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Radial point collocation method (RPCM) for solving convection-diffusion problems*

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Abstract: In this paper, Radial point collocation method (RPCM), a kind of meshfree method, is applied to solve convection-diffusion problem. The main feature of this approach is to use the interpolation schemes in local supported domains based on radial basis functions. As a result, this method is local and hence the system matrix is banded which is very attractive for practical engineering problems. In the numerical examination, RPCM is applied to solve non-linear convection-diffusion 2D Burgers equations. The results obtained by RPCM demonstrate the accuracy and efficiency of the proposed method for solving transient fluid dynamic problems. A fictitious point scheme is adopted to improve the solution accuracy while Neumann boundary conditions exist. The meshfree feature of the present method is very attractive in solving computational fluid problems.

Key words: Radial basis functions, Radial point collocation method (RPCM), Collocation, Meshfree, Convection-diffusion

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INTRODUCTION

Meshfree methods are increasingly becoming popular as they are effective for dealing with computational mechanics problems including both solid and fluid problems. Most of various approaches proposed in (Liu, 2002) are based on Galerkin form (weak form) which need background meshes for numerical integrations, and are actually not truly meshfree methods. Other approaches are based on collocation form (strong form), such as finite point method (FPM) (Onate *et al*., 1996), radial basis functions (RBFs) (Kansa, 1990; Kansa and Hon, 2000; Liu and Gu, 2002; 2003; Lee *et al*., 2003; Liu *et al*., 2005a; 2005b; Wu and Liu, 2003). The strong-form method needs no meshes and is truly meshfree. In addition, its implementation is simple and straightforward. Specialized numerical stable processes should be used in FPM, namely additional stable terms must be imposed in the procedure of collocation schemes, which is not very convenient. Another commonly used collocation method is RBFs which has undergone intensive research and enjoyed considerable success as a technique for interpolating multivariate data and solving various kinds of partial differential equations (PDEs) (Kansa, 1990; Kansa and Hon, 2000). However, the primary disadvantage of traditional RBF approach is that it is global, and as a result, its coefficient matrices obtained from this discretization scheme are full. Full matrices tend to become progressively more ill-conditioned as the rank increases. In (Kansa and Hon, 2000), several techniques are explored to improve the conditioning of the coefficient matrix and the solution accuracy. A local multiquadrics (MQ) approach, which can assure that its coefficient matrices have sufficient bandwidth, has been presented to solve successfully boundary value problems, and its accuracy and convergence have been well investigated in (Lee *et al*., 2003).

This work applies a collocation scheme based on radial point interpolation method (PIM), termed as

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radial point collocation method (RPCM), to solve transient state convection diffusion equations. PIM was originally proposed by Liu and Gu (2001), and was further studied for fluid problems (Wu and Liu, 2003). So far, most application of PIM is based on Galerkin or Petrov-Galerkin weak forms, and needs numerical integrations. Its collocation form RPCM has also been applied to solve PDEs with normal derivative on Neumann boundaries (Liu *et al*., 2005b).

The current work is aimed at constructing point interpolation approximation with radial basis functions. Then, collocation schemes are proposed to solve transient-state convection-diffusion equations. Several linear and nonlinear convection diffusion problems are tested in our numerical computations using the present method. The fictitious point approach has been also presented to improve the accuracy for solving PDEs while Neumann boundary conditions exist. Some concluding remarks are given finally.

RADIAL POINT INTERPOLATION

The approximation of a function $u(\boldsymbol{x})$, using radial basis functions, may be written as a linear combination of n radial basis functions, viz.,

$$u(\boldsymbol{x}) \cong \hat{u}(\boldsymbol{x}) = \sum_{i=1}^{n} a_i \phi(\| \boldsymbol{r} - \boldsymbol{r}_i \|, c_i), \quad (1)$$

where n is the number of points in the supported domain near $\boldsymbol{x}$, a_i are coefficients to be determined and ϕ are the Multiquadrics (MQ) in this paper, i.e.,

$$\phi(\| \boldsymbol{r} - \boldsymbol{r}_i \|, c_i) = \left(\sqrt{\| \boldsymbol{r} - \boldsymbol{r}_i \|^2 + c_i^2} \right)^Q .$$

The shape parameter can be defined as $c_i=c=(\alpha_c r_c)$ (Liu, 2002).

For 2D problems:

$$\| \boldsymbol{r} - \boldsymbol{r}_i \| = \sqrt{(x - x_i)^2 + (y - y_i)^2}. \quad (2)$$

Due to its good accuracy and convergence, MQ is chosen as interpolation basis function in the following numerical calculations. When MQ function is adopted, the selection of the shape parameter c is an open issue. In a given interpolation, as c value increases, the shape of the MQ interpolants becomes flatter, and the collocation matrix becomes more singular. The accuracy of the approximation, however, gets better, until the numerical inversion breaks down due to round off error. How to choose the optimal shape parameter is a problem that has received the attention of many researchers (Liu, 2002; Lee *et al*., 2003). So far, this is an open question and no mathematical theory has been developed for determining the optimal value. Here, the form of dimensionless shape parameter α_c will be employed. r_c is the characteristic length that is related to the nodal space in the local supported domain of the collocation point and it is usually the average nodal spacing for all the nodes in this supported domain. For uniform discrete model, it is chosen to be the distance between uniformly equally distributed points. For scattered point model, it is chosen to be the average size of the supported domain. In addition, according to our computations, the additional polynomials in MQ function cannot enhance the accuracy and stability, although the thin plate spline (TPS) radial point collocation method can enhance the accuracy and stability when augmented with some lower order polynomials (Liu *et al*., 2005b).

The coefficients a_i in Eq.(1) can be determined by forcing the function interpolations to pass through all n nodes within the supported domain. The interpolations of the function at the kth point have the form:

$$\hat{u}(\boldsymbol{x}_k) = a_1 \phi(\| \boldsymbol{r}_k - \boldsymbol{r}_1 \|) + \cdots + a_n \phi(\| \boldsymbol{r}_k - \boldsymbol{r}_n \|), \quad k=1, \ldots, n. \quad (3)$$

They can be expressed by matrix formulations as follows:

$$\hat{\boldsymbol{u}}^{\mathrm{e}} = \boldsymbol{\Phi a}, \quad (4)$$

$$\hat{\boldsymbol{u}}^{\mathrm{e}} = [\hat{u}(\boldsymbol{x}_1) \ \cdots \ \hat{u}(\boldsymbol{x}_k) \ \cdots \ \hat{u}(\boldsymbol{x}_n)]^{\mathrm{T}}, \quad (5)$$

$$\boldsymbol{a} = [a_1 \ \cdots \ a_i \ \cdots \ a_n]^{\mathrm{T}}. \quad (6)$$

The expression of matrix $\boldsymbol{\Phi}$ can be found in (Liu *et al*., 2005b).

Thus the unknown coefficients vector is

$$\boldsymbol{a} = \boldsymbol{\Phi}^{-1} \hat{\boldsymbol{u}}^{\mathrm{e}}. \quad (7)$$

Finally, the approximation form of the function can be obtained as follows:

$$\hat{u}(\boldsymbol{x}) = \boldsymbol{\varphi a} = \boldsymbol{\varphi \Phi}^{-1}\hat{\boldsymbol{u}}^{\mathrm{e}} = \boldsymbol{\psi}\hat{\boldsymbol{u}}^{\mathrm{e}}, \tag{8}$$

$$\boldsymbol{\varphi} = [\phi(\| \boldsymbol{r} - \boldsymbol{r}_1 \|, c_1) \ \dots \ \phi(\| \boldsymbol{r} - \boldsymbol{r}_n \|, c_n)], \tag{9}$$

in which $\boldsymbol{\psi}$ are shape functions vector.

NONLINEAR CONVECTIVE EQUATIONS

Consider the following non-linear pure convective equation:

$$u_t + \nabla \cdot \boldsymbol{F}(u) = q, \tag{10}$$

$$\nabla \cdot \boldsymbol{F}(u) = \boldsymbol{a}(u) \cdot \nabla u, \ \boldsymbol{a}(u) = \mathrm{d}\boldsymbol{F}(u)/\mathrm{d}u. \tag{11}$$

Assuming $u^{m+1}=u(\boldsymbol{x}, t^{m+1})=u(\boldsymbol{x}, t^m+\Delta t)$, so that its Taylor series expansion can be obtained with respect to time variant t:

(1) Explicit form:

$$u^{m+1} = u^m + \Delta t \cdot u_t^m + \frac{1}{2}\Delta t^2 \cdot u_{tt}^m + o(\Delta t^3). \tag{12}$$

From Eq.(10), we obtain

$$u_t = q - \nabla \cdot \boldsymbol{F}(u), \tag{13}$$

$$\begin{aligned} u_{tt} &= -\partial_t[\nabla \cdot \boldsymbol{F}(u)] = -\nabla \cdot [\partial_t \boldsymbol{F}(u)] = -\nabla \cdot [\boldsymbol{a}(u)u_t] \\ &= -\nabla \cdot \{\boldsymbol{a}(u)[-\nabla \cdot \boldsymbol{F}(u)]\} = \nabla \cdot [\boldsymbol{a}(u)\nabla \cdot \boldsymbol{F}(u)]. \end{aligned} \tag{14}$$

Substituting Eqs.(13) and (14) into Eq.(12) yields the following Eq.(15). Here 3-order time derivatives have been omitted.

$$\begin{aligned} u^{m+1} = u^m &+ \Delta t[q - \nabla \cdot \boldsymbol{F}(u^m)] \\ &+ \frac{1}{2}\Delta t^2 \nabla \cdot [\boldsymbol{a}(u^m)\nabla \cdot \boldsymbol{F}(u^m)]. \end{aligned} \tag{15}$$

(2) Implicit form (2-order):

$$u^{m+1} = u^m + \Delta t \cdot u_t^{m+1}. \tag{16}$$

(3) Crank-Nicholson:

$$u^{m+1} = u^m + \frac{\Delta t}{2}(u_t^m + u_t^{m+1}). \tag{17}$$

Substituting approximate function $\hat{u}(\boldsymbol{x})$ in Eq.(8) into Eq.(15) or Eq.(16) or Eq.(17) yields the final discrete simultaneous equations.

NUMERICAL SIMULATIONS

To illustrate the performance of the method described in previous sections, several time-dependent convection-diffusion examples known as Rotating Gaussian wave problem and 2D Burgers equations are considered. The results were obtained by MQ-RPCM. The errors in computations are defined as follows:

$$\begin{aligned} e &= \sqrt{\sum_{i=1}^{N}(u_i^{\mathrm{ex}} - \hat{u}_i)^2 \Big/ \sum_{i=1}^{N}(u_i^{\mathrm{ex}})^2}, \\ e_x &= \sqrt{\sum_{i=1}^{N}\left(\frac{\partial u_i^{\mathrm{ex}}}{\partial x} - \frac{\partial \hat{u}_i}{\partial x}\right)^2 \Big/ \sum_{i=1}^{N}\left(\frac{\partial u_i^{\mathrm{ex}}}{\partial x}\right)^2}, \\ e_y &= \sqrt{\sum_{i=1}^{N}\left(\frac{\partial u_i^{\mathrm{ex}}}{\partial y} - \frac{\partial \hat{u}_i}{\partial y}\right)^2 \Big/ \sum_{i=1}^{N}\left(\frac{\partial u_i^{\mathrm{ex}}}{\partial y}\right)^2}. \end{aligned} \tag{18}$$

Example 1 2D Burgers equation

Burgers equation in 2D is

$$u_t + uu_x + uu_y = \varepsilon \Delta u, \ (x, y) \in [0,1] \times [0,1]. \tag{19}$$

The exact solution is:

$$u^{\mathrm{ex}}(x, y, t) = \left\{1 + \exp\left[\frac{x + y - (t + 0.25)}{2\varepsilon}\right]\right\}^{-1}. \tag{20}$$

The initial and boundary conditions are determined by the above exact solution as follows:

$$\begin{aligned} &u(x, y, 0) = u^{\mathrm{ex}}(x, y, 0), \\ &u(x, 0, t) = u^{\mathrm{ex}}(x, 0, t), \ u(x, 1, t) = u^{\mathrm{ex}}(x, 1, t), \\ &u(0, y, t) = u^{\mathrm{ex}}(0, y, t), \ u(1, y, t) = u^{\mathrm{ex}}(1, y, t). \end{aligned} \tag{21}$$

We ran the problem with ε=0.01. A uniformly distributed 41×41 nodes model over the unit square of solution was employed in the calculations, and MQ radial point interpolation approximation scheme was used for the spatial discretization. In the procedure of implicit time integration, 100-time-step with time interval of 0.01 is adopted. The supported domains are chosen according to the 25 nearest nodes in every supported domain for uniform and non-uniform models. Fig.1 shows the 3D graphs for the values of function and the absolute error at t=1.0 using 41×41 nodes model with shape parameters: $c=\alpha_c\times(1.0/40)$, α_c=6.0, Q=1.0. In addition, Fig.2 shows the 3D graphs for the values of the function and the absolute error at t=1.0 for the 1681 Halton scattered point model shown in Fig.3 (given in the next page).

Example 2 Possion equation with Neumann boundary condition

$$\nabla^2 u + u = (2+3x)\mathrm{e}^{x-y},\ (x,y)\in\Omega=[0,1]\times[0,1], \quad (22)$$

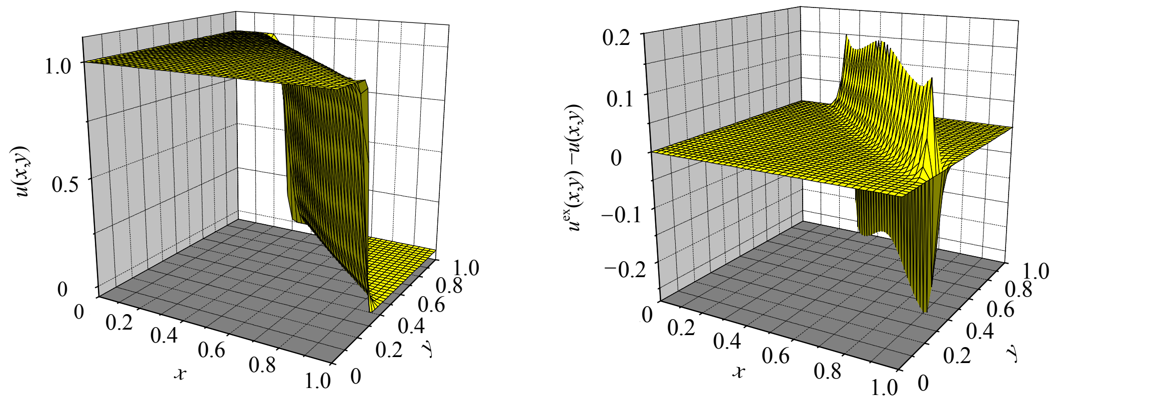
with the boundary conditions

$$\begin{aligned} &u(x,y)\big|_{\substack{x=0\\y=0}}=0,\\ &\partial u/\partial x\big|_{x=0}=\mathrm{e}^{-y},\quad \partial u/\partial y\big|_{y=0}=-x\mathrm{e}^{x},\\ &\partial u/\partial x\big|_{x=1}=2\mathrm{e}^{1-y},\ \partial u/\partial y\big|_{y=1}=-x\mathrm{e}^{x-1}. \end{aligned} \quad (23)$$

The exact solution is given by

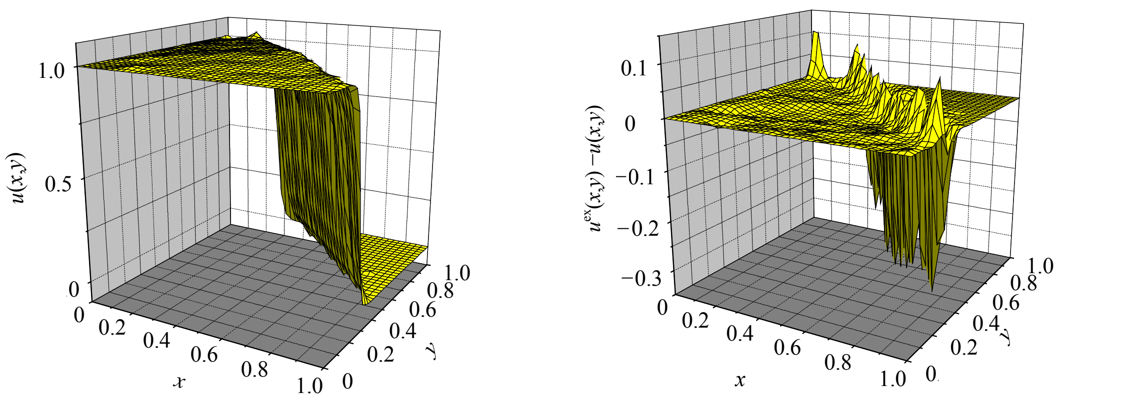
$$u^{\mathrm{ex}}(x,y)=x\mathrm{e}^{x-y}. \quad (24)$$

Uniform and scattered point models are employed to investigate the efficiency of fictitious point



(a) (b)

Fig.1 Solution for 2D Burger equation at t=1.0 with 81×81 uniform model for Example 1. (a) Numerical solution for $u(x,y)$; (b) Absolute error



(a) (b)

Fig.2 Solution for 2D Burgers equation at t=1.0 with 1681 Halton scattered points model for Example 1. (a) Numerical solution for $u(x,y)$; (b) Absolute error

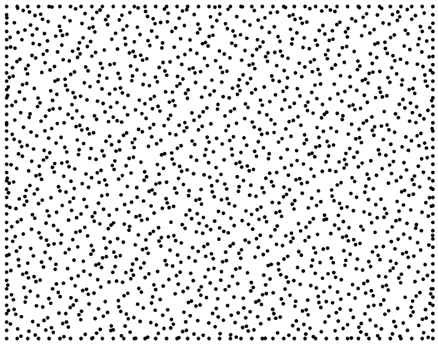


Fig.3 1681-node Halton scattered points model for Example 1

scheme for dealing with Neumann boundary conditions. The results obtained by the uniform 11×11 nodes model shown in Fig.4a are listed in Table 1. It is obvious that the accuracy has been greatly improved by using fictitious point scheme for Neumann boundaries. The same conclusion can be obtained for scattered point model shown in Fig.4b. Table 2 shows the results obtained for scattered point model.

Example 3 Non-linear convection-diffusion problem in 2D with Neumann boundary conditions: Double-variant Burgers equation (Donea *et al*., 2000).

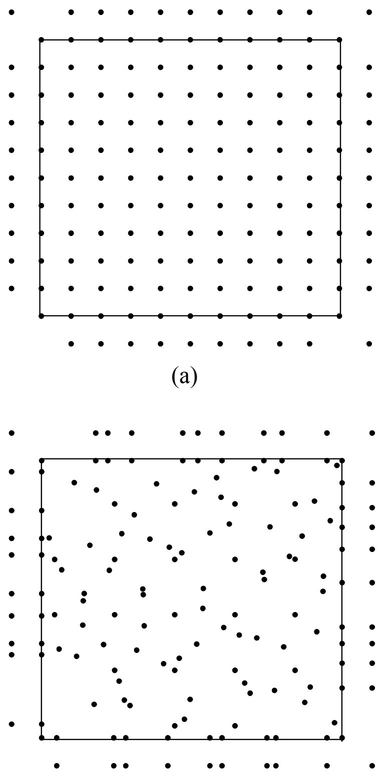
Consider the solution of a 2D Burgers problem over the square domain $\boldsymbol{x}\in\Omega=[0,1]\times[0,1]$ for which an analytical solution can be derived, thus allowing direct assessment of the quality of the numerical results obtained with the RPCM.

Table 1 Uniform 11×11 nodes model, 25 nearest nodes in every supported domain (r_c=0.1)

α_c	Without fictitious point near Neumann BC			39 fictitious points near Neumann BC		
	e (%)	e_x (%)	e_y (%)	e (%)	e_x (%)	e_y (%)
3.0	149.80	60.00	134.40	6.380	1.490	4.280
4.0	189.20	75.30	170.90	1.920	0.480	1.370
5.0	134.60	51.60	116.40	0.620	0.200	0.520
6.0	86.00	32.10	72.50	0.220	0.110	0.250
7.0	10.90	4.91	15.10	0.096	0.078	0.160
8.0	3.71	1.94	3.37	0.056	0.060	0.120
9.0	1.46	1.28	3.09	0.047	0.048	0.100
10.0	1.13	0.82	1.77	0.046	0.040	0.089
11.0	8.33	4.33	10.02	0.070	0.037	0.088
12.0	13.28	6.69	15.62	0.026	0.034	0.070
13.0	–	–	–	0.150	0.060	0.140
14.0	–	–	–	0.095	0.355	0.550

Table 2 121-node scattered point model, 25 nearest nodes in every supported domain

α_c	Without fictitious point near Neumann BC			39 fictitious points near Neumann BC		
	e (%)	e_x (%)	e_y (%)	e (%)	e_x (%)	e_y (%)
1.0	38.07	22.13	30.63	44.20	14.150	32.73
2.0	7.47	1.88	5.04	2.91	0.950	2.37
3.0	5.84	1.29	3.38	0.16	0.048	0.14
4.0	9.21	2.05	5.26	0.70	0.179	0.43
5.0	21.19	4.84	11.80	0.49	0.497	0.93



(a)

(b)

Fig.4 Uniform and scattered models with 160 nodes for Example 2 (121 real points within the solution domain and 39 fictitious points outside the solution domain). (a) Uniform model; (b) Scattered point model

$$\begin{cases} \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}=\varepsilon\nabla^2 u, \\ \dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}=\varepsilon\nabla^2 v, \end{cases} \quad \varepsilon=0.01, \qquad (25)$$

which are coupled through their non-linear convective terms, and by the following initial and boundary conditions

$$u(x,y,0)=\sin(\pi x)\cos(\pi y),$$
$$v(x,y,0)=\cos(\pi x)\sin(\pi y),$$
$$u(0,y,t)=u(1,y,t)=0,$$
$$v(x,0,t)=v(x,1,t)=0,$$
$$\frac{\partial u}{\partial n}(x,0,t)=\frac{\partial u}{\partial n}(x,1,t)=0,$$
$$\frac{\partial v}{\partial n}(0,y,t)=\frac{\partial v}{\partial n}(1,y,t)=0.$$

The analytical solution to the above Burgers problem has been given by Donea *et al.*(2000).

A uniformly distributed 31×31 nodes model with adding 120 fictitious points for dealing Neumann boundary conditions over the unit square of solution has been employed in the calculations and MQ-RPCM has been used for the spatial discretization. The supported domains are chosen according to 25 nearest nodes in every supported domain for uniform and scattered point models. In the procedure of time integration, 100-time-step with time interval 0.01 is adopted. Fig.5 shows the value of function at diagonal line $x=y$ at t=0.2, 0.6, 1.0. Fig.7 shows the 3D graphs for the values of function and its contour lines at t=1.0. Fig.8 shows the 3D graphs for the values of function and its contour lines at t=1.0 for 961 Halton scattered points model as shown in Fig.6.

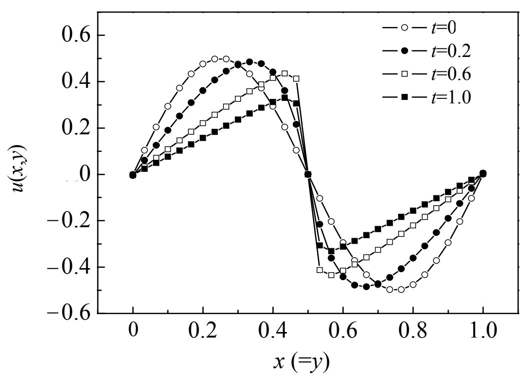


Fig.5 Value of *u*(*x*,*y*) on the diagonal at times *t*=0, 0.2, 0.6, 1.0 for Example 3

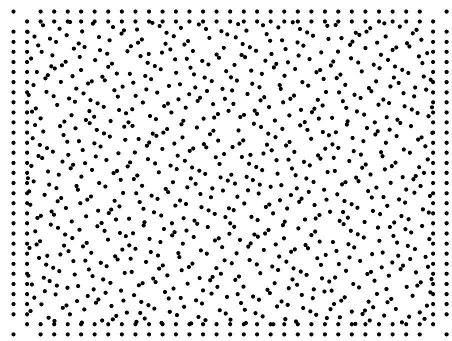


Fig.6 Halton scattered point model for Example 3 (961-node Halton point model+120 fictitious points)

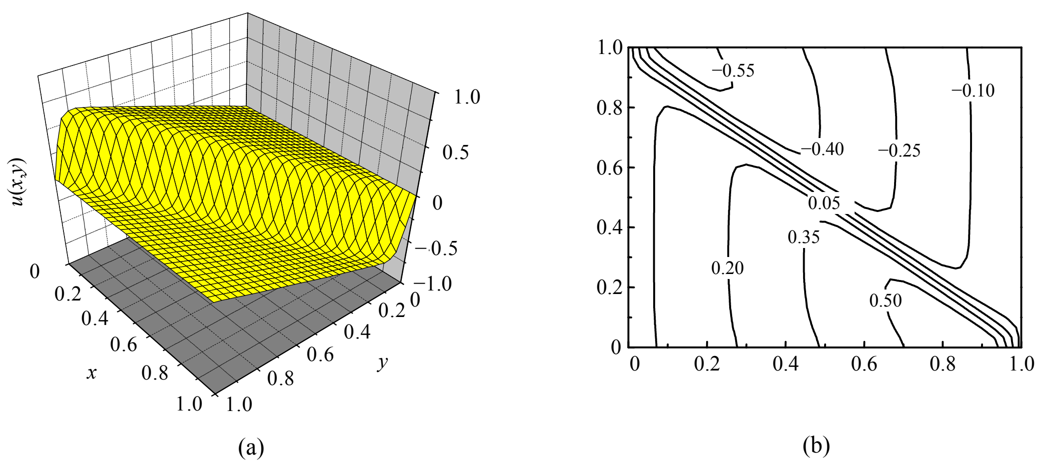
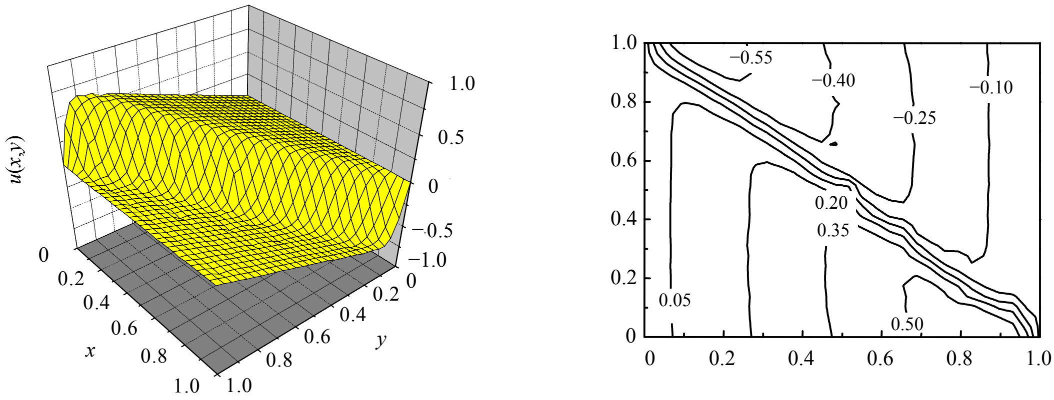


Fig.7 Value of *u*(*x*,*y*) and its contour line at *t*=1.0 obtained with 31×31 uniform model for Example 3. (a) Value of *u*(*x*,*y*); (b) Contour line of *u*(*x*,*y*)



(a) (b)

Fig.8 Value of *u*(*x*,*y*) and its contour line at *t*=1.0 obtained with 961 Halton scattered point model for Example 3. (a) Value of *u*(*x*,*y*); (b) Contour line of *u*(*x*,*y*)

CONCLUSION

Radial point collocation method (RPCM) is presented for solving convection-diffusion equations in this paper. The method has the following advantages: (1) truly meshfree method; (2) local interpolation that leads to banded algebraic equation; (3) no need for special numerical stable processes like those used in FPM. The accuracy of the numerical solutions indicates that the present method is well suited for transient convection-diffusion problems.

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