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Reconstruction algorithm in lattice-invariant signal spaces

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Abstract: In this paper, we mainly pay attention to the weighted sampling and reconstruction algorithm in lattice-invariant signal spaces. We give the reconstruction formula in lattice-invariant signal spaces, which is a generalization of former results in shift-invariant signal spaces. That is, we generalize and improve Aldroubi, Gröchenig and Chen's results, respectively. So we obtain a general reconstruction algorithm in lattice-invariant signal spaces, which the signal spaces is sufficiently large to accommodate a large number of possible models. They are maybe useful for signal processing and communication theory.

Key words: Lattice-invariant space, Reconstruction algorithm, Irregular sampling
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INTRODUCTION

In digital signal and image processing, digital communication, etc., a continuous signal is usually represented and processed by using its discrete samples. A finite energy bandlimited signal is completely characterized by its samples, and described by the famous classical Shannon sampling theorem with wide application in signal processing and communication theory.

However, in many real applications sampling points are not always regular. If a weighted sampling is considered, the system will be made to be more efficient. It is well known that in the sampling and reconstruction problem of non-bandlimited spaces, the signal is often assumed to belong to shift-invariant spaces (Aldroubi, 2002; Aldroubi and Feichtinger, 1998; Aldroubi and Gröchenig, 2000; 2001; Chen *et al*., 2002; Sun and Zhou, 2002; Xian and Lin, 2004; Xian and Qiang, 2003; Luo and Lin, 2004; Xian *et al*., 2004).

Clearly we hope signal spaces be sufficiently large to accommodate a large number of possible models. So Feichtinger introduced lattice-invariant space of the form $V_m^p(\varphi)=\left\{\sum_{k\in R^d} c(k)\varphi(\cdot-\boldsymbol{L}k): c\in\ell_m^p\right\}$ in (Aldroubi and Feichtinger, 2002), where $\boldsymbol{L}$ is a $d\times d$ non-singular matrix, $1<p<\infty$. The matrix $\boldsymbol{L}$ transforms the lattice Z^d into the lattice $\wedge$. When $\boldsymbol{L}$ is the identity matrix, we can obtain the standard shift-invariant spaces. Feichtinger pointed out that a combination of generator φ (e.g. radial symmetric ones) with suitable lattice $\wedge$ (related to sphere packing) is a good alternative for the usual voxel representation of volume data also observed in (Lewitt, 1992). So the lattice-invariant spaces are a sufficiently large value family of signal spaces. We will show the reconstruction formula for the lattice-invariant spaces. This result is generalized and improved form of Chen's result in (Chen *et al*., 2002). At the same time, my results are a generalization of Proposition 1(iv) in (Aldroubi and Gröchenig, 2000) and Eq.(5.1) in (Aldroubi and Gröchenig, 2001).

RECONSTRUCTION IN LATTICE-INVARIANT SPACES

Let X be a countable separated index set in R^d, that is, $\inf_{x,y\in X,x\neq y}|x-y|\geq\delta>0$. A weight is a non-negative function on R^d which we may assume to be continuous. A weight m is called polynominaly s-moderate, if there are constant C, $s>0$ such that $m(t+x)\le C(1+|t|)^s m(x)$ for all $t,x\in R^d$. It is also called Sobolev weight. The weighted ℓ^p -space $\ell_m^p(Z^d)$ is defined by the norm $\|c\|_{\ell_m^p}=(\sum|c(x)|^p m(x)^p)^{1/p}$ with the usual modification for $p=\infty$. A function f belongs to $L_m^p(R^d)$ with weight function m if mf belong to $L_m^p(R^d)$. Equipped with the norm $\|f\|_{L_m^p}=\|mf\|_{L^p}$, $L_m^p(R^d)$ is a Banach space. Weighted lattice invariant spaces can be described as:

$$V_m^p(\varphi)=\left\{\sum_{k\in R^d} c(k)\varphi(\cdot-\boldsymbol{L}k): c\in\ell_m^p\right\},$$

where φ is a suitable generator, $\boldsymbol{L}$ is a $d\times d$ non-singular matrix, $1<p<\infty$. When $\boldsymbol{L}$ is the identity matrix, we obtain the standard shift-invariant spaces.

We impose the following standard assumptions on the generator φ:

(i) $\{\varphi(\cdot-\boldsymbol{L}k):k\in Z^d\}$ form a Riesz basis for $V^2(\varphi)$.

(ii) φ is continuous.

(iii) φ satisfies the decay condition $|\varphi|\le C(1+|x|)^{-d-s-\varepsilon}$ for any $s>d$ and some $\varepsilon>0$.

Since the Riesz basis $\{\varphi(\cdot-\boldsymbol{L}k):k\in Z^d\}$ is lattice-invariant, the dual basis must again be of the form $\{\tilde\varphi(\cdot-\boldsymbol{L}k):k\in Z^d\}$. The dual $\tilde\varphi$ satisfies the relation $\langle\varphi(\cdot-\boldsymbol{L}k),\tilde\varphi(\cdot-\boldsymbol{L}l)\rangle=\delta_{kl}$. So $V^2(\varphi)$ is a reproducing kernel Hilbert space. Its kernel functions are given explicitly by $k_x(y)=\sum_{k\in Z^d}\overline{\varphi(x-\boldsymbol{L}k)}\tilde\varphi(y-\boldsymbol{L}k)$.

If $\sum_{x\in X}|f(x)|^2\approx\|f\|^2$ is satisfied for $X\subset R^d$, then X is called a set of sampling for $V^2(\varphi)$. That is, it is equivalent to saying that $\{k_x:x\in X\}$ is a frame for $V^2(\varphi)$.

Lemma 1 (Gröchenig, 2004) If $X\subset R^d$ is separated, then for any $s>d$, $\sup_{v\in R^d}\sum_{x\in X}(1+|x-v|)^{-s}=C_s<\infty$.

Corollary 1 If $X\subset R^d$ is separated, then for any $s>d$, $\sup_{z\in Z^d}\sum_{x\in X}(1+|x-\boldsymbol{L}z|)^{-s}=C_s'<\infty$, where $\boldsymbol{L}$ is matrix.

Lemma 2 If $X\subset R^d$ is separated and $s>d$, then $\sum_{x\in X}(1+|x-\boldsymbol{L}n|)^{-s}(1+|x-\boldsymbol{L}m|)^{-s}\le C(1+|\boldsymbol{L}(n-m)|)^{-s}$, $(\forall m,n\in Z^d)$.

Proof By Corollary 1, we can easily obtain Lemma 2.

Lemma 3 Assume that $d\times d$ matrix $\boldsymbol{L}$, that weight function m satisfies $m(\boldsymbol{L}n)\le m(n)$ for all $n\in Z^d$ and that X is separated. Let $A_{kn}=(1+|\boldsymbol{L}n|)^{-d-s-\varepsilon}$ for some $\varepsilon>0$, $s>d$ ($n\in Z^d$, $x\in X$). Then the operator A defined on finite sequences $(c_n)_{n\in Z^d}$ by matrix multiplication $(Ac)_x=\sum_{n\in Z^d}A_{xn}c_n$ is a bounded operator from $\ell_m^p(Z^d)$ to $\ell_m^p(X)$ for all $p\in(1,\infty)$ and all s-moderate weight m.

Remark Some methods of (Bergh and Löfström, 1976) are used in the proof of Lemma 3.

The following Theorem 1 is Jaffard's Theorem.

Theorem 1 (Jaffard's Theorem (Gröchenig, 2004; Jaffard, 1990; Lewitt, 1992)) Assume that the matrix $\boldsymbol{G}=(G_{kl})_{k,l\in Z^d}$ satisfies the following properties:

(a) $\boldsymbol{G}$ is invertible as an operator on $L^2(Z^d)$, and

(b) $|G_{kl}|\le C(1+|\boldsymbol{L}(k-l)|^r$, $k,l\in Z^d$ for some constant $C>0$, $d\times d$ matrix and some $r>d$. Then the inverse matrix $\boldsymbol{H}=\boldsymbol{G}^{-1}$ satisfies the same decay, that is,

$$|H_{kl}|\le C(1+|\boldsymbol{L}(k-l)|^r,\ k,l\in Z^d.$$

Lemma 4 Given $1\le p<\infty$, $s\ge 0$ and polynominaly s-moderate weight m. Assume that the generator φ satisfies the assumptions (i)~(iii), and that $X=\{x_j\}$ is a set of sampling for $V^2(\varphi)$.

(a) Then the coefficient operator defined by $Cf=(\langle f,k_{x_j}\rangle)_{x_j\in X}$ is bounded from $V_m^p(\varphi)$ to $\ell_m^p(X)$.

(b) The synthesis operator defined by $Dc=\sum c_j k_{x_j}$ extend to a bounded mapping from $\ell_m^p(X)$ to $V_m^p(\varphi)$.

(c) The frame operator $S=\sum_{x_j\in X}\langle f,k_{x_j}\rangle k_{x_j}$ maps $V_m^p(\varphi)$ into $V_m^p(\varphi)$, and the series converges unconditionally for $1\le p<\infty$.

Proof It is easy to prove (a) and (b) from results of (Aldroubi and Gröchenig, 2001) and Lemma 3.

In the following, we will prove (c). The boundedness of $S=DC$ follows by combining (a) and (b). As for the unconditional convergence of the series, let

$\varepsilon>0$ and choose $N_0=N_0(\varepsilon)$ such that $\left\|\langle f,k_x\rangle_{x\notin N_0}\right\|_{\ell_m^p}\le\varepsilon$.

Then for any finite set $N_1\supset N_0$, assertions (a) and (b) imply that

$$\left\|Sf-\sum_{x\in N_1}\langle f,k_x\rangle k_x\right\|_{L_m^p}\le\|D\|_{op}\left\|\langle f,k_x\rangle_{x\notin N_1}\right\|_{\ell_m^p}\le\varepsilon\|D\|_{op}$$

This means that $\sum_{x_j\in X}\langle f,k_{x_j}\rangle k_{x_j}$ converges unconditionally in $V_m^p(\varphi)$.

Lemma 5 If $\{\varphi(\cdot-\boldsymbol{L}k):k\in Z^d\}$ is a Riesz basis for $V^2(\varphi)$ and satisfies $|\varphi|\le C(1+|\boldsymbol{L}x|^{-r})$ for some $r>d$, then the dual generator satisfies $\tilde{\varphi}(x)|\le C'(1+|\boldsymbol{L}x|)^{-r}$.

Proof For every $f\in V^2(\varphi)$, we have the series expansion $f=\sum_{k\in Z^d}\langle f,\tilde{\varphi}(\cdot-\boldsymbol{L}k)\rangle\varphi(\cdot-\boldsymbol{L}k)$.

Since $\tilde{\varphi}\in V^2(\varphi)$, it has the series expansion $\tilde{\varphi}=\sum_{k\in Z^d}b_k\varphi(\cdot-\boldsymbol{L}k)$. The coefficients b_n are determined by the biorthogonality condition

$$\langle\tilde{\varphi}(\cdot-\boldsymbol{L}k),\varphi(\cdot-\boldsymbol{L}l)\rangle=\delta_{kl},$$
$$\delta_{l0}=\langle\tilde{\varphi},\varphi(\cdot-\boldsymbol{L}l)\rangle=\sum_{m\in Z^d}b_m\langle\varphi,\varphi(\cdot-\boldsymbol{L}(l-m))\rangle.$$

This convolution can be written with the (infinite) matrix $\boldsymbol{\Phi}$ with entries $\Phi_{lm}=\langle\varphi,\varphi(\cdot-\boldsymbol{L}(l-m))\rangle=\gamma_{l-m}$. The assumption on the decay of φ and Lemma 2 imply that $|\Phi_{lm}|\le C(1+|\boldsymbol{L}(l-m)|)^{-r}$, $l,m\in Z^d$. Since $\{\varphi(\cdot-\boldsymbol{L}k):k\in Z^d\}$ is a Riesz basis for matrix $\boldsymbol{\Phi}$ is invertible on $\ell^2(Z^d)$, and $\boldsymbol{\Phi}^{-1}$ is again a convolution with a sequence β. Since $r>d$, Jaffard Theorem yields the decay estimate $|(\boldsymbol{\Phi}^{-1})_{lm}|=|\beta_{l-m}|\le C'(1+|\boldsymbol{L}(l-m)|)^{-r}$.

Consequently $b=\boldsymbol{\Phi}^{-1}\delta=\beta*\delta=\beta$. Thus $|b_l|\le C(1+|\boldsymbol{L}l|)^{-r}$, $l\in Z^d$, and invoking Lemma 2 once again we obtain that $|\tilde{\varphi}(x)|\le C'(1+|\boldsymbol{L}x|)^{-r}$.

Lemma 6 If the generator φ satisfies (i)~(iii), then $|\langle k_x,\varphi(\cdot-\boldsymbol{L}k)\rangle|\le C(1+|x-\boldsymbol{L}k|)^{-s}$ and $|\langle k_x,\tilde{\varphi}(\cdot-\boldsymbol{L}k)\rangle|\le C(1+|x-\boldsymbol{L}k|)^{-s}$ with decay $s>d$. Moreover, $|T_{mn}|\le C(1+|\boldsymbol{L}(m-n)|)^{-s}$ for $m,n\in Z^d$, where $T_{mn}=\sum_{x\in X}\langle\varphi(\cdot-\boldsymbol{L}n),k_x\rangle\langle k_x,\tilde{\varphi}(\cdot-\boldsymbol{L}m)\rangle$ $=\langle S\varphi(\cdot-\boldsymbol{L}n),\tilde{\varphi}(\cdot-\boldsymbol{L}m)\rangle$.

Proof Using the decay properties of generator φ and $\tilde{\varphi}$ (Lemma 5 with $r=s+d+\varepsilon$) and $f(x)=\langle f,k_x\rangle$, we estimate $|\langle k_x,\varphi(\cdot-\boldsymbol{L}k)\rangle|=|\varphi(x-\boldsymbol{L}k)|\le C(1+|x-\boldsymbol{L}k|^{-s}$ and $|\langle k_x,\tilde{\varphi}(\cdot-\boldsymbol{L}k)\rangle|=|\tilde{\varphi}(x-\boldsymbol{L}k)|\le C(1+|x-\boldsymbol{L}k|)^{-s}$.

By the above discussions and Lemma 2, we have

$$\begin{aligned}|T_{mn}|&\le\sum_{x\in X}|\langle k_x,\varphi(\cdot-\boldsymbol{L}n)\rangle||\langle k_x,\tilde{\varphi}(\cdot-\boldsymbol{L}m)\rangle|\\&\le\sum_{x\in X}(1+|x-\boldsymbol{L}n|)^{-s}(1+|x-\boldsymbol{L}m|)^{-s}\\&\le C(1+|\boldsymbol{L}(m-n)|)^{-s}\ (\forall m,n\in Z^d).\end{aligned}$$

Theorem 2 Assume that the generator φ satisfies the assumptions (i)~(iii), that m is polynominaly s-moderate, and that X is a set of sampling for $V^2(\varphi)$ with dual frame $\tilde{k}_x$.

(a) Then we have for every $f\in V_m^p(\varphi)$ ($1\le p\le\infty$),

$$A_p\|f\|_{L_m^p}\le\left(\sum_{x\in X}|f(x)|^p\,m(x)^p\right)^{1/p}\le B_p\|f\|_{L_m^p}.$$

(b) Each $\tilde{k}_x$ satisfies the following estimate $|\tilde{k}_x(t)|\le C(1+|t-x|)^{-d-s-\varepsilon}$ for all $x\in X$, $t\in Z^d$, with a constant C independent of x.

(c) The reconstruction formula $f=\sum_{x\in X}f(x)\tilde{k}_x$ holds for any $f\in V_m^p(\varphi)$ and the reconstruction series converges unconditionally in $f\in V_m^p(\varphi)$ ($1\le p\le\infty$).

Proof At first, we will show the frame operator S is invertible on $V_m^p(\varphi)$, where $1\le p\le\infty$ and m is an s-moderate weight.

Let $T_{mn}=\sum_{x_j\in X}\langle\varphi(\cdot-\boldsymbol{L}n),k_{x_j}\rangle\langle k_{x_j},\tilde{\varphi}(\cdot-\boldsymbol{L}m)\rangle=$ $\langle S\varphi(\cdot-\boldsymbol{L}n),\tilde{\varphi}(\cdot-\boldsymbol{L}m)\rangle$ and $(\Gamma f)_n=\langle f,\tilde{\varphi}(\cdot-\boldsymbol{L}n)\rangle$.

It is easy to show that T is invertible. Then from Jaffard's Theorem we conclude that the entries of the inverse matrix also satisfy $(T^{-1})_{mn}\le C(1+|\boldsymbol{L}(m-n)|)^{-s-d-\varepsilon}$.

By Lemma 3, T^{-1} is a bounded operator on sequence spaces $\ell_m^p(Z^d)$. Since T is invertible and $S=\Gamma^{-1}T\Gamma$, S is invertible on $V_m^p(\varphi)$.

Since $f=S^{-1}Sf=S^{-1}DCf$, we have

$$\|f\|_{L_m^p} \le \|S^{-1}\|_{op} \|D\|_{op} \|C\|_{op} \|f\|_{L_m^p} .$$

By the definition of norm $\|\cdot\|_{\ell_m^p(X)}$ and operator C, we can obtain the following result from the above inequalities:

$$A\|f\|_{L_m^p} \le \left(\sum_{x\in X} |f(x)|^p \; m(x)^p \right)^{1/p} \le B\|f\|_{L_m^p} ,$$

where $A = \dfrac{1}{\|S^{-1}\|_{op} \|D\|_{op}}$, $B = \|C\|_{op}$.

(b) By $\tilde{k}_x = S^{-1}k_x$, we know $|\langle \tilde{k}_x, \varphi(\cdot - \boldsymbol{L}n)\rangle| = |\langle S^{-1}k_x, \varphi(\cdot - \boldsymbol{L}n)\rangle| = |\langle k_x, S^{-1}\varphi(\cdot - \boldsymbol{L}n)\rangle| \le C_2(1+|x-\boldsymbol{L}n|)^{-d-s-\varepsilon}$.

Similarly $|\langle k_t, \varphi(\cdot - \boldsymbol{L}k)\rangle| \le C(1+|t-\boldsymbol{L}k|)^{-d-s-\varepsilon}$. So $|\tilde{k}_x(t)| = \sum_{k\in Z^d} \left|\langle \tilde{k}_x, \varphi(\cdot - \boldsymbol{L}k)\rangle\right| \left|\langle \tilde{\varphi}(\cdot - \boldsymbol{L}k), k_t\rangle\right| \le C_2(1+|x-t|)^{-d-s-\varepsilon}$.

(c) Since the series $\sum_{x\in X} \langle f, k_x\rangle k_x$ converges unconditionally by Lemma 4(c), the series $S^{-1}\left(\sum_{x\in X} \langle f, k_x\rangle k_x\right)$ also converges unconditionally. So $f = \sum_{x\in X} \langle f, k_x\rangle S^{-1}k_x = \sum_{x\in X} f(x)\tilde{k}_x$ and the reconstruction series $\sum_{x\in X} f(x)\tilde{k}_x$ converges unconditionally in $V_m^p(\varphi)$ for $1<p<\infty$.

In contrast to sampling theorem in (Aldroubi, 2002; Aldroubi and Gröchenig, 2000; 2001; Chen *et al*., 2002), lattice-invariant space $V_m^p(\varphi)$ is more common than shift-invariant spaces, and the sampling density in lattice-invariant space $V_m^p(\varphi)$ is determined entirely by the required density in the Hilbert spaces $V^2(\varphi)$ (and suitable decay of φ), and the space is high dimension.

CONCLUSION

In this paper, we mainly pay attention to the weighted sampling and reconstruction in lattice-invariant subspaces. We give the reconstruction formula in lattice-invariant spaces, which is generalization of results in shift-invariant spaces (Aldroubi, 2002; Aldroubi and Gröchenig, 2000; 2001; Chen *et al*., 2002). Due to the limitation of paper space, we omit some numerical examples and detail of proofs of lemmas and theorems.

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