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Symmetric alteration of four knots of B-spline and NURBS surfaces*

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Abstract: Modifying the knots of a B-spline curve, the shape of the curve will be changed. In this paper, we present the effect of the symmetric alteration of four knots of the B-spline and the NURBS surfaces, i.e., symmetrical alteration of the knots of surface, the extended paths of points of the surface will converge to a point which should be expressed with several control points. This theory can be used in the constrained shape modification of B-spline and NURBS surfaces.

Key words: B-spline surface, NURBS surface, Knot modification, Path
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INTRODUCTION

Modification of knots of a B-spline curve changes the shape of the curve. In some recent publications (Juhász and Hoffmann, 2001; 2003; Hoffmann and Juhász, 2004; Li and Wang, 2005), the authors studied the effect of the alteration of a single knot on the shape of the curve. Hoffmann (2004) and Li & Wang (2006) applied these theories into the constrained shape modification of B-spline and NURBS curves. Juhász (2001) and Hoffmann & Juhász (2005) studied the effect of symmetric alteration of two knots of B-spline curves and obtained some theoretical results to describe the characteristics of the restricted part of the path.

We present the effect of symmetric alteration of four knots of B-spline and NURBS surfaces. And in the conclusion we describe the characteristics of the restricted part of the path of the surfaces, which can be a theoretical base of surface modification.

This paper is organized as follows: In Section 2, the basic definitions and notations will be presented. Then in Section 3 we will present the effect of symmetrically modifying two knots of NURBS curves. Symmetric alteration of four knots of B-spline and NURBS surfaces will be presented in Section 4 and Section 5 respectively.

PRELIMINARY KNOWLEDGE

B-spline basis

B-spline curves are well-known tools in computer-aided geometric design with optimal shape preserving properties. They are polynomial curves defined as linear combination of the control points by some basis functions over a closed interval, whose subdivision values are called knots. The basic definitions of the basis functions and the curve are as follows.

Definition 1 Let $U=\{u_l\}_{-\infty}^{+\infty}$ be a given knot sequence with $u_l \le u_{l+1}$, then the normalized B-spline basis functions of order k (degree k–1) are defined recursively by the following equations:

$$N_{l,1}=\begin{cases}1, & \text{if } u\in[u_l,u_{l+1}),\\ 0, & \text{otherwise.}\end{cases}$$

$$N_{l,k}(u)=\frac{u-u_l}{u_{l+k-1}-u_l}N_{l,k-1}(u)+\frac{u_{l+k}-u}{u_{l+k}-u_{l+1}}N_{l+1,k-1}(u).$$

Here we define 0/0=0.

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Definition 2 Let $\{N_{0,k}(u), N_{1,k}(u), \ldots, N_{n,k}(u)\}$ be the normalized B-spline basis functions of order k defined on a given knot sequence U as in Definition 1, then a B-spline curve can be defined as follows:

$$\boldsymbol{b}(u)=\sum_{l=0}^{n} N_{l,k}(u)\boldsymbol{p}_l,\quad u\in[u_{k-1},u_{n+1}],$$

where $\boldsymbol{p}_l$ (l=0, 1, …, n) are control points.

The ith arc can be written as

$$\boldsymbol{b}_i(u)=\sum_{l=i-k+1}^{i} N_{l,k}(u)\boldsymbol{p}_l,\ u\in[u_i,u_{i+1}],\ k-1\le i\le n.$$

Definition 3 The surface $\boldsymbol{s}(u, v)$ defined by

$$\boldsymbol{s}(u,v)=\sum_{l=0}^{n}\sum_{g=0}^{m} N_{l,k}(u)N_{g,h}(v)\boldsymbol{p}_{lg},$$
$$u\in[u_{k-1},u_{n+1}],\ v\in[v_{h-1},v_{m+1}]$$

is called B-spline surface of order (k,h), where $N_{l,k}(u)$ and $N_{g,h}(v)$ are B-spline basis functions of order k and h, which are defined on given knot sequences $U=\{u_l\}_{-\infty}^{+\infty}$ and $V=\{v_g\}_{-\infty}^{+\infty}$ respectively, with $\boldsymbol{p}_{lg}$ being control points. The (i,j)th patch can be written as

$$\boldsymbol{s}_{ij}(u,v)=\sum_{l=i-k+1}^{i}\sum_{g=j-h+1}^{j} N_{l,k}(u)N_{g,h}(v)\boldsymbol{p}_{lg},$$
$$u\in[u_i,u_{i+1}],\ v\in[v_j,v_{j+1}].$$

NURBS curves and surfaces

Using the notations above, we give the definitions of the NURBS curve and surface:

Definition 4 With $\{N_{0,k}(u), N_{1,k}(u), \ldots, N_{n,k}(u)\}$ being the normalized B-spline basis functions of order k defined on a given knot sequence U, and w_l being weights, $\boldsymbol{p}_l$ being control points, a NURBS curve can be defined by:

$$\boldsymbol{r}(u)=\frac{\sum_{l=0}^{n} N_{l,k}(u)w_l\boldsymbol{p}_l}{\sum_{l=0}^{n} N_{l,k}(u)w_l},\ u\in[u_{k-1},u_{n+1}].$$

Definition 5 The surface $\boldsymbol{f}(u, v)$ defined by

$$\boldsymbol{f}(u,v)=\frac{\sum_{l=0}^{n}\sum_{g=0}^{m} N_{l,k}(u)N_{g,h}(v)w_{lg}\boldsymbol{p}_{lg}}{\sum_{l=0}^{n}\sum_{g=0}^{m} N_{l,k}(u)N_{g,h}(v)w_{lg}},$$
$$u\in[u_{k-1},u_{n+1}],\ v\in[v_{h-1},v_{m+1}] \tag{1}$$

is called the NURBS surface of order (k,h), where $\boldsymbol{p}_{lg}$, w_{lg} are control points and weights, respectively.

Symmetric alteration of two knots of B-spline curves

When a knot u_i is altered, points of the B-spline curve will move on special rational curves $\boldsymbol{b}(u,u_i)$ called paths (Juhász and Hoffmann, 2001). In order to preserve the monotony of knot values, u_i cannot take any value but has to be within the range $[u_{i-1},u_{i+1}]$. Analogously, symmetrically modifying two knots u_i and u_j, $i<j$ to the values $u_i+\lambda$ and $u_i-\lambda$, the value of λ must be within the range $[-c,c]$, $c=\min(u_i-u_{i+1}, u_{i+1}-u_i, u_j-u_{j+1}, u_{j+1}-u_j)$. So paths obtained by the modification of one or more knots are relatively short arcs due to the limited range of the modification. Hoffmann and Juhász (2004) extended the paths by extending the value of the modified knots, that is, during the modification of u_i, $u_i<u_{i-1}$ and $u_i<u_{i+1}$ are allowed. For the symmetric alteration of two knots u_i and u_j, Hoffmann and Juhász (2005) extended the paths by extending the value of λ:

Lemma 1 Symmetrically altering the knots u_i and u_{i+z} ($z\in\{1, 2, \ldots, k\}$, where k is the order of the original B-spline curve), extended paths of points of the arcs $\boldsymbol{b}_j$ ($j=i, i+1, \ldots, i+z-1$), converge to the midpoint of the segment bounded by the control points $\boldsymbol{p}_i$ and $\boldsymbol{p}_{i+z-k}$ when $\lambda\to-\infty$, i.e.,

$$\lim_{\lambda\to-\infty}\boldsymbol{b}(u,\lambda)=\lim_{\lambda\to-\infty}\sum_{l=0}^{n} N_{l,k}(u,\lambda)\boldsymbol{p}_l=\frac{\boldsymbol{p}_i+\boldsymbol{p}_{i+z-k}}{2},$$
$$u\in[u_i,u_{i+z}). \tag{2}$$

SYMMETRIC ALTERATION OF TWO KNOTS OF NURBS CURVES

In the same way, if we symmetrically modify two knots of a NURBS curve, we get the extended paths:

$$\boldsymbol{r}(u,\lambda)=\frac{\sum_{l=0}^{n}N_{l,k}(u,\lambda)w_l\boldsymbol{p}_l}{\sum_{l=0}^{n}N_{l,k}(u,\lambda)w_l},\ u\in[u_{k-1},u_{n+1}].$$

Denoting $w_l\boldsymbol{p}_l$ by $\boldsymbol{q}_l$ and letting $\lambda\to-\infty$, by Lemma 1, we have

$$\lim_{\lambda\to-\infty}\sum_{l=0}^{n}N_{l,k}(u,\lambda)w_l=\frac{w_i+w_{i+z-k}}{2},$$

$$\lim_{\lambda\to-\infty}\sum_{l=0}^{n}N_{l,k}(u,\lambda)\boldsymbol{q}_l=\frac{\boldsymbol{q}_i+\boldsymbol{q}_{i+z-k}}{2}=\frac{w_i\boldsymbol{q}_i+w_{i+z-k}\boldsymbol{q}_{i+z-k}}{2}.$$

So we have the following theorem (Figs.1 and 2):

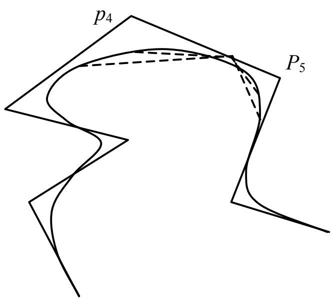

Fig.1 Extended paths of the points of a cubic NURBS curve obtained by symmetric modification of u_5 and u_7 with the weights w_3:w_5=3:4

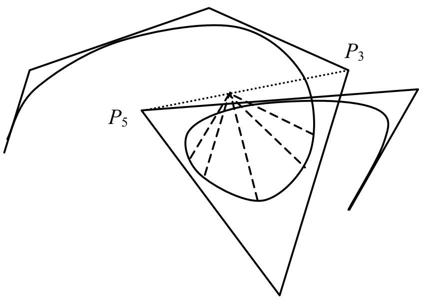

Fig.2 Extended paths of the points of a cubic NURBS curve obtained by symmetric modification of u_5 and u_8 with the weights w_4:w_5=2:4

Theorem 1 Symmetrically altering the knots u_i and u_{i+z}, ($z\in\{1, 2, \ldots, k\}$, where k is the order of the original NURBS curve), extended paths of points of the arcs $\boldsymbol{r}_j$ ($j=i, i+1, \ldots, i+z-1$) converge to a point which should be expressed with barycentric coordinates

$$\left(\frac{w_i}{w_i+w_{i+z-k}},\ 1-\frac{w_i}{w_i+w_{i+z-k}}\right)$$

with respect to the endpoints of the line segment from $\boldsymbol{p}_i$ to $\boldsymbol{p}_{i+z-k}$ when $\lambda\to-\infty$, i.e.,

$$\lim_{\lambda\to-\infty}\boldsymbol{r}(u,\lambda)=\frac{w_i\boldsymbol{p}_i+w_{i+z-k}\boldsymbol{p}_{i+z-k}}{w_i+w_{i+z-k}},\ u\in[u_i,u_{i+z}).\quad(3)$$

SYMMETRIC ALTERATION OF FOUR KNOTS OF B-SPLINE SURFACES

Symmetrically modifying four knots of a B-spline surface, that is, changing the values of u_i, u_{i+a}, and v_j, v_{j+b}, $a\in\{1, 2, \ldots, k\}$, $b\in\{1, 2, \ldots, h\}$ to $u_i+\lambda$, $u_{i+a}-\lambda$ and $v_j+\mu$, $v_{j+b}-\mu$ respectively, the extended paths of patches $\boldsymbol{s}(u, v, \lambda, \mu)$, $u\in[u_i, u_{i+a}]$, $v\in[v_j, v_{j+b}]$ can be expressed as:

$$\boldsymbol{s}(u,v,\lambda,\mu)=\sum_{l=i-k+1}^{i+a-1}\sum_{g=j-h+1}^{j+b-1}N_{l,k}(u,\lambda)N_{g,h}(v,\mu)\boldsymbol{p}_{lg}.$$

Here we prove the following theorem (Fig.3):

Theorem 2 Symmetrically altering four knots u_i, u_{i+a} and v_j, v_{j+b}, $a\in\{1, 2, \ldots, k\}$, $b\in\{1, 2, \ldots, h\}$ of a B-spline surface, the extended paths of points of patches $\boldsymbol{s}(u, v, \lambda, \mu)$ converge to a point when $\lambda\to-\infty$ and $\mu\to-\infty$:

$$\lim_{\substack{\lambda\to-\infty\\ \mu\to-\infty}}\boldsymbol{s}(u,v,\lambda,\mu)=\frac{1}{4}(\boldsymbol{p}_{ij}+\boldsymbol{p}_{i+a-k,j}+\boldsymbol{p}_{i,j+b-h}+\boldsymbol{p}_{i+a-k,j+b-h}).\quad(4)$$

Proof From Section 2 of (Hoffmann and Juhász, 2005), we know that the following equations hold for $u\in[u_i, u_{i+a}]$:

$$\lim_{\lambda\to-\infty}N_{i,k}(u,\lambda)=\lim_{\lambda\to-\infty}N_{i+a-k,k}(u,\lambda)=\frac{1}{2},\ a\in\{1,\ldots,k-1\}$$

$$\lim_{\lambda\to-\infty}N_{i,k}(u,\lambda)=\lim_{\lambda\to-\infty}N_{i+a-k,k}(u,\lambda)=1,\ a=k$$

$$\lim_{\lambda\to-\infty}N_{l,k}(u,\lambda)=0,\ l\neq i,\ i+a-k;\ a\in\{1,\ldots,k\}$$

It is the same for the v-direction:

$$\lim_{\mu\to-\infty}N_{j,h}(v,\mu)=\lim_{\mu\to-\infty}N_{j+b-h,h}(v,\mu)=\frac{1}{2},\ b\in\{1,\ldots,h-1\},$$

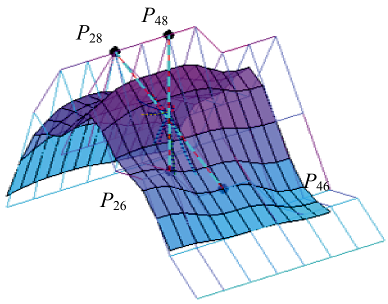

Fig.3 Extended paths of the points of a (3,3) B-spline surface obtained by symmetric modification of pairs of knots u_4, u_6 and v_8, v_{10}

$$\lim_{\mu\to-\infty} N_{j,h}(v,\mu)=\lim_{\mu\to-\infty} N_{j+b-h,h}(v,\mu)=1,\ b=h,$$

$$\lim_{\mu\to-\infty} N_{g,h}(v,\mu)=0,\ g\neq j,\ j+b-h;\ b\in\{1,\ ...,\ h\}.$$

The above equations yield Eq.(4).

SYMMETRIC ALTERATION OF FOUR KNOTS OF NURBS SURFACES

Substituting $w_{lg}\boldsymbol{p}_{lg}$ by $\boldsymbol{q}_{lg}$ in Eq.(1) and symmetrically modifying four knots of a NURBS surface, as in Section 4, we get the extended paths of the segments $\boldsymbol{f}(u, v, \lambda, \mu)$, $u\in[u_i, u_{i+a}]$, $v\in[v_j, v_{j+b}]$ of the NURBS surface

$$\boldsymbol{f}(u,v,\lambda,\mu)=\frac{\sum_{l=i-k+1}^{i+a-1}\sum_{g=j-h+1}^{j+b-1}N_{l,k}(u,\lambda)N_{g,h}(v,\mu)\boldsymbol{q}_{lg}}{\sum_{l=i-k+1}^{i+a-1}\sum_{g=j-h+1}^{j+b-1}N_{l,k}(u,\lambda)N_{g,h}(v,\mu)w_{lg}}.$$

When taking the limits $\lambda\to-\infty$ and $\mu\to-\infty$ of the above paths, by Theorem 2 the numerator and the denominator of these limits become:

$$\lim_{\substack{\lambda\to-\infty\\ \mu\to-\infty}}\sum_{l=i-k+1}^{i+a-1}\sum_{g=j-h+1}^{j+b-1}N_{l,k}(u,\lambda)N_{g,h}(v,\mu)\boldsymbol{q}_{lg}=\frac{1}{4}(\boldsymbol{q}_{ij}+\boldsymbol{q}_{i+a-k,j}+\boldsymbol{q}_{i,j+b-h}+\boldsymbol{q}_{i+a-k,j+b-h}),$$

$$\lim_{\substack{\lambda\to-\infty\\ \mu\to-\infty}}\sum_{l=i-k+1}^{i+a-1}\sum_{g=j-h+1}^{j+b-1}N_{l,k}(u,\lambda)N_{g,h}(v,\mu)w_{lg}=\frac{1}{4}(w_{ij}+w_{i+a-k,j}+w_{i,j+b-h}+w_{i+a-k,j+b-h}).$$

Therefore, we get the theorem for the symmetric alteration of four knots of NURBS surfaces (Fig.4):

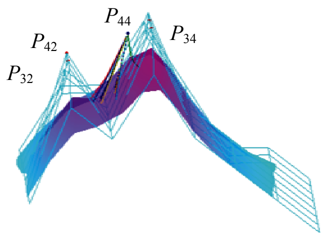

Fig.4 Extended paths of the points of a (3,3) NURBS surface obtained by symmetric modification of pairs of knots u_4, u_7 and v_4, v_6

Theorem 3 Symmetrically altering four knots u_i, u_{i+a} and v_j, v_{j+b}, $a\in\{1, 2, \ldots, k\}$, $b\in\{1, 2, \ldots, h\}$ of a NURBS surface, the extended paths of points of patches $\boldsymbol{f}(u, v, \lambda, \mu)$ converge to a point when $\lambda\to-\infty$ and $\mu\to-\infty$, that is:

$$\lim_{\substack{\lambda\to-\infty\\ \mu\to-\infty}}\boldsymbol{f}(u,v,\lambda,\mu)=\frac{\boldsymbol{q}_{ij}+\boldsymbol{q}_{i+a-k,j}+\boldsymbol{q}_{i,j+b-h}+\boldsymbol{q}_{i+a-k,j+b-h}}{w_{ij}+w_{i+a-k,j}+w_{i,j+b-h}+w_{i+a-k,j+b-h}},$$

where $w_{lg}\boldsymbol{p}_{lg}=\boldsymbol{q}_{lg}$; $l=i, i+a-k$; $g=j, j+b-h$.

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