldsymbol{J}_{\mathrm{u},j}) = \boldsymbol{\delta}_{k^n}^{p_*}\}$ hold.

Case 2: $\boldsymbol{\delta}_{k^n}^{p_\mathrm{o}} \notin Z(1)$. This case corresponds to step 4. On one hand, if there exist an $s \in \{2, 3, \ldots, t\}$ and a $z_j(s) \in Z_j(s) \subseteq Z(s)$, such that $z_j(s) \in Z_j(s) \subseteq Z(s)$ and $z_j(s) \notin \{\boldsymbol{\delta}_{k^n}^{r}|\mathit{\Gamma}(\mathrm{Col}_r(\boldsymbol{J}_{\mathrm{u},j})) \subset Z(s-1)\}$, the strategy profile $z_j(s)$ cannot be controlled to the strategy profile $z_j(s-1) \in Z(s-1)$ at step 1 with probability 1. Subsequently, $\boldsymbol{X}(t-s) \in Z(s)$ cannot be guaranteed to reach $\boldsymbol{X}(t)$ at step s with probability 1 because $Z(s-1)$ contains all strategy profiles that can be controlled to $\boldsymbol{X}(t)$ at step $s-1$ with probability 1. Therefore, no matter whether $\boldsymbol{X}(t-s)$ is reachable from $\boldsymbol{X}(0)$, it cannot ensure that $\boldsymbol{X}(t)$ is reachable from $\boldsymbol{X}(0)$ with probability 1. On the other hand, if there exists $s \in \{2, 3, \ldots, t\}$ such that $Z_j(s) \subset \{\boldsymbol{\delta}_{k^m}^{j}|\mathrm{Col}_{p_\mathrm{o}}(\boldsymbol{J}_{\mathrm{u},j}) = \boldsymbol{\delta}_{k^n}^{p_*}\}$ holds, it cannot be guaranteed that $Z(s)$ contains all strategy profiles that can be controlled to $\boldsymbol{X}(t)$ at step s with probability 1. As a result, the evolutionary process and control sequence are incomplete.

In addition, Algorithm 1 starts from the target strategy profile $\boldsymbol{\delta}_{k^n}^{p_*}$ and relies on a backward search method to find the set $Z(t)$ that contains all the strategy profiles that can be controlled to $\boldsymbol{\delta}_{k^n}^{p_*}$ at step t with probability 1. Hence, if the initial strategy profile satisfies $\boldsymbol{\delta}_{k^n}^{p_\mathrm{o}} \in Z(t)$ and $\boldsymbol{\delta}_{k^n}^{p_\mathrm{o}} \notin Z(s)$ for $s = 1, 2, \ldots, t-1$, then t is the length of the shortest evolutionary process.

Remark 5 (Complexity analysis) According to Algorithm 1, it is obvious that the complexity depends mainly on lines 2, 4, and 5. Now, the time and space complexities of Algorithm 1 are both $O(k^{2n+m})$.

Example 5 Recall Example 2. We explore its finite-step reachability to the target Pareto equilibrium. According to Eqs. (16) and (17), it can be determined that

$$\bar{\boldsymbol{x}}_i(t+1) = \boldsymbol{R}_i \boldsymbol{X}_{-i}(t), \quad i = 1, 2, 3, 4,$$

where

$$\boldsymbol{R}_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$
$$\boldsymbol{R}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix},$$
$$\boldsymbol{R}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$
$$\boldsymbol{R}_4 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Furthermore, Eqs. (18) and (19) hold with

$$\boldsymbol{J}_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$
$$\boldsymbol{J}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix},$$
$$\boldsymbol{J}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$
$$\boldsymbol{J}_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{J}=\begin{bmatrix}
1&0&0&0&1&0&0&1&0&0&0&1&0&0&0&1\\
0&0&0&1&0&0&0&1&1&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&1&1&0&0&0&1&0&0&0\\
1&0&0&0&1&0&0&1&0&0&0&1&0&0&0&1\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&1&0&0&0&1&0&0&0&1&0\\
0&0&1&0&0&0&1&0&0&1&0&0&0&1&0&0\\
0&1&0&0&0&1&1&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0
\end{bmatrix}.$$

Based on Theorem 3, we find that the number of Pareto equilibria is trace($\boldsymbol{J}$)=6, which is in agreement with Example 4. In this example, we set the length of the evolution process to 3 and $\boldsymbol{X}(0)=\boldsymbol{\delta}_{16}^2\sim(1,1,1,2)$. By calculation, we find that there are $J_{4,2}^3=4$ and $J_{14,2}^3=0$ paths to the strategy profiles $\boldsymbol{\delta}_{16}^4\sim(1,1,2,2)$ and $\boldsymbol{\delta}_{16}^{14}\sim(2,2,1,2)$ with a positive probability, respectively.

Example 6 Recall Example 2 and assume that player 3 is a pseudo-player. Hence, the set of players is $N=\{1,2,4\}$ and the payoffs of players 1,2, and 4 under the strategy profile $\boldsymbol{s}=(s_1,s_2,\ldots,s_8)$ are shown in Table 2. Based on Table 1, we can deduce the payoff matrices of players 1, 2, and 4 under control $\boldsymbol{\delta}_2^1$ and $\boldsymbol{\delta}_2^2$, which are shown in Tables 3 and 4, respectively. Combining Table 2 and Theorem 2, it can be calculated that $\boldsymbol{\delta}_8^2\sim(1,1,2)$ and $\boldsymbol{\delta}_8^8\sim(2,2,2)$ are Pareto equilibria. In this example, the length of the evolution process is still 3, the target strategy profile is $\boldsymbol{\delta}_8^2\sim(1,1,2)$, and $\boldsymbol{X}(0)=\boldsymbol{\delta}_8^4\sim(1,2,2)$. Based on Eq. (19), it can be computed that

$$\bar{\boldsymbol{X}}(t+1)=\boldsymbol{J}\boldsymbol{X}(t),$$

where

$$\boldsymbol{J}=\begin{bmatrix}
0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0\\
0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1
\end{bmatrix}.$$

Then, it can be calculated that $J_{2,4}^3=0$, meaning that the EMOG cannot naturally evolve to $\boldsymbol{\delta}_8^2\sim(1,1,2)$ at step 3. Thus, taking player 3 in Example 2 as a pseudo-player, the controlled one-step evolutionary equation is

$$\bar{\boldsymbol{X}}_{\mathrm{u}}(t+1)=\boldsymbol{J}_{\mathrm{u}}\boldsymbol{u}(t)\boldsymbol{X}_{\mathrm{u}}(t), \tag{23}$$

where

$$\boldsymbol{J}_{\mathrm{u}}=\begin{bmatrix}
1&0&1&0&0&0&0&0&0&0&0&1&0&1&0&1\\
0&0&0&0&1&0&1&0&0&1&0&1&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&1&0&1&0&1&0\\
0&0&0&0&0&1&0&1&1&0&1&0&0&0&0&0
\end{bmatrix}.$$

Furthermore, from Eq. (22) we have

$$\bar{\boldsymbol{J}}_{\mathrm{u}}=\begin{bmatrix}
1&0&1&1&0&1&0&1\\
0&1&0&1&1&0&1&0\\
1&0&1&1&0&1&0&1\\
0&1&0&1&0&0&0&0\\
0&1&1&1&0&0&0&0\\
1&0&1&0&0&0&0&0\\
0&1&1&1&1&0&1&0\\
1&0&1&0&0&1&0&1
\end{bmatrix}.$$

Based on Theorem 5, it can be verified that $(\bar{\boldsymbol{J}}_{\mathrm{u}}^3)_{2,4}=4$, meaning that there are four evolutionary processes that can be controlled from $\boldsymbol{\delta}_8^4$ to $\boldsymbol{\delta}_8^2$ at step 3 with a positive probability.

Next, we find the shortest evolutionary process and control sequence from $\boldsymbol{\delta}_8^4$ to $\boldsymbol{\delta}_8^2$ with probability 1. According to Algorithm 1, the backward search process is shown in Fig. 4. Therefore, the evolutionary processes and control sequences are $\boldsymbol{\delta}_8^4\xrightarrow{\boldsymbol{u}(0)=\boldsymbol{\delta}_2^1}\boldsymbol{\delta}_8^5\xrightarrow{\boldsymbol{u}(1)=\boldsymbol{\delta}_2^1}\boldsymbol{\delta}_8^2$ and $\boldsymbol{\delta}_8^4\xrightarrow{\boldsymbol{u}(0)=\boldsymbol{\delta}_2^1}\boldsymbol{\delta}_8^7\xrightarrow{\boldsymbol{u}(1)=\boldsymbol{\delta}_2^1}\boldsymbol{\delta}_8^2$.

4 Conclusions

This paper has investigated the existence, finite-step reachability, and finite-step controllability of Pareto equilibria for MOGs. STP was used to convert the payoff function into its algebraic form, which is more concise and facilitates computations. The main results have been presented through static and dynamic analyses. In the static analysis, two necessary and sufficient conditions were proposed to test

Table 2 Payoffs of players 1, 2, and 4 in Example 6

	$\boldsymbol{c}_i(\boldsymbol{s})$							
	$\boldsymbol{s}_1=(1,1,1)$	$\boldsymbol{s}_2=(1,1,2)$	$\boldsymbol{s}_3=(1,2,1)$	$\boldsymbol{s}_4=(1,2,2)$	$\boldsymbol{s}_5=(2,1,1)$	$\boldsymbol{s}_6=(2,1,2)$	$\boldsymbol{s}_7=(2,2,1)$	$\boldsymbol{s}_8=(2,2,2)$
c_1	5	8	2	3	1	6	4	5
$\boldsymbol{c}_2$	$[4,8]^{\mathrm{T}}$	$[6,5]^{\mathrm{T}}$	$[2,7]^{\mathrm{T}}$	$[4,3]^{\mathrm{T}}$	$[2,1]^{\mathrm{T}}$	$[5,3]^{\mathrm{T}}$	$[5,4]^{\mathrm{T}}$	$[6,6]^{\mathrm{T}}$
c_4	3	6	7	7	10	4	8	9

The three values of $\boldsymbol{s}_i$ in the brackets represent the corresponding selected strategies of the three players

Table 3 Payoffs of players 1, 2, and 4 under control δ_2^1 in Example 6

	$\boldsymbol{c}_i(\boldsymbol{s})$							
	$\boldsymbol{s}_1=(1,1,1)$	$\boldsymbol{s}_2=(1,1,2)$	$\boldsymbol{s}_3=(1,2,1)$	$\boldsymbol{s}_4=(1,2,2)$	$\boldsymbol{s}_5=(2,1,1)$	$\boldsymbol{s}_6=(2,1,2)$	$\boldsymbol{s}_7=(2,2,1)$	$\boldsymbol{s}_8=(2,2,2)$
c_1	7	7	8	8	3	9	5	2
$\boldsymbol{c}_2$	$[5,4]^{\mathrm{T}}$	$[3,2]^{\mathrm{T}}$	$[3,9]^{\mathrm{T}}$	$[6,1]^{\mathrm{T}}$	$[5,3]^{\mathrm{T}}$	$[6,2]^{\mathrm{T}}$	$[1,3]^{\mathrm{T}}$	$[8,4]^{\mathrm{T}}$
c_4	9	2	10	7	2	3	7	12

The three values of $\boldsymbol{s}_i$ in the brackets represent the corresponding selected strategies of the three players

Table 4 Payoffs of players 1, 2, and 4 under control δ_2^2 in Example 6

	$\boldsymbol{c}_i(\boldsymbol{s})$							
	$\boldsymbol{s}_1=(1,1,1)$	$\boldsymbol{s}_2=(1,1,2)$	$\boldsymbol{s}_3=(1,2,1)$	$\boldsymbol{s}_4=(1,2,2)$	$\boldsymbol{s}_5=(2,1,1)$	$\boldsymbol{s}_6=(2,1,2)$	$\boldsymbol{s}_7=(2,2,1)$	$\boldsymbol{s}_8=(2,2,2)$
c_1	1	9	2	4	7	6	8	3
$\boldsymbol{c}_2$	$[7,4]^{\mathrm{T}}$	$[10,4]^{\mathrm{T}}$	$[9,2]^{\mathrm{T}}$	$[9,6]^{\mathrm{T}}$	$[3,5]^{\mathrm{T}}$	$[5,2]^{\mathrm{T}}$	$[5,7]^{\mathrm{T}}$	$[3,6]^{\mathrm{T}}$
c_4	5	8	5	5	7	3	6	4

The three values of $\boldsymbol{s}_i$ in the brackets represent the corresponding selected strategies of the three players

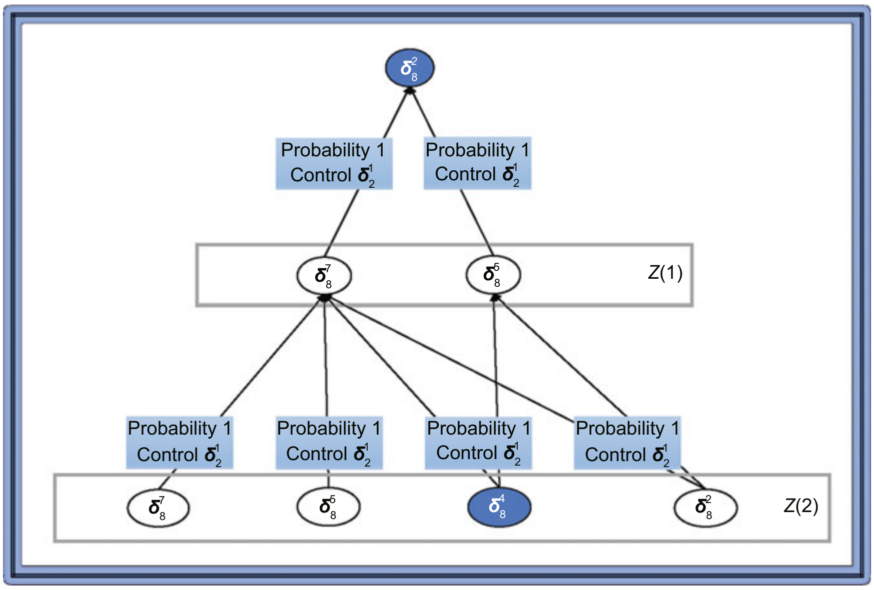

Fig. 4 The backward search process of Example 6

whether all players can meet their expectations and whether the strategy profile is a Pareto equilibrium, separately. In the dynamic analysis, a strategy updating rule was introduced, and based on that rule, the one-step evolutionary equation was obtained and used to find the next feasible strategy profiles. Meanwhile, conclusions regarding the number of Pareto equilibria and the finite-step reachability were provided. Particularly, motivated by the control theory, a finite number of pseudo-players were used as external controls, and the finite-step controllability of the EMOG was researched. To find the shortest evolutionary process and control sequence, a backward search algorithm was designed and its effectiveness was proved.

MOGs with network topology structure and unknown disturbances can describe more complex situations in reality. Thus, further work on equilibrium analysis for this type of game is promising.

Contributors

Fanyueyang ZHANG and Jun'e FENG designed the research. Fanyueyang ZHANG processed the data and drafted the paper. Jun'e FENG helped organize the paper, and revised and finalized the paper.

Conflict of interest

Both authors declare that they have no conflict of interest.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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