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Application of Hamiltonian system for two-dimensional transversely isotropic piezoelectric media*

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Abstract: This paper presents a symplectic method for two-dimensional transversely isotropic piezoelectric media with the aid of Hamiltonian system. A symplectic system is established directly by introducing dual variables and a complete space of eigensolutions is obtained. The solutions of the problem can be expressed by eigensolutions. Some solutions, which are local and are neglected usually by Saint Venant principle, are shown. Curves of non-zero-eigenvalues and their eigensolutions are given by the numerical results.

Key words: Hamiltonian system, Piezoelectric media, Eigensolution

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INTRODUCTION

The traditional method of solving solutions of the transversely isotropic piezoelectric media belongs to Lagrange formulation so that the method of separation of variables cannot be applied due to the involvement of high order partial differentiation in the Euclidian space and is difficult for some problems. New theoretical system or method is a key for the investigation of piezoelectric materials.

Since the foundational equations (Sosa and Pak, 1990) in transversely isotropic piezoelectric materials are as similar as ones in elasticity, the method of elasticity can be generalized to the linear piezoelectric problem. Tzou (1993) and Tzou *et al.*(2002) presented shells method and Dunn and Wienecke (1996) presented Green's functions method based on the linear theory of piezoelectricity. Ding *et al.*(1996; 2002) and Ding and Chen (2001) conducted many researches on the exact analyses of three-dimensional piezoelasticity problems and obtained some solutions. Chen (1999) and Chen and Ding (2004) discussed crack and contact problems in terms of potential theory method. The theories, models and methods advanced the developing of the subject. However the researches above are under the Lagrange system. Zhong (1995) introduced a symplectic space method based on the conservative Hamiltonian system to solve the elastic problem, which is different from the traditional semi-inverse solution method. This paper introduces a symplectic method to piezoelasticity problems, which is then reduced to solving eigenvalues and deriving eigensolutions.

HAMILTONIAN SYSTEM

Consider a homogeneous anisotropic piezoelectric medium of the strip plane-domain, which is transversely isotropic and the longitudinal direction is anisotropic. The Cartesian coordinate (x, z) is selected

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such that the z-axis is along the longitudinal direction with origin at the central point of the cross section ($x=\pm a$) where $2a$ is the width of the strip. Let σ_{ij}, D_i, and $\boldsymbol{q}=\{u, w, \varphi\}^{\mathrm{T}}$ be the components of stress, electric displacement and displacements (φ is the electrical potential function) respectively. The linear relations (Sosa and Pak, 1990) of piezoelectricity can be described as

$$\begin{cases} \sigma_x = c_{11}\partial_x u + c_{13}\dot{w} + e_{31}\dot{\varphi}; \\ \sigma_{xz} = c_{44}(\partial_x w + \dot{u}) + e_{15}\partial_x \varphi; \\ \sigma_{zz} = c_{13}\partial_x u + c_{33}\dot{w} + e_{33}\dot{\varphi} \\ D_x = e_{15}(\partial_x w + \dot{u}) - \varepsilon_{11}\partial_x \varphi; \\ D_z = e_{31}\partial_x u + e_{33}\dot{w} - \varepsilon_{33}\dot{\varphi} \end{cases} \tag{1}$$

where c_{ij}, e_{ij} and ε_{ij} are elastic stiffness, piezoelectric and dielectric constants, respectively, and the over-dot represents differential with respect to z, namely $\dot{f} = (\partial / \partial z) f$, which z coordinate is taken in analogy to the time coordinate. And define $f'=\partial_x f=\partial f/\partial x$. The Lagrange function, the potential energy density and work, are

$$\begin{aligned} L(u,w,\varphi) = {} & [c_{11}(u')^2 + c_{33}\dot{w}^2 + 2c_{13}u'\dot{w} + c_{44}(w' + \dot{u})^2 \\ & + 2e_{31}u'\dot{\varphi} + 2e_{33}\dot{w}\dot{\varphi} + 2e_{15}(w' + \dot{u})\varphi' \\ & - \varepsilon_{11}(\varphi')^2 - \varepsilon_{33}\dot{\varphi}^2]/2 - u f_x - w f_z + \varphi q \end{aligned} \tag{2}$$

where $\{f_x, f_z\}^{\mathrm{T}}$ are the external body forces and q the density of free charges. The dual vector $\boldsymbol{p}$ of the Hamiltonian system should be introduced first as

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \begin{Bmatrix} c_{44}(w' + \dot{u}) + e_{15}\varphi' \\ c_{13}u' + c_{33}\dot{w} + e_{15}\dot{\varphi} \\ e_{31}u' + e_{33}\dot{w} - \varepsilon_{33}\dot{\varphi}] \end{Bmatrix} = \begin{Bmatrix} \sigma_{xz} \\ \sigma_{zz} \\ D_z \end{Bmatrix} \tag{3}$$

Explicitly, the physical meaning of the dual variable $\boldsymbol{p}$ are the stresses and electric displacement in the z-direction. Based on the mutually dual vectors $\boldsymbol{q}$ and $\boldsymbol{p}$, the Hamiltonian function and the variational equation can be introduced as

$$\begin{aligned} & H(\boldsymbol{q}, \boldsymbol{p}) = \boldsymbol{p}^{\mathrm{T}}\dot{\boldsymbol{q}} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ & \delta \int_{\Omega} [\boldsymbol{p}^{\mathrm{T}}\dot{\boldsymbol{q}} - H(\boldsymbol{q}, \boldsymbol{p})]\, \mathrm{d}\Omega = 0 \end{aligned} \tag{4}$$

The dual equations for the Hamiltonian system can be obtained directly as

$$\begin{aligned} & \begin{bmatrix} \dot{u} & \dot{w} & \dot{\varphi} & \dot{p}_1 & \dot{p}_2 & \dot{p}_3 \end{bmatrix}^{\mathrm{T}} = \\ & \begin{bmatrix} 0 & -\partial_x & -a_1\partial_x & a_2 & 0 & 0 \\ -a_3\partial_x & 0 & 0 & 0 & a_4 & a_5 \\ -a_6\partial_x & 0 & 0 & 0 & a_5 & -a_7 \\ a_8\partial_x^2 & 0 & 0 & 0 & -a_3\partial_x & -a_6\partial_x \\ 0 & 0 & 0 & -\partial_x & 0 & 0 \\ 0 & 0 & a_9\partial_x^2 & -a_1\partial_x & 0 & 0 \end{bmatrix} \\ & \times \begin{bmatrix} u & w & \varphi & p_1 & p_2 & p_3 \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} 0 & 0 & 0 & f_x & f_z & q \end{bmatrix}^{\mathrm{T}} \end{aligned} \tag{5}$$

where $a_0=1/(e_{33}^2+c_{33}\varepsilon_{33})$, $a_1=e_{15}/c_{44}$, $a_2=1/c_{44}$, $a_3=(c_{13}\varepsilon_{33}+e_{31}e_{33})a_0$, $a_4=\varepsilon_{33}a_0$, $a_5=e_{33}a_0$, $a_6=(e_{31}c_{33}-c_{13}e_{33})a_0$, $a_7=c_{33}a_0$, $a_8=-c_{11}+(c_{13}^2\varepsilon_{33}+2e_{31}e_{33}c_{13}-e_{31}^2c_{33})a_0$ and $a_9=\varepsilon_{11}+e_{15}^2/c_{44}$. Suppose the boundary conditions along the contour ($x=\pm a$) are free from traction, with the external normal stress, shear stress and electric displacement being equal to zero at the surface of the boundary. From Eq.(4), the corresponding conditions of the lateral boundary are

$$\begin{aligned} & [a_8\partial_x u - a_3 p_2 + a_6 p_3]_{x=\pm a} = 0; \\ & p_1|_{x=\pm a} = 0; \quad \partial_x \varphi|_{x=\pm a} = 0 \end{aligned} \tag{6}$$

In fact, if Lagrange function variants from Eq.(2), the governing equations and the conditions of the boundary can be obtained by displacement method belonging to the Lagrangian system.

ADJOINT SYMPLECTIC ORTHOGONALITY RELATIONSHIP

Let Eq.(6) be rewritten as

$$\dot{\boldsymbol{\psi}} = \boldsymbol{H}\boldsymbol{\psi} + \boldsymbol{f} \tag{7}$$

where the state vetor $\boldsymbol{\psi}=\{\boldsymbol{q}, \boldsymbol{p}\}^{\mathrm{T}}$ and $\boldsymbol{H}$ is Hamiltonian operator matrix. The solution of Eq.(7) can be divided into two parts, the general solution of the homogeneous equations and a special solution of non-homogeneous equations, respectively. The solution of the homogeneous equation of Eq.(7) can be solved as

$$\boldsymbol{\psi}=\boldsymbol{\psi}_j(x)\,\mathrm{e}^{\mu_j z} \tag{8}$$

where ψ_j and μ_j are eigenpair, or eigenvector and eigenvalue, which, if μ_j is an eigenvalue, then $-\mu_j$ is an eigenvalue also and there is an adjoint symplectic orthogonality relationship (Zhong, 1995; Xu *et al.*, 1997) among eigenvalues and eigenvectors. Introduce the operation

$$<\boldsymbol{\psi}_i^{\mathrm{T}},\boldsymbol{J},\boldsymbol{\psi}_j>=\int_{-a}^{a}\boldsymbol{\psi}_i^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{\psi}_j(x)\,\mathrm{d}x \tag{9}$$

where $\boldsymbol{J}$ is an identical symplectic matrix, so that

$$\begin{aligned}&<\boldsymbol{\psi}_{\alpha_i}^{\mathrm{T}},\boldsymbol{J},\boldsymbol{\psi}_{\beta_j}>=\delta_{ij},\ <\boldsymbol{\psi}_{\beta_i}^{\mathrm{T}},\boldsymbol{J},\boldsymbol{\psi}_{\alpha_j}>=-\delta_{ij},\\&<\boldsymbol{\psi}_{\alpha_i}^{\mathrm{T}},\boldsymbol{J},\boldsymbol{\psi}_{\alpha_j}>=<\boldsymbol{\psi}_{\beta_i}^{\mathrm{T}},\boldsymbol{J},\boldsymbol{\psi}_{\beta_j}>=0\end{aligned} \tag{10}$$

here μ_{α_j} and μ_{β_j} (ψ_{α_j} and ψ_{β_j}) are an adjoint pair of eigenvalues (eigenvectors). Any state vector $\boldsymbol{\psi}$ can always be expanded by a linear combination of the eigenvectors.

ZERO EIGENVALUE SOLUTIONS

In this section, the homogeneous equations and the traction free natural boundary conditions are considered only. Consider the problem of zero-eigenvalue, or μ=0. Besides direct eigensolutions, $\boldsymbol{\psi}_i^{(0)}$, of $\boldsymbol{H\psi}$=0, Jordan form principal vectors of various orders comprise an important part that can be determinated by $\boldsymbol{H\psi}_i^{(n+1)}=\boldsymbol{\psi}_i^{(n)}$ and the solution of the original problem is $\boldsymbol{\eta}_i^{(n+1)}$. The solutions can be obtained as

$$\begin{cases}\boldsymbol{\eta}_1^{(0)}=\boldsymbol{\psi}_1^{(0)}=\{1\quad 0\quad 0\quad 0\quad 0\quad 0\}^{\mathrm{T}},\\ \boldsymbol{\eta}_2^{(0)}=\boldsymbol{\psi}_2^{(0)}=\{0\quad 1\quad 0\quad 0\quad 0\quad 0\}^{\mathrm{T}},\\ \boldsymbol{\eta}_3^{(0)}=\boldsymbol{\psi}_3^{(0)}=\{0\quad 0\quad 1\quad 0\quad 0\quad 0\}^{\mathrm{T}},\end{cases} \tag{11}$$

$$\begin{cases}\boldsymbol{\eta}_1^{(1)}=\boldsymbol{\psi}_1^{(1)}+z\boldsymbol{\psi}_1^{(0)}=\{z\quad -x\quad 0\quad 0\quad 0\quad 0\}^{\mathrm{T}}\\ \boldsymbol{\eta}_2^{(1)}=\boldsymbol{\psi}_2^{(1)}+z\boldsymbol{\psi}_2^{(0)}=\{a_{11}x\quad z\quad 0\quad 0\quad a_{12}\quad a_{13}\}^{\mathrm{T}},\\ \boldsymbol{\eta}_3^{(1)}=\boldsymbol{\psi}_3^{(1)}+z\boldsymbol{\psi}_3^{(0)}=\{a_{14}x\quad z\quad 0\quad 0\quad a_{15}\quad a_{16}\}^{\mathrm{T}},\end{cases} \tag{12}$$

$$\begin{aligned}\boldsymbol{\eta}_1^{(2)}&=\boldsymbol{\psi}_1^{(2)}+z\boldsymbol{\psi}_1^{(1)}+z^2\boldsymbol{\psi}_1^{(0)}/2\\&=\left\{(z^2-a_{11}x^2)/2\quad -xz\quad 0\quad 0\quad -a_{12}x\quad -a_{13}x\right\}^{\mathrm{T}},\end{aligned} \tag{13}$$

$$\begin{aligned}\boldsymbol{\eta}_1^{(3)}&=\boldsymbol{\psi}_1^{(3)}+z\boldsymbol{\psi}_1^{(2)}+z^2\boldsymbol{\psi}_1^{(1)}/2+z^3\boldsymbol{\psi}_1^{(0)}/6\\&=\{z^3/6-a_{11}x^2z/2,\,a_{18}x^3+a_{19}x-xz^2/2,\\&\quad a_{17}(x^3-3a^2x),\,a_{12}(x^2-a^2)/2,\,-a_{12}xz,\,a_{13}xz\}^{\mathrm{T}}.\end{aligned} \tag{14}$$

The geometrical interpretations of these solutions are two rigid translations of coordinates, a translation of electric displacement, a rigid body rotation in the plane, two simple extension deformations (the external electric displacement does not act on the ends but forces and the displacement induced by uniform electric field in which there dose not exist any external force), a pure bending and a shearing-bending in the *x-z* plane respectively. In the solutions, parameters are defined as $a_{10}=a_5^2a_8+a_3^2a_7+a_6^2a_4+2a_3a_5a_6+a_4a_7a_8$, $a_{11}=(a_3a_7-a_5a_6)/a_{10}$, $a_{12}=(a_7a_8+a_6^2)/a_{10}$, $a_{13}=(a_3a_6+a_5a_8)/a_{10}$, $a_{14}=(a_3a_5+a_4a_6)/a_{10}$, $a_{15}=(a_3a_6+a_5a_8)/a_{10}$, $a_{16}=(a_3^2-a_4a_6)/a_{10}$, $a_{17}=(a_1a_{12}-a_{13})/(6a_9)$, $a_{18}=(a_2a_{12}-a_{11}-6a_{17})/6$ and $a_{19}=-(a_{11}+6a_{18})a^2/2$. These solutions are Saint Venant type and satisfy relationships of the adjoint symplectic orthogonal. It can be verified that other principal vectors of the Jordan form do not exist.

NON-ZERO EIGENVALUE SOLUTIONS

The non-zero eigenvalue solutions are covered by the Saint-Venant principle. Consider the equation $(\boldsymbol{H}-\mu\boldsymbol{I})\boldsymbol{\psi}=0$ ($\mu\neq0$). The general solution of the equation can be expressed by trigonometric functions and can be divided into two parts, symmetry (first term on the right side of Eq.(15)) and anti-symmetry (second term), as

$$\begin{aligned}u&=-\mu\sum_{n=1}^{3}B_n\alpha_{1n}\sin(\mu x/s_n)\,\mathrm{e}^{\mu z}\\&\quad+\mu\sum_{n=1}^{3}B_{n+3}\alpha_{1n}\cos(\mu x/s_n)\,\mathrm{e}^{\mu z}\\ w&=\mu\sum_{n=1}^{3}B_n\alpha_{2n}\cos(\mu x/s_n)\,\mathrm{e}^{\mu z}\end{aligned}$$

$$+\mu\sum_{n=1}^{3}B_{n+3}\alpha_{2n}\sin(\mu x/s_n)\mathrm{e}^{\mu z}$$

$$\varphi=\mu\sum_{n=1}^{3}B_n\alpha_{31}\cos(\mu x/s_n)\mathrm{e}^{\mu z}$$

$$+\mu\sum_{n=1}^{3}B_{n+3}\alpha_{3n}\sin(\mu x/s_n)\mathrm{e}^{\mu z}$$

$$p_1=-\mu^2\sum_{n=1}^{3}B_n\beta_{1n}\sin(\mu x/s_n)\mathrm{e}^{\mu z}$$

$$+\mu^2\sum_{n=1}^{3}B_{n+3}\beta_{1n}\cos(\mu x/s_n)\mathrm{e}^{\mu z}$$

$$p_2=\mu^2\sum_{n=1}^{3}B_n\beta_{2n}\cos(\mu x/s_n)\mathrm{e}^{\mu z}$$

$$+\mu^2\sum_{n=1}^{3}B_{n+3}\beta_{2n}\sin(\mu x/s_n)\mathrm{e}^{\mu z}$$

$$p_3=\mu^2\sum_{n=1}^{3}B_n\beta_{3n}\cos(\mu x/s_n)\mathrm{e}^{\mu z}$$

$$+\mu^2\sum_{n=1}^{3}B_{n+3}\beta_{3n}\sin(\mu x/s_n)\mathrm{e}^{\mu z} \tag{15}$$

where B_i (i=1, 2, 3, 4, 5, 6) are constants to be determined and parameters are defined as

$$a_{20}=c_{44}/a_0,$$
$$a_{21}=-c_{33}[c_{44}\varepsilon_{11}+(e_{15}+e_{31})^2]-\varepsilon_{33}[c_{11}c_{33}+c_{44}^2-(c_{13}+c_{44})^2]-e_{33}[2c_{44}e_{15}+c_{11}e_{33}-2(c_{13}+c_{44})(e_{15}+e_{31})],$$
$$a_{22}=c_{44}[c_{11}\varepsilon_{33}+(e_{15}+e_{31})^2]+\varepsilon_{11}[c_{11}\varepsilon_{33}+c_{44}^2-(c_{13}+c_{44})^2]+e_{15}[2c_{11}e_{33}+c_{44}e_{15}-2(c_{13}+c_{44})(e_{15}+e_{31})],$$
$$a_{23}=-c_{11}(e_{15}^2+c_{44}\varepsilon_{11}),$$
$$a_{24}=(c_{13}+c_{44})\varepsilon_{11}+(e_{15}+e_{31})e_{15},$$
$$a_{25}=(c_{13}+c_{44})\varepsilon_{33}+(e_{15}+e_{31})e_{33},$$
$$a_{26}=c_{11}\varepsilon_{33}+c_{44}\varepsilon_{11}+(e_{15}+e_{31})^2,$$
$$a_{27}=c_{11}e_{33}-c_{44}e_{31}-c_{13}(e_{15}+e_{31}),$$
$$\alpha_{1i}=1/s_i,$$
$$\alpha_{2i}=(c_{11}\varepsilon_{11}-a_{26}s_i^2+c_{44}\varepsilon_{33}s_i^4)/(a_{24}-a_{25}s_i^2)s_i^2,$$
$$\alpha_{3i}=(c_{11}e_{15}-a_{27}s_i^2+c_{44}e_{33}s_i^4)/(a_{24}-a_{25}s_i^2)s_i^2,$$
$$\beta_{1i}=c_{44}\alpha_{1i}+c_{44}\alpha_{2i}/s_i+e_{15}\alpha_{3i}/s_i,$$
$$\beta_{2i}=-c_{13}\alpha_{1i}/s_i+c_{33}\alpha_{2i}+e_{33}\alpha_{3i},$$
$$\beta_{3i}=-e_{31}\alpha_{1i}/s_i+e_{33}\alpha_{2i}-\varepsilon_{33}\alpha_{3i},\quad (i=1,2,3)$$

and defines s_i^2 ($i=1,2,3$), which are three roots of $a_{20}r^3+a_{21}r^2+a_{22}r+a_{23}=0$. Substituting the solutions Eq.(15) into conditions of the lateral boundary Eq.(6), one has

$$\boldsymbol{Ac}=0 \tag{16}$$

where $\boldsymbol{c}=\{B_1, B_2, B_3, B_4, B_5, B_6\}^{\mathrm{T}}$ and the matrix $\boldsymbol{A}$ are function of eigenvalues. Consider the condition of non-zero-solution of Eq.(16), the eigenvalue μ (decay rate) can be determined from

$$|\boldsymbol{A}|=0 \tag{17}$$

Thus eigenvalues and eigensolutions can be obtained by Eqs.(15)~(17). Finally, the solutions can be linear combinations of eigenfunctions and adjoint eigenfunctions, or

$$\boldsymbol{\psi}=\sum\left[D_n\boldsymbol{\psi}_{\alpha_n}(x)\mathrm{e}^{\mu_{\alpha_n}z}+G_n\boldsymbol{\psi}_{\beta_n}(x)\mathrm{e}^{\mu_{\beta_n}z}\right] \tag{18}$$

SOLUTIONS OF NONHOMOGENEOUS EQUATIONS

Since the general solution of the homogeneous equations has been obtained by Eq.(18), one special solution of non-homogeneous Eq.(7) is needed only to be discussed in this section. Let the form of the special solution be

$$\boldsymbol{\psi}_p=\sum\left[D_n^*(z)\boldsymbol{\psi}_{\alpha_n}(x)+G_n^*(z)\boldsymbol{\psi}_{\beta_n}(x)\right]$$

and the non-homogeneous term of Eq.(7) expansed with eigensolutions, or

$$\boldsymbol{f}=\sum\left[d_n(z)\boldsymbol{\psi}_{\alpha_n}(x)+g_n(z)\boldsymbol{\psi}_{\beta_n}(x)\right]$$

where $d_n(z)=<\boldsymbol{f}^{\mathrm{T}},\boldsymbol{J},\boldsymbol{\psi}_{\beta_n}(x)>$ and $g_n(z)=<\boldsymbol{f}^{\mathrm{T}},\boldsymbol{J},\boldsymbol{\psi}_{\alpha_n}(x)>$. Substituting the expansion into Eq.(7) and using the adjoint symplectic orthogonality relationship Eq.(10), the equations are obtained as

$$\dot{D}_n^*(z)=\mu_{\alpha_n}D_n^*(z)+d_n(z);\ \dot{G}_n^*(z)=-\mu_{\beta_n}G_n^*(z)+g_n(z) \tag{19}$$

The solutions of Eq.(19) are

$$D_n^*(z) = \int_0^z d_n(\xi) e^{\mu_{\alpha_n}(z-\xi)} d\xi; \quad G_n^*(z) = \int_0^z g_n(\xi) e^{-\mu_{\alpha_n}(z-\xi)} d\xi \tag{20}$$

Therefore, the solution of the problem can be expressed as

$$\psi = \sum \left\{ \left[D_n e^{\mu_{\alpha_n} z} + D_n^*(z) \right] \psi_{\alpha_n}(x) + \left[G_n e^{\mu_{\beta_n} z} + G_n^*(z) \right] \psi_{\beta_n}(x) \right\} \tag{21}$$

NUMERICAL RESULTS

Let $X=x/a$, $Z=z/a$, $U=u/a$, $W=w/a$, $\Phi=\varphi/(a\times10^{10}$ N/c), $P_1=p_1/c_{11}$, $P_2=p_2/c_{11}$ and $P_3=p_3/(c_{11}\times10^{-10}$ c/N) are non-dimensional forms. Consider transversely isotropic elastic parameters $c_{11}=12\times10^{10}$ N/m^2, $c_{13}=0.6c_{11}$, $c_{33}=0.9c_{11}$ and $c_{44}=0.3c_{11}$. Since there are approximate proportional relations $\varepsilon_{11}:\varepsilon_{33}\approx1:1$ and $-e_{31}:e_{15}:e_{33}\approx1:3:3$, take

$$e_{31} = -0.1e[c_{11}/(a\times10^{10}\text{ N/c})],$$
$$e_{15} = 0.3e[c_{11}/(a\times10^{10}\text{ N/c})],$$
$$e_{33} = 0.3e[c_{11}/(a\times10^{10}\text{ N/c})],$$
$$\varepsilon_{11} = \varepsilon_{33} = 0.1\varepsilon(c_{11}\text{ Fm/c}^2)$$

into which a piezoelectric constant e (3 to 8) and a dielectric characteristic constant ε (30 to 150) are introduced respectively. According to the adjoint characteristic of eigenvalues, we discuss the real parts of eigenvalue $-\mu$, which is the decaying coefficient and shows the local effect, so the smaller absolute value of eigenvalues is most significant. Because eigensolutions and lateral boundaries are divided into two kinds, symmetry and anti-symmetry solutions, corresponding eigenvalues can be obtained independently. Fig.1 gives numerical results of real parts of eigenvalues (symmetry and anti-symmetry case) and shows the first five eigenvalues, the smallest eigenvalues, with respect to different piezoelectric parameter e (ε=60). Fig.2 shows eigenvalues with the dielectric parameter ε (e=5), where, the solid line, thick line, dashed line, dotted line and dot-dash line show the first order eigenvalue to the fifth order eigenvalues respectively. The results indicated that piezoelectric and dielectric modulus influence decaying coefficients.

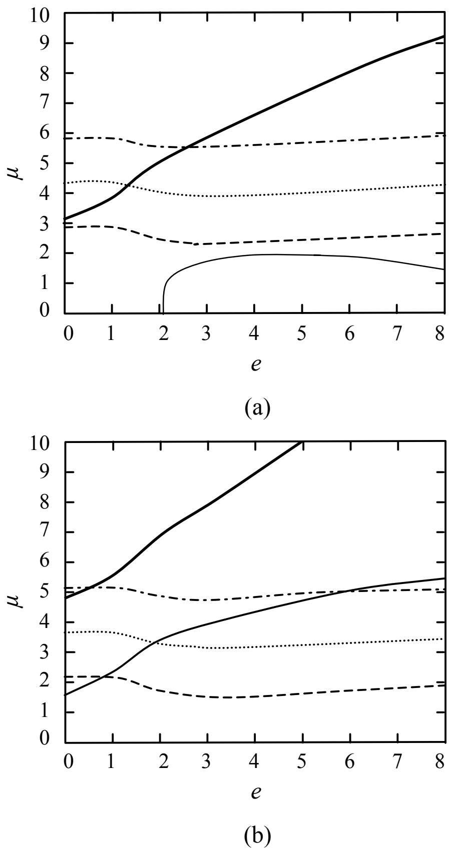

Fig.1 The first five eigenvalues with piezoelectric parameter. (a) Symmetry; (b) Anti-symmetry

In terms of eigenvalues obtained by Eq.(17), the corresponding eigensolutions are shown by Eq.(15). It can be proved that both the real part and imaginary part of eigensolutions are the problem solutions and the eigensolutions can be divided into symmetry and anti-symmetry solutions. Since the electrical potential function φ and the electric displacement $p_3=D_z$ are important in the problem, their graphs are specially noticed in this paper and graphs for other components of eigensolutions are as similar. Fig.3 and Fig.4 give two components of eigensolutions respectively, which correspond to the first five eigenvalues that are shown in Fig.1 or Fig.2. Graphs of eigensolutions are depicted clearly in the figures.

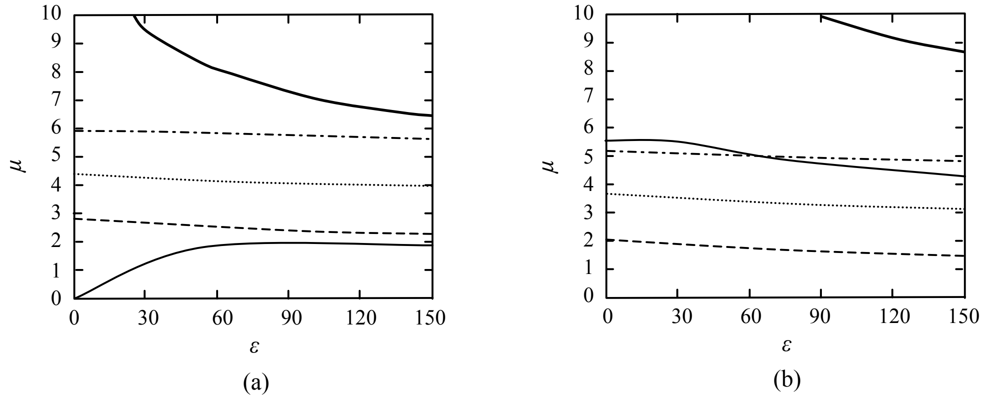

Fig.2 The first five eigenvalues with dielectric parameter. (a) Symmetry; (b) Anti-symmetry

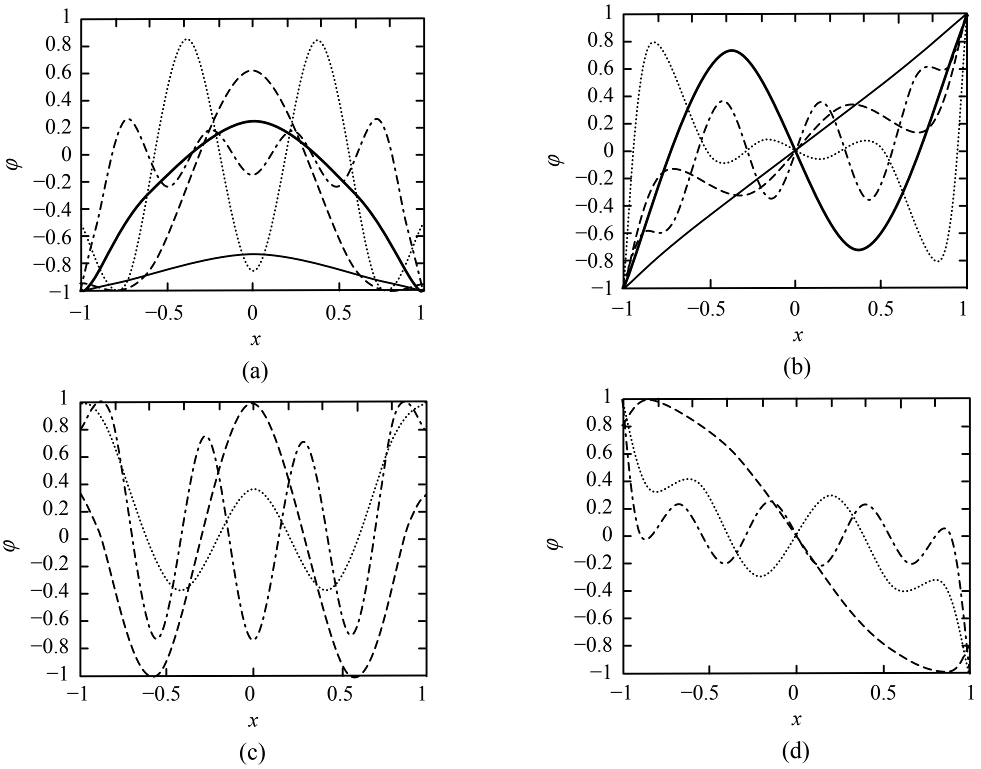

Fig.3 The component (the electrical potential function φ) of the first five eigensolutions. (a) Symmetry (real part); (b) Anti-symmetry (real part); (c) Symmetry (imaginary part); (d) Anti-symmetry (imaginary part)

CONCLUSION

Application of Hamiltonian system can change the method of studying problem of transversely isotropic piezoelectric media, which is updated to solve the problem in the symplectic space under the Hamiltonian system but in traditional Euclidian space under Lagrange system. General solutions and the particular solution can be expressed in terms of eigensolutions. The problem is reduced to the zero eigenvalues with their Jordan forms and the non-zero eigenvalue solutions and symplectic solutions space is complete. The symplectic method can be generalized to other subjects and is effective for mixed boundary conditions specially.

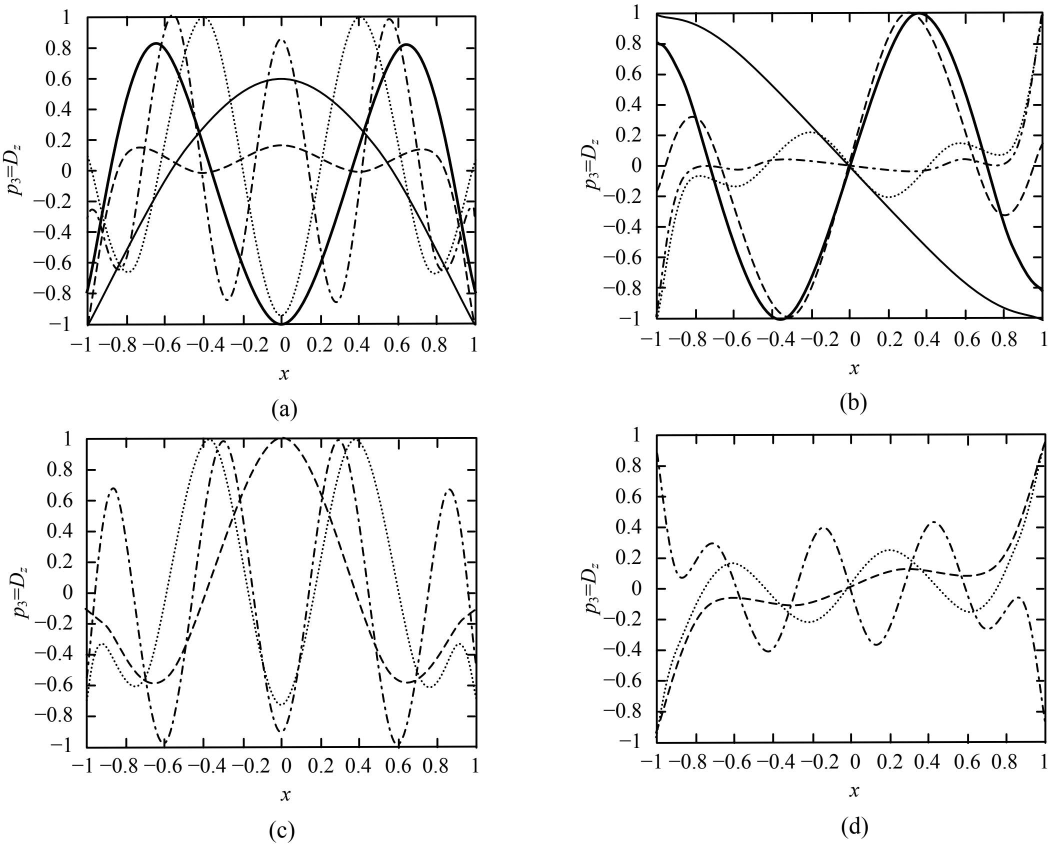

Fig.4 The component (the electrical displacement $p_3=D_z$) of the first five eigensolutions. (a) Symmetry (real part); (b) Anti-symmetry (real part); (c) Symmetry (imaginary part); (d) Anti-symmetry (imaginary part)

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