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Computation of lower derivatives of rational triangular Bézier surfaces and their bounds estimation*

ZHANG Lei (张 磊)[1,2], WANG Guo-jin (王国瑾)[†‡1,2]

([1]Department of Mathematics, Zhejiang University, Hangzhou 310027, China)
([2]State Key Laboratory of CAD & CG, Zhejiang University, Hangzhou 310027, China)
[†]E-mail: gjwang@hzcnc.com

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Abstract: By introducing the homogenous coordinates, degree elevation formulas and combinatorial identities, also by using multiplication of Bernstein polynomials and identity transformation on equations, this paper presents some explicit formulas of the first and second derivatives of rational triangular Bézier surface with respect to each variable (including the mixed derivative) and derives some estimations of bound both on the direction and magnitude of the corresponding derivatives. All the results above have value not only in surface theory but also in practice.

Keywords: Computer Aided Geometric Design, Derivative, Rational triangular Bézier surface, Bound
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INTRODUCTION

In Computer Aided Geometric Design (CAGD), series of researches on the computation of derivatives of parametric curve and surface as well as bound estimation of the corresponding derivatives have been done worldwide owing to their essentiality (Sederberg and Wang, 1987; Floater, 1992; Saito *et al*., 1995; Wang and Wang, 1995; Wang *et al*., 1997; Kim *et al*., 2001). But there exist two limits of these research works. First, for a rational parametric curve or surface, only computation of the first derivative and its bound estimation are provided; second, for a rational surface, the evaluation formulas and bound estimation are only derived on the surface defined on a rectangle domain, e.g. rational quadrangular Bézier surface. As for a rational surface defined on a triangle domain, because the deduction process is complicated and the computation algorithm is less effective, evaluating its first and second derivatives and estimating the corresponding bounds have not been presented yet.

However, the surface defined on a rectangle domain cannot be used to solve many problems in CAGD. For example, interpolation of the surfaces over scattered data without regular distribution is often by using a triangular surface; a part of the control points of some surfaces represented by tensor-product Bézier patches must be degenerate, but they are non-degenerate if the surfaces are represented by some triangular Bézier patches (Farin *et al*., 1987). Many published articles on the triangular surface (Farin, 1986; Tian, 1988; 1990; Hu, 1996; Hu *et al*., 1996a; 1996b) imply the surface is an important tool in the field of CAGD. The rational triangular Bézier surface has many advantages of the triangular surfaces by polynomial form, and it can also exactly represent a conic section and control the shape with weights, so it is a hot subject in CAGD at present. Farin *et al*.(1987) constructed an octant of a sphere exactly by a quartic rational triangular Bernsterin-Bézier surface, which is non-degenerate. Tian

‡Corresponding author
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(1990) presented a recursive algorithm and a subdivisional algorithm for evaluating rational triangular surfaces. Hu *et al*.(1996a; 1996b) derived a method for subdividing a rational quadrangular surface into three rational triangular surface. All the works above indicate that rational triangular Bézier surface plays an important role in CAGD, so it is necessary to solve the remaining suspensive problems mentioned above, i.e., the computation of lower derivatives of rational triangular Bézier surfaces and their bounds estimation.

Facing the challenge, in this paper, by using the direction operator between the Cartesian vectors of two homogeneous points as well as degree elevation formulas of rational triangular Bézier surface, also by executing the identity transformation on the linear combinations of Bernstein basis, a successful method for computing the derivatives is presented. We can represent the αth (α=1,2) derivatives of a rational triangular Bézier surface with degree n in another rational triangular Bézier surface with degree $2^{\alpha}n$. So the lower derivatives of any rational triangular Bézier surface can be evaluated and displayed through computer programming by the recursive algorithm and subdivisional algorithm designed by Tian (1990). Furthermore, by using some combinatorial identities and multiplication of Bernstein polynomials, bounds both on the direction and magnitude of the lower derivatives of rational triangular Bézier surface can also be derived.

COMPUTATION OF LOWER DERIVATIVES OF RATIONAL TRIANGULAR BÉZIER SURFACE

A degree n rational Bézier surface on a triangular domain (rational triangular B-B surface) can be defined as follows:

$$\boldsymbol{R}(u,v,w)=\frac{\boldsymbol{N}(u,v,w)}{D(u,v,w)},\qquad (u,v,w)\in T. \tag{1}$$

where

$$D(u,v,w)=\sum_{i+j+k=n}\omega_{i,j,k}B_{i,j,k}^{n}(u,v,w), \tag{2}$$

$$\boldsymbol{N}(u,v,w)=\sum_{i+j+k=n}\omega_{i,j,k}\boldsymbol{R}_{i,j,k}B_{i,j,k}^{n}(u,v,w), \tag{3}$$

and $B_{i,j,k}^{n}(u,v,w)=\dfrac{n!}{i!\,j!\,k!}u^{i}v^{j}w^{k}$ is Bernstein basis with degree n, $\boldsymbol{R}_{i,j,k}=(x_{i,j,k},y_{i,j,k},z_{i,j,k})\in\mathbb{R}^3$ is control point of the surface, $\omega_{i,j,k}$ is weight, T:{(u,v,w) |$u+v+w=1$, $0\le u,v,w\le 1$} is the parametric domain of the surface.

If $\boldsymbol{R}_{i,j,k}$ and $\omega_{i,j,k}$ are defined the same as above, an equivalent representation of the surface Eq.(1) can be written as follows:

$$\boldsymbol{R}(u,v)=\frac{\sum_{i+j=0}^{n}\omega_{i,j,n-i-j}\boldsymbol{R}_{i,j,n-i-j}B_{i,j,n-i-j}^{n}(u,v)}{\sum_{i+j=0}^{n}\omega_{i,j,n-i-j}B_{i,j,n-i-j}^{n}(u,v)}$$
$$\equiv\frac{\sum_{i+j=0}^{n}\omega_{i,j}\boldsymbol{R}_{i,j}B_{i,j}^{n}(u,v)}{\sum_{i+j=0}^{n}\omega_{i,j}B_{i,j}^{n}(u,v)}=\frac{\boldsymbol{N}(u,v)}{D(u,v)},$$
$$(u,v)\in D=\{(u,v)\,|\,u,v\ge 0,u+v\le 1\}. \tag{4}$$

Eq.(1) depends on three parameters u, v, w; Eq.(4) gives a less complicated representation of the rational triangular Bézier surface which can be easily used to compute the derivatives. Tian (1990) suggested a recursive algorithm and subdivisional algorithm for evaluating the rational triangular B-B surface, so the main idea here is to represent the first and second derivatives (including the mixed derivatives) with respect to each parameter of a rational triangular B-B surface Eq.(4) in another rational triangular B-B surface with appropriate degrees, hence the final computation of derivatives can be obtained with the evaluation algorithm of surfaces through computer programming.

Let us now start with the computation of the first derivative with respect to the parameter u of the rational triangular B-B surface Eq.(4). The differentiation of Eq.(1) with respect to the parameter u yields the following equation:

$$\frac{\partial\boldsymbol{R}(u,v)}{\partial u}=\frac{\dfrac{\partial\boldsymbol{N}(u,v)}{\partial u}D(u,v)-\dfrac{\partial D(u,v)}{\partial u}\boldsymbol{N}(u,v)}{\{D(u,v)\}^2}. \tag{5}$$

As the denominator $\{D(u,v)\}^2$ is the product of polynomials, it can be rearranged in the form of

Bernstein polynomials of degree $2n$ as follows:

$$D^2(u,v)=\sum_{l+m=0}^{2n}\kappa_{l,m}B_{l,m}^{2n}(u,v)\,. \tag{6}$$

where

$$\kappa_{l,m}=\sum_{i+j=\max\{0,l+m-n\}}^{\min\{n,l+m\}}\omega_{i,j}\omega_{l-i,m-j}\frac{\binom{l}{i}\binom{m}{j}\binom{2n-l-m}{n-i-j}}{\binom{2n}{n}} \tag{7}$$

To represent the numerator of Eq.(5) in a triangular B-B surface of degree $2n$ succinctly, we manipulate the control points in a homogeneous coordinate system. Rewriting the control point $\boldsymbol{R}_{i,j}=(x_{i,j},y_{i,j},z_{i,j})$ in the form of homogenous coordinate yields $\tilde{\boldsymbol{R}}_{i,j}=(X_{i,j},Y_{i,j},Z_{i,j},\omega_{i,j})=(\omega_{i,j}x_{i,j},\omega_{i,j}y_{i,j},\omega_{i,j}z_{i,j},\omega_{i,j})$, $0\le i+j\le n$. Then define the direction (Satio *et al*., 1995) of Cartesian vector between two homogenous points $\tilde{\boldsymbol{R}}_{i,j}$ and $\tilde{\boldsymbol{R}}_{p,q}$ as

$$\begin{aligned}\mathrm{Dir}(\tilde{\boldsymbol{R}}_{i,j},\tilde{\boldsymbol{R}}_{p,q})&=\omega_{i,j}\omega_{p,q}(\boldsymbol{R}_{p,q}-\boldsymbol{R}_{i,j})\\&=(\omega_{i,j}X_{p,q}-\omega_{p,q}X_{i,j},\ \omega_{i,j}Y_{p,q}\\&\quad-\omega_{p,q}Y_{i,j},\ \omega_{i,j}Z_{p,q}-\omega_{p,q}Z_{i,j}),\\&\qquad\omega_{i,j}\omega_{p,q}\neq0,\end{aligned}$$

and according to the differential equation $\frac{\partial B_{i,j}^n(u,v)}{\partial u}=n\left\{B_{i-1,j}^{n-1}(u,v)-B_{i,j}^{n-1}(u,v)\right\}$, the numerator of Eq.(5) can be rewritten as follows:

$$\begin{aligned}&\frac{\partial\boldsymbol{N}(u,v)}{\partial u}D(u,v)-\frac{\partial D(u,v)}{\partial u}\boldsymbol{N}(u,v)\\&=\sum_{p+q=0}^{n}\sum_{i+j=0}^{n}\omega_{i,j}\omega_{p,q}(\boldsymbol{R}_{i,j}-\boldsymbol{R}_{p,q})B_{p,q}^n(u,v)\frac{\partial B_{i,j}^n(u,v)}{\partial u}\\&=\sum_{p+q=0}^{n}\left\{n\sum_{i+j=0}^{n-1}\mathrm{Dir}\left(\tilde{\boldsymbol{R}}_{i,j}-\tilde{\boldsymbol{R}}_{i+1,j},\tilde{\boldsymbol{R}}_{p,q}\right)B_{i,j}^{n-1}(u,v)\right\}B_{p,q}^n(u,v).\end{aligned} \tag{8}$$

Furthermore, Eq.(8) can be represented by degree elevation (Farin, 1990; Hoschek and Lasser, 1992) as follows:

$$\begin{aligned}&\frac{\partial\boldsymbol{N}(u,v)}{\partial u}D(u,v)-\frac{\partial D(u,v)}{\partial u}\boldsymbol{N}(u,v)\\&=\sum_{p+q=0}^{n}\sum_{i+j=0}^{n}\Big\{i\mathrm{Dir}\left(\tilde{\boldsymbol{R}}_{i-1,j}-\tilde{\boldsymbol{R}}_{i,j},\tilde{\boldsymbol{R}}_{p,q}\right)\\&\quad+j\mathrm{Dir}\left(\tilde{\boldsymbol{R}}_{i,j-1}-\tilde{\boldsymbol{R}}_{i+1,j-1},\tilde{\boldsymbol{R}}_{p,q}\right)+(n-i-j)\\&\quad\times\mathrm{Dir}\left(\tilde{\boldsymbol{R}}_{i,j}-\tilde{\boldsymbol{R}}_{i+1,j},\tilde{\boldsymbol{R}}_{p,q}\right)\Big\}B_{i,j}^n(u,v)B_{p,q}^n(u,v)\\&=\sum_{l+m=0}^{2n}\boldsymbol{D}_{l,m}B_{l,m}^{2n}(u,v),\end{aligned} \tag{9}$$

where

$$\boldsymbol{D}_{l,m}=\sum_{i+j=\max\{0,l+m-n\}}^{\min\{n,l+m\}}\boldsymbol{H}_{i,j;l-i,m-j}\frac{\binom{l}{i}\binom{m}{j}\binom{2n-l-m}{n-i-j}}{\binom{2n}{n}}, \tag{10}$$

$$\begin{aligned}\boldsymbol{H}_{i,j;l-i,m-j}&=i\mathrm{Dir}\left(\tilde{\boldsymbol{R}}_{i-1,j}-\tilde{\boldsymbol{R}}_{i,j},\tilde{\boldsymbol{R}}_{l-i,m-j}\right)\\&\quad+j\mathrm{Dir}\left(\tilde{\boldsymbol{R}}_{i,j-1}-\tilde{\boldsymbol{R}}_{i+1,j-1},\tilde{\boldsymbol{R}}_{l-i,m-j}\right)+(n-i-j)\\&\quad\times\mathrm{Dir}\left(\tilde{\boldsymbol{R}}_{i,j}-\tilde{\boldsymbol{R}}_{i+1,j},\tilde{\boldsymbol{R}}_{l-i,m-j}\right).\end{aligned} \tag{11}$$

Consequently, substituting the denominator and numerator with Eq.(6) and Eq.(9) respectively in Eq.(5) yields

$$\frac{\partial\boldsymbol{R}(u,v)}{\partial u}=\frac{\sum_{l+m=0}^{2n}\boldsymbol{D}_{l,m}B_{l,m}^{2n}(u,v)}{\sum_{l+m=0}^{2n}\kappa_{l,m}B_{l,m}^{2n}(u,v)}=\frac{\sum_{l+m=0}^{2n}\kappa_{l,m}\boldsymbol{J}_{l,m}B_{l,m}^{2n}(u,v)}{\sum_{l+m=0}^{2n}\kappa_{l,m}B_{l,m}^{2n}(u,v)},\quad (u,v)\in D, \tag{12}$$

where

$$\boldsymbol{J}_{l,m}=D_{l,m}/\kappa_{l,m} \tag{13}$$

Thus, the first derivative with respect to the parameter u of the degree n rational triangular B-B surface Eq.(4) can be represented in the degree $2n$ rational triangular B-B surface Eq.(12) with the control points $\boldsymbol{J}_{l,m}$ and the weights $\kappa_{l,m}$ ($0\le l+m\le 2n$). Additionally, if $\boldsymbol{J}_{l,m}$ and $\kappa_{l,m}$ are still defined as above, an equivalent expression of Eq.(12), i.e., the first derivative with respect to the parameter u of the surface Eq.(1), is obtained as follows:

$$\frac{\partial \boldsymbol{R}(u,v)}{\partial u}=\frac{\sum_{l+m=0}^{2n}\kappa_{l,m,2n-l-m}\boldsymbol{J}_{l,m,2n-l-m}B_{l,m,2n-l-m}^{2n}(u,v)}{\sum_{l+m=0}^{2n}\kappa_{l,m,2n-l-m}B_{l,m,2n-l-m}^{2n}(u,v)}=\frac{\sum_{l+m+k=2n}\kappa_{l,m,k}\boldsymbol{J}_{l,m,k}B_{l,m,k}^{2n}(u,v,w)}{\sum_{l+m+k=2n}\kappa_{l,m,k}B_{l,m,k}^{2n}(u,v,w)},\quad (u,v,w)\in T. \tag{14}$$

Next, we compute the second derivative with respect to the parameter u of the surface Eq.(4). Differentiating Eq.(12) with respect to u again and repeating the same above procedure yields

$$\frac{\partial^2 \boldsymbol{R}(u,v)}{\partial u^2}=\frac{\sum_{r+s=0}^{4n}\mu_{r,s}\boldsymbol{L}_{r,s}B_{r,s}^{4n}(u,v)}{\sum_{r+s=0}^{4n}\mu_{r,s}B_{r,s}^{4n}(u,v)},\quad (u,v)\in D \tag{15}$$

where

$$\mu_{r,s}=\sum_{l+m=\max\{0,r+s-2n\}}^{\min\{2n,r+s\}}\kappa_{l,m}\kappa_{r-l,s-m}\frac{\binom{r}{l}\binom{s}{m}\binom{4n-r-s}{2n-l-m}}{\binom{4n}{2n}} \tag{16}$$

$$\boldsymbol{L}_{r,s}=\boldsymbol{E}_{r,s}/\mu_{r,s}, \tag{17}$$

$$\boldsymbol{E}_{r,s}=\sum_{l+m=\max\{0,r+s-2n\}}^{\min\{2n,r+s\}}\boldsymbol{I}_{l,m;r-l,s-m}\frac{\binom{r}{l}\binom{s}{m}\binom{4n-r-s}{2n-l-m}}{\binom{4n}{2n}}, \tag{18}$$

$$\boldsymbol{I}_{l,m;r-l,s-m}=l\mathrm{Dir}\left(\tilde{\boldsymbol{J}}_{l-1,m}-\tilde{\boldsymbol{J}}_{l,m},\tilde{\boldsymbol{J}}_{r-l,s-m}\right)+m\mathrm{Dir}\left(\tilde{\boldsymbol{J}}_{l,m-1}-\tilde{\boldsymbol{J}}_{l+1,m-1},\tilde{\boldsymbol{J}}_{r-l,s-m}\right)+\left(2n-l-m\right)\times\mathrm{Dir}\left(\tilde{\boldsymbol{J}}_{l,m}-\tilde{\boldsymbol{J}}_{l+1,m},\tilde{\boldsymbol{J}}_{r-l,s-m}\right). \tag{19}$$

Therefore, the second derivative with respect to the parameter u of the rational triangular B-B surface Eq.(4) can be represented in degree $4n$ rational triangular B-B surface Eq.(15) with the control points $\boldsymbol{L}_{r,s}$ and the weights $\mu_{r,s}$ ($0\le r+s\le 4n$). Similarly, if $\boldsymbol{L}_{r,s}$ and $\mu_{r,s}$ are defined the same as above, we can obtain an equivalent expression of Eq.(15), i.e., the second derivative with respect to the parameter u of the surface Eq.(1), as follows:

$$\frac{\partial^2 \boldsymbol{R}(u,v)}{\partial u^2}=\frac{\sum_{r+s=0}^{4n}\mu_{r,s,4n-r-s}\boldsymbol{L}_{r,s,4n-r-s}B_{r,s,4n-r-s}^{4n}(u,v)}{\sum_{r+s=0}^{4n}\mu_{r,s,4n-r-s}B_{r,s,4n-r-s}^{4n}(u,v)}=\frac{\sum_{r+s+t=4n}\mu_{r,s,t}\boldsymbol{L}_{r,s,t}B_{r,s,t}^{4n}(u,v,w)}{\sum_{r+s+t=4n}\mu_{r,s,t}B_{r,s,t}^{4n}(u,v,w)},\quad (u,v,w)\in T. \tag{20}$$

At the same time, to compute the second mixed derivative of the rational triangular B-B surface Eq.(4), differentiation with respect to v of Eq.(12) is implemented. Noting that the appropriate differential equation is now

$$\frac{\partial B_{i,j}^{n}(u,v)}{\partial v}=n\left\{B_{i,j-1}^{n-1}(u,v)-B_{i,j}^{n-1}(u,v)\right\},$$

the result of which is as follows:

$$\frac{\partial^2 \boldsymbol{R}(u,v)}{\partial u\partial v}=\frac{\sum_{r+s=0}^{4n}\eta_{r,s}\boldsymbol{K}_{r,s}B_{r,s}^{4n}(u,v)}{\sum_{r+s=0}^{4n}\eta_{r,s}B_{r,s}^{4n}(u,v)},\quad (u,v)\in D. \tag{21}$$

where

$$\eta_{r,s}=\sum_{l+m=\max\{0,r+s-2n\}}^{\min\{2n,r+s\}}\kappa_{l,m}\kappa_{r-l,s-m}\frac{\binom{r}{l}\binom{s}{m}\binom{4n-r-s}{2n-l-m}}{\binom{4n}{2n}} \tag{22}$$

$$\boldsymbol{K}_{r,s}=\boldsymbol{F}_{r,s}/\eta_{r,s}, \tag{23}$$

$$\boldsymbol{F}_{r,s}=\sum_{l+m=\max\{0,r+s-2n\}}^{\min\{2n,r+s\}}\boldsymbol{G}_{l,m;r-l,s-m}\frac{\binom{r}{l}\binom{s}{m}\binom{4n-r-s}{2n-l-m}}{\binom{4n}{2n}} \tag{24}$$

$$\boldsymbol{G}_{l,m;r-l,s-m}=l\mathrm{Dir}\left(\tilde{\boldsymbol{J}}_{l-1,m}-\tilde{\boldsymbol{J}}_{l-1,m+1},\tilde{\boldsymbol{J}}_{r-l,s-m}\right)+m\mathrm{Dir}\left(\tilde{\boldsymbol{J}}_{l,m-1}-\tilde{\boldsymbol{J}}_{l,m},\tilde{\boldsymbol{J}}_{r-l,s-m}\right)+\left(2n-l-m\right)\times\mathrm{Dir}\left(\tilde{\boldsymbol{J}}_{l,m}-\tilde{\boldsymbol{J}}_{l,m+1},\tilde{\boldsymbol{J}}_{r-l,s-m}\right). \tag{25}$$

Thus, we have represented the second mixed derivatives of the rational triangular B-B surface

Eq.(4) by the degree 4n rational triangular B-B surface Eq.(21) with control points $\boldsymbol{K}_{r,s}$ and weights $\eta_{r,s}$ ($0\le r+s\le 4n$). Then, if $\boldsymbol{K}_{r,s}$ and $\eta_{r,s}$ are similarly defined as above, we can also obtain the equivalent equation of Eq.(21) which is just the second mixed derivative of the surface Eq.(1) as follows:

$$\frac{\partial^2 \boldsymbol{R}(u,v)}{\partial u \partial v}=\frac{\sum_{r+s=0}^{4n}\eta_{r,s,4n-r-s}\boldsymbol{K}_{r,s,4n-r-s}B_{r,s,4n-r-s}^{4n}(u,v)}{\sum_{r+s=0}^{4n}\eta_{r,s,4n-r-s}B_{r,s,4n-r-s}^{4n}(u,v)}=\frac{\sum_{r+s+t=4n}\eta_{r,s,t}\boldsymbol{K}_{r,s,t}B_{r,s,t}^{4n}(u,v,w)}{\sum_{r+s+t=4n}\eta_{r,s,t}B_{r,s,t}^{4n}(u,v,w)},\quad (u,v,w)\in T. \tag{26}$$

The same procedure can be used to compute the first and second derivatives with respect to the parameter v of the surface Eq.(4).

BOUND ESTIMATION OF LOWER DERIVATIVES OF RATIONAL TRIANGULAR B-B SURFACE

In this Section, the first step is to rewrite each equation of the derivative of the rational triangular B-B surface Eq.(4) in Section 2 in a different expression, and then estimate the bound of each derivative based on the new expression. Eq.(8) yields that

$$\frac{\partial \boldsymbol{N}(u,v)}{\partial u}D(u,v)-\frac{\partial D(u,v)}{\partial u}\boldsymbol{N}(u,v)=\sum_{l+m=0}^{2n-1}\boldsymbol{H}_{l,m}B_{l,m}^{2n-1}(u,v) \tag{27}$$

where

$$\boldsymbol{H}_{l,m}=\frac{n}{\binom{2n}{n}}\sum_{i+j=\max\{0,l+m-n\}}^{\min\{n-1,l+m\}}\left\{\binom{l}{i}\binom{m}{j}\binom{2n-1-l-m}{n-1-i-j}\times\mathrm{Dir}\left(\tilde{\boldsymbol{R}}_{i,j}-\tilde{\boldsymbol{R}}_{i+1,j},\tilde{\boldsymbol{R}}_{l-i,m-j}\right)\right\} \tag{28}$$

Therefore,

$$\frac{\partial \boldsymbol{R}(u,v)}{\partial u}=\frac{\sum_{l+m=0}^{2n-1}\boldsymbol{H}_{l,m}B_{l,m}^{2n-1}(u,v)}{\{D(u,v)\}^2},\quad (u,v)\in D. \tag{29}$$

This means that the first derivative with respect to the parameter u of the surface Eq.(4) has the same direction as that of the surface Eq.(27), which is a degree $2n-1$ triangular Bézier surface with the control points $\boldsymbol{H}_{l,m}$ ($0\le l+m\le 2n-1$), but the magnitude of the first derivative is only $\{D(u,v)\}^{-2}$ that of the latter [see Eq.(29)].

Because every coefficient of the vectors

$$\mathrm{Dir}\left(\tilde{\boldsymbol{R}}_{l-i,m-j},\Delta_1\tilde{\boldsymbol{R}}_{i,j}\right),\ \text{where}\ \left(\max\{0,l+m-n+1\}\le i+j\le\min\{n-1,l+m\}\right)$$

in Eq.(28) is positive and the coefficients of the vectors $\boldsymbol{H}_{l,m}$ in Eq.(29) are also positive, the direction of the first derivative with respect to the parameter u of the surface Eq.(1) or Eq.(4) is bounded in the cone generated by the following vectors:

$$\boldsymbol{l}=\{\mathrm{Dir}(\tilde{\boldsymbol{R}}_{l-i,m-j},\Delta_1\tilde{\boldsymbol{R}}_{i,j})|\max\{0,l+m-n+1\}\le i+j\le\min\{n-1,l+m\}\}$$

where Δ_1 is the shift operator with respect to the parameter u, i.e., $\Delta_1\boldsymbol{R}_{i,j}=\boldsymbol{R}_{i+1,j}-\boldsymbol{R}_{i,j}$.

Next, we will compute the bound on the magnitude of the first derivative. First, an inequality is obtained:

$$\begin{aligned}\|\boldsymbol{H}_{l,m}\|&=\frac{n}{\binom{2n}{n}}\left\|\sum_{i+j=\max\{0,l+m-n\}}^{\min\{n-1,l+m\}}\left\{\binom{l}{i}\binom{m}{j}\binom{2n-1-l-m}{n-1-i-j}\mathrm{Dir}\left(\tilde{\boldsymbol{R}}_{i,j}-\tilde{\boldsymbol{R}}_{i+1,j},\tilde{\boldsymbol{R}}_{l-i,m-j}\right)\right\}\right\|\\&\le\frac{n}{\binom{2n}{n}}\max_{\substack{\max\{0,l+m-n\}\\ \le i+j\le\\ \min\{n-1,l+m\}}}\left\|\mathrm{Dir}\left(\tilde{\boldsymbol{R}}_{i,j}-\tilde{\boldsymbol{R}}_{i+1,j},\tilde{\boldsymbol{R}}_{l-i,m-j}\right)\right\|\times\sum_{i+j=\max\{0,l+m-n\}}^{\min\{n-1,l+m\}}\binom{l}{i}\binom{m}{j}\binom{2n-1-l-m}{n-1-i-j},\\&\qquad\left(0\le l+m\le 2n-1\right).\end{aligned} \tag{30}$$

One can easily verify that the third factor of the right-hand side of the expression Eq.(30) satisfies the following equation on condition that either $l+m<n$ or $l+m\geq n$ (in this case, both $l<n-1$ and $l\geq n$ are allowed):

$$\sum_{i+j=\max\{0,l+m-n\}}^{\min\{n-1,l+m\}}\binom{l}{i}\binom{m}{j}\binom{2n-1-l-m}{n-1-i-j} =\sum_{i=\max\{0,l-n\}}^{\min\{l,n-1\}}\binom{l}{i}\sum_{j=\max\{0,l+m-n-i\}}^{\min\{m,n-1-i\}}\binom{m}{j}\binom{2n-1-l-m}{n-1-i-j}.$$

Furthermore, by the following combinatorial identities,

$$\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}=\binom{m+n}{r},$$

we can obtain

$$\sum_{i+j=\max\{0,l+m-n\}}^{\min\{n-1,l+m\}}\binom{l}{i}\binom{m}{j}\binom{2n-1-l-m}{n-1-i-j} \leq\sum_{i=0}^{n-1}\binom{l}{i}\sum_{j=0}^{n-1-i}\binom{m}{j}\binom{2n-1-l-m}{n-1-i-j}=\binom{2n-1}{n-1} \tag{31}$$

The following inequality can also be derived:

$$\left\|\mathrm{Dir}\left(\tilde{\boldsymbol{R}}_{i,j}-\tilde{\boldsymbol{R}}_{i+1,j},\tilde{\boldsymbol{R}}_{l-i,m-j}\right)\right\|=\left\|\omega_{i+1,j}\omega_{l-i,m-j}(\boldsymbol{R}_{i+1,j}-\boldsymbol{R}_{l-i,m-j})-\omega_{i,j}\omega_{l-i,m-j}(\boldsymbol{R}_{i,j}-\boldsymbol{R}_{l-i,m-j})\right\| \leq 2\left(\max_{0\leq i+j\leq n}\omega_{i,j}\right)^2\cdot\max_{\substack{0\leq i+j\leq n\\0\leq p+q\leq n}}\left\|\boldsymbol{R}_{i,j}-\boldsymbol{R}_{p,q}\right\|,$$

$$\left(\max\{0,l+m-n+1\}\leq i+j\leq\min\{n-1,l+m\}\right). \tag{32}$$

Substituting the corresponding factors in Eq.(30) with Eqs.(31) and (32), it yields the bound on the magnitude of the vector $\boldsymbol{H}_{l,m}$ as:

$$\left\|\boldsymbol{H}_{l,m}\right\|\leq n\left(\max_{0\leq i+j\leq n}\omega_{i,j}\right)^2\cdot\max_{\substack{0\leq i+j\leq n\\0\leq p+q\leq n}}\left\|\boldsymbol{R}_{i,j}-\boldsymbol{R}_{p,q}\right\|, \quad \left(0\leq l+m\leq 2n-1\right). \tag{33}$$

Hence, from Eqs.(29) and (33), we have

$$\left\|\frac{\partial\boldsymbol{R}(u,v)}{\partial u}\right\|\leq n\left(\frac{\max_{0\leq i+j\leq n}\omega_{i,j}}{\min_{0\leq i+j\leq n}\omega_{i,j}}\right)^2\cdot\max_{\substack{0\leq i+j\leq n\\0<p+q\leq n}}\left\|\boldsymbol{R}_{i,j}-\boldsymbol{R}_{p,q}\right\|, \quad (u,v)\in D. \tag{34}$$

or the equivalent expression with respect to the parameters u, v, w:

$$\left\|\frac{\partial\boldsymbol{R}(u,v,w)}{\partial u}\right\|\leq n\left(\frac{\max_{i+j+k=n}\omega_{i,j,k}}{\min_{i+j+k=n}\omega_{i,j,k}}\right)^2\cdot\max_{\substack{i+j+k=n\\p+q+r=n}}\left\|\boldsymbol{R}_{i,j,k}-\boldsymbol{R}_{p,q,r}\right\|, \quad (u,v,w)\in T. \tag{35}$$

Estimation of the first derivative with respect to the parameter v can also be done using the same method mentioned above.

To get the bound estimation on the second derivative of the surface Eq.(4), the differentiation of Eq.(29) as well as by multiplication of Bernstein polynomials yields the following equation:

$$\frac{\partial^2\boldsymbol{R}(u,v)}{\partial u^2}=\frac{1}{\{D(u,v)\}^3}\left\{(2n-1)\sum_{l+m=0}^{n}\omega_{l,m}B_{l,m}^{n}(u,v)\times\sum_{i+j=0}^{2n-2}\left(\boldsymbol{H}_{i+1,j}-\boldsymbol{H}_{i,j}\right)B_{i,j}^{2n-2}(u,v)-2n\sum_{i+j=0}^{n-1}\left\{\left(\omega_{i+1,j}-\omega_{i,j}\right)\times B_{i,j}^{n-1}(u,v)\right\}\sum_{l+m=0}^{2n-1}\boldsymbol{H}_{l,m}B_{l,m}^{2n-1}(u,v)\right\}$$
$$=\frac{1}{\{D(u,v)\}^3}\sum_{s+t=0}^{3n-2}\boldsymbol{Q}_{s,t}B_{s,t}^{3n-2}(u,v),\quad (u,v)\in D, \tag{36}$$

where

$$\boldsymbol{Q}_{s,t}=\frac{2n-1}{\binom{3n-2}{n}}\sum_{i+j=\max\{0,s+t-n\}}^{\min\{2n-2,s+t\}}\left\{\omega_{s-i,t-j}\left(\boldsymbol{H}_{i+1,j}-\boldsymbol{H}_{i,j}\right)\times\binom{s}{i}\binom{t}{j}\binom{3n-2-s-t}{2n-2-i-j}\right\}-\frac{2n}{\binom{3n-2}{n-1}}\times\sum_{i+j=\max\{0,s+t-2n-1\}}^{\min\{n-1,s+t\}}\left\{\left(\omega_{i+1,j}-\omega_{i,j}\right)\boldsymbol{H}_{l,m}\times\binom{s}{i}\binom{t}{j}\binom{3n-2-s-t}{n-1-i-j}\right\} \tag{37}$$

It shows that the direction of the second derivative with respect to u of the surface Eq.(4) is parallel to a triangular B-B surface of degree $3n-2$ whose control points are $\boldsymbol{Q}_{s,t}$ ($0\le s+t\le 3n-2$), and the magnitude is only $\{D(u,v)\}^{-3}$ that of the latter [see Eq.(36)]. For estimation of bound on the second derivative, first by Eq.(36), we have the following inequality:

$$\begin{aligned}&\left\|\frac{\partial^2 \boldsymbol{R}(u,v)}{\partial u^2}\right\| \\ &\le \frac{1}{D^3(u,v)}\left\{2(2n-1)\left(\max_{0\le i+j\le n}\omega_{i,j}\right)\left(\max_{0\le i+j\le 2n-1}\|\boldsymbol{H}_{i,j}\|\right)\right. \\ &\left.+2n\left(\max_{0\le i+j\le n}\left|\Delta_1\omega_{i,j}\right|\right)\left(\max_{0\le i+j\le 2n-1}\|\boldsymbol{H}_{i,j}\|\right)\right\} \\ &\le \frac{2}{\left(\min\limits_{0\le i+j\le n}\omega_{i,j}\right)^3}\left\{(2n-1)\max_{0\le i+j\le n}|\omega_{i,j}|\right. \\ &\left.+n\max_{0\le i+j\le n}|\Delta_1\omega_{i,j}|\right\}\max_{0\le i+j\le 2n-1}\left\|\boldsymbol{H}_{i,j}\right\|,\end{aligned}$$

then substituting the factor $\max\limits_{0\le i+j\le 2n-1}\|\boldsymbol{H}_{i,j}\|$ in the expression above into the bound estimation of the vector $\boldsymbol{H}_{l,m}$, i.e., Eq.(33), thus the final bound estimation of the derivative is obtained as follows:

$$\begin{aligned}\left\|\frac{\partial^2 \boldsymbol{R}(u,v)}{\partial u^2}\right\| &\le 2n\frac{\left(\max\limits_{0\le i+j\le n}\omega_{i,j}\right)^2}{\left(\min\limits_{0\le i+j\le n}\omega_{i,j}\right)^3}\left\{(2n-1)\max_{0\le i+j\le n}\omega_{i,j}\right. \\ &\left.+n\max_{0\le i+j\le n}|\Delta_1\omega_{i,j}|\right\}\max_{\substack{0\le i+j\le n\\0\le p+q\le n}}\left\|\boldsymbol{R}_{i,j}-\boldsymbol{R}_{p,q}\right\|,\ (u,v)\in D. \quad (38)\end{aligned}$$

Then, the expression above can be simplified into a less strict form, that is

$$\begin{aligned}\left\|\frac{\partial^2 \boldsymbol{R}(u,v)}{\partial u^2}\right\| &\le 2n(4n-1)\left(\frac{\max\limits_{0\le i+j\le n}\omega_{i,j}}{\min\limits_{0\le i+j\le n}\omega_{i,j}}\right)^3 \\ &\times\max_{\substack{0\le i+j\le n\\0\le p+q\le n}}\|\boldsymbol{R}_{i,j}-\boldsymbol{R}_{p,q}\|,\ (u,v)\in D. \quad (39)\end{aligned}$$

Finally, we try to find the bound on the second mixed derivative of the surface Eq.(29). Differentiating Eq.(29) with respect to the parameter v and using multiplication of Bernstein polynomials, we can derive the second mixed derivative as follows:

$$\begin{aligned}&\frac{\partial^2 \boldsymbol{R}(u,v)}{\partial u\partial v} \\ &=\frac{1}{\{D(u,v)\}^3}\left\{(2n-1)\sum_{i+j=0}^{2n-2}\left(\boldsymbol{H}_{i,j+1}-\boldsymbol{H}_{i,j}\right)B_{i,j}^{2n-2}(u,v)\right. \\ &\times\sum_{l+m=0}^{n}\omega_{l,m}B_{l,m}^{n}(u,v)-2n\sum_{i+j=0}^{n-1}\left(\omega_{i,j+1}-\omega_{i,j}\right)B_{i,j}^{n-1}(u,v) \\ &\left.\times\sum_{l+m=0}^{2n-1}\boldsymbol{H}_{l,m}B_{l,m}^{2n-1}(u,v)\right\} \\ &=\frac{1}{\{D(u,v)\}^3}\sum_{s+t=0}^{3n-2}\boldsymbol{P}_{s,t}B_{s,t}^{3n-2}(u,v),\quad (u,v)\in D, \quad (40)\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{P}_{s,t}&=\frac{2n-1}{\binom{3n-2}{n}}\sum_{i+j=\max\{0,s+t-n\}}^{\min\{2n-2,s+t\}}\left\{\omega_{s-i,t-j}\left(\boldsymbol{H}_{i,j+1}-\boldsymbol{H}_{i,j}\right)\right. \\ &\left.\times\binom{s}{i}\binom{t}{j}\binom{3n-2-s-t}{2n-2-i-j}\right\} \\ &-\frac{2n}{\binom{3n-2}{n-1}}\sum_{i+j=\max\{0,s+t-2n-1\}}^{\min\{n-1,s+t\}}\left\{\left(\omega_{i,j+1}-\omega_{i,j}\right)\boldsymbol{H}_{l,m}\right. \\ &\left.\times\binom{s}{i}\binom{t}{j}\binom{3n-2-s-t}{n-1-i-j}\right\}. \quad (41)\end{aligned}$$

The result above illustrates that the second mixed derivative of the surface Eq.(4) has the same direction as that of a degree $3n-2$ triangular B-B surface with control points $\boldsymbol{P}_{s,t}$ ($0\le s+t\le 3n-2$), and magnitude of only $\{D(u,v)\}^{-3}$ that of the latter. The following inequality is obtained from Eq.(40):

$$\begin{aligned}&\left\|\frac{\partial^2 \boldsymbol{R}(u,v)}{\partial u\partial v}\right\| \\ &\le \frac{1}{D^3(u,v)}\left\{2(2n-1)\left(\max_{0\le i+j\le 2n-1}\left\|\boldsymbol{H}_{i,j}\right\|\right)\left(\max_{0\le i+j\le n}\omega_{i,j}\right)\right. \\ &\left.+2n\left(\max_{0\le i+j\le n}|\Delta_2\omega_{i,j}|\right)\left(\max_{0\le i+j\le 2n-1}\left\|\boldsymbol{H}_{i,j}\right\|\right)\right\} \\ &\le \frac{2}{\left(\min\limits_{0\le i+j\le n}\omega_{i,j}\right)^3}\left\{(2n-1)\max_{0\le i+j\le n}\omega_{i,j}\right.\end{aligned}$$

$$+n \max_{0\le i+j\le n}\left|\Delta_2\omega_{ij}\right|\Big\} \max_{0\le i+j\le 2n-1}\left\|\boldsymbol{H}_{i,j}\right\|.$$

where Δ_2 is the shift operator with respect to the parameter v, i.e., $\Delta_2\omega_{i,j}=\omega_{i,j+1}-\omega_{i,j}$. In the same way, by the bound estimation Eq.(33) of the vector $\boldsymbol{H}_{l,m}$, we can finally obtain

$$\left\|\frac{\partial^2\boldsymbol{R}(u,v)}{\partial u\partial v}\right\| \le 2n\frac{\left(\max\limits_{0\le i+j\le n}\omega_{i,j}\right)^2}{\left(\min\limits_{0\le i+j\le n}\omega_{i,j}\right)^3}\left\{(2n-1)\max_{0\le i+j\le n}\omega_{i,j}+n\max_{0\le i+j\le n}\left|\Delta_2\omega_{i,j}\right|\right\} \times \max_{\substack{0\le i+j\le n\\0\le p+q\le n}}\left\|\boldsymbol{R}_{i,j}-\boldsymbol{R}_{p,q}\right\|, \qquad (u,v)\in D. \tag{42}$$

This is a bound estimation on the second mixed derivative of the surface Eq.(4), and the result can be simplified as follows:

$$\left\|\frac{\partial^2\boldsymbol{R}(u,v)}{\partial u\partial v}\right\| \le 2n(4n-1)\left(\frac{\max\limits_{0\le i+j\le n}\omega_{i,j}}{\min\limits_{0\le i+j\le n}\omega_{i,j}}\right)^3 \times \max_{\substack{0\le i+j\le n\\0\le p+q\le n}}\left\|\boldsymbol{R}_{i,j}-\boldsymbol{R}_{p,q}\right\|,\ (u,v)\in D. \tag{43}$$

In addition, the results Eqs.(42) and (43) can be rewritten to an equivalent expression with respect to the parameters u, v, and w. The same method is also used for the bound estimation on the second derivative with respect to the parameter v and the second mixed derivative with respect to the parameter (v,u), which are $\frac{\partial^2\boldsymbol{R}(u,v)}{\partial v^2}$ and $\frac{\partial^2\boldsymbol{R}(u,v)}{\partial v\partial u}$ respectively.

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