Journal of Zhejiang University SCIENCE A
ISSN 1009-3095 (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

An efficient method for tracing planar implicit curves

YU Zheng-sheng[†1,2], CAI Yao-zhi[3], OH Min-jae[2], KIM Tae-wan[†‡2], PENG Qun-sheng[4]

([1]Computer Science School, Hangzhou Dianzi University, Hangzhou 310018, China)

([2]Department of Naval Architecture and Ocean Engineering, Seoul National University, Seoul 151-744, Korea)

([3]Applied Mathematics Department, Zhejiang University, Hangzhou 310027, China)

([4]State Key Laboratory of CAD & CG, Zhejiang University, Hangzhou 310027, China)

[†]E-mail: yuzhengsheng@tom.com; taewan@snu.ac.kr

Received Apr. 3, 2006; revision accepted Apr. 17, 2006

Abstract: This paper presents a method for tracing a planar implicit curve $f(x, y)=0$ on a rectangular region based on continuation scheme. First, according to the starting track-point and the starting track-direction of the curve, make a new function $F(x, y)=0$ where the same curve with $f(x, y)=0$ is defined. Then we trace the curve between the two domains where $F(x, y)>0$ and $F(x, y)<0$ alternately, according to the two rules presented in this paper. Equal step size or adaptive step size can be used, when we trace the curve. An irregular planar implicit curve (such as the curve with large curvatures at some points on the curve), can be plotted if an adaptive step size is used. Moreover, this paper presents a scheme to search for the multiple points on the curve. Our method has the following advantages: (1) it can plot C^0 planar implicit curves; (2) it can plot the planar implicit curves with multiple points; (3) by the help of using the two rules, our method does not need to compute the tangent vector at the points on the curve, and directly searches for the direction of the tracing curve; (4) the tracing procedure costs only one of two evaluations of function $f(x, y)=0$ per moving step, while most existing similar methods cost more evaluations of the function.

Key words: Planar implicit curve, Curve tracing, Continuation method, Geometric modeling
doi:10.1631/jzus.2006.A1115 **Document code:** A **CLC number:** TP39

INTRODUCTION

Tracing a planar implicit curve $f(x, y)=0$ on a rectangular region $[x_l, x_r]\times[y_b, y_t]$ is of great interest in Computer-Aided Design and Computer Graphics. While parametric curves are easy to plot, plotting implicit curves is a challenging problem. Planar implicit curve plotting method can be classified into two categories (Shou *et al*., 2005; Martin *et al*., 2002; Lopes *et al*., 2002). In the first category are subdivision methods (Shou *et al*., 2005; Martin *et al*., 2002) which are also called enumeration methods by Lopes *et al*.(2002). In the other category are continuation methods (Cai, 1990; Chandler, 1988; van Aken and Novak, 1985; van Aken, 1984; Bresenham, 1965; 1977; Cohen, 1976; Lennon *et al*., 1973). Subdivision methods begin with the whole plotting rectangular region itself as a start cell. If the cell is empty, namely it is not passed by the plotting curve, it is discarded; otherwise it is subdivided into smaller cells, which are then visited recursively, until these cells reach pixel size. There are two main problems with subdivision methods: the selection of the grid, so that we do not lose small components of the curve, and determination of whether a cell in the grid intersects the curve. Continuation methods are usually cheap because they use one or more seed pixels on the curve and then trace the curve continuously.

Here we introduce existing continuation methods. Bresenham (1965) drew a line by deciding the next tracing point using midpoint method. Bresenham's circle algorithm (Bresenham, 1977) takes advantage of symmetry and the equation of the circle to simplify greatly the arithmetic of the decision procedure for selecting the next pixel at each step of the represen-

‡ Corresponding author

tation. Both of the above Bresenham's methods cannot be adapted to general planar implicit curves. Lennon's method (Lennon *et al.*, 1973) and Cohen's method (Cohen, 1976) trace the implicit curves by computing the tangent vectors at the tracking points of every step. Therefore, the two methods also cannot plot C^0 planar implicit curves. van Aken (1984) presented an incremental ellipse generator which is only used in plotting an ellipse. van Aken and Novak (1985) presented a midpoint method for deriving curve-drawing algorithms for generating nonparametric curves. The method may not be successful, however, in a region where two edges of a curve actually cross each other. Chandler (1988) proposed a tracking algorithm for implicitly defined curve [described by an equation of the form $f(x, y)=0$], which produces the next approximating pixel by looking for a sign difference in function evaluations at midpoints between the eight nearest neighboring pixels. It is possible for the algorithm to track the implicit curve when there are multiple points (where the curve crosses itself). However, the procedure costs between two and eight evaluations of function $f(x, y)$ per moving step for the curves without multiple point.

The continuation method presented here as compared to earlier approaches has main advantages which include:

(1) Generality: The algorithm can plot general planar implicit curves.

(2) Efficiency: By the use of the two rules given in this paper, the tracking procedure directly searches for the tracing direction. For the curves without multiple points, the procedure costs only one of two evaluations of function $f(x, y)$ for each moving step. The algorithm does not need to compute the tangent vector at points on the curve.

(3) Robustness: The algorithm can plot C^0 planar implicit curves and be used to plot the planar implicit curves with multiple points.

(4) Simplicity: The algorithm is relatively simple to implement.

The rest of our paper is structured as follows. Section 2 presents our method of plotting planar implicit curve. Section 3 gives 4 examples. Section 4 presents conclusions and future work.

PLOTTING PLANAR IMPLICIT CURVES

Generally, in an xy-plane, an implicit curve $f(x, y)=0$ has the property of partitioning a plane into three point sets (Cai, 1990): the first set, f^0 (on the curve) satisfies $f(x, y)=0$; the second f^+ (positive domain) satisfies $f(x, y)>0$; and the third f^- (negative domain) satisfies $f(x, y)<0$. If the curve defined by the equation $f(x, y)=0$ has no such property, the equation becomes the formula that can partition a plane into positive and negative domains. For example, although, the equation $(x^2+y^2-r^2)^2=0$ does not have the property, the curve defined by the equation, can be translated into the same curve defined by another equation $x^2+y^2-r^2=0$.

In this section, the whole procedure of plotting a planar implicit curve is discussed in detail. First, we discuss how to plot planar implicit curves without multiple points. Second, the method for tracing the planar implicit curves with multiple points is presented.

Plotting planar implicit curves without multiple points

To plot planar implicit curves without multiple points, we first form the initial state for tracing the curves, then trace the curves according to two rules to obtain a ladder polyline, and then generate an exact approaching polyline by recursive binary subdivision.

1. Forming the initial state for tracing a planar implicit curve

To trace the planar implicit curve $f(x, y)=0$, we must give or compute the following data: the ranges of arguments x and y ($x_l \le x \le x_r$ and $y_b \le y \le y_t$), step size dx ($dx>0$) in x direction, step size dy ($dy>0$) in y direction, starting point (x_0, y_0) on the curve, starting direction (Δx, Δy).

We can give starting direction (Δx, Δy) as follows. Give the quadrant number G of the starting direction (for example, if the tangent direction on the starting point of the curve is $(-1, 2)$, then G is equal to 2), then compute the starting direction by the following equation:

$$\begin{cases}\Delta x = dx \cdot \mathrm{sign}((G-1.5)(G-3.5)),\\ \Delta y = dy \cdot \mathrm{sign}(2.5-G),\end{cases} \tag{1}$$

where $\mathrm{sign}(x)=\begin{cases}1, & \text{if } x\ge 0,\\ -1, & \text{if } x<0.\end{cases}$

Apparently, for an arbitrarily given expression of a planar implicit curve $f(x, y)=0$ which can partition a plane into three domains (f^+, f^0, and f^-), first, we generally do not know which domain is f^+, and which domain is f^-. Therefore, we make a new expression $F(x, y)$ for the curve as shown in Eq.(2) according to the expression $f(x, y)=0$:

$$F(x, y)=\sigma f(x, y), \tag{2}$$

where in the monotonous segment of the curve beginning with the starting point (x_0, y_0), we give σ the value below:

$$\sigma=\begin{cases}-1, & \text{if } f(x_0+\Delta x, y_0)\ge 0,\\ 1, & \text{if } f(x_0+\Delta x, y_0)<0.\end{cases}$$

Obviously, from Eq.(2), we have $F(x_0+\Delta x, y_0)<0$. Suppose the starting direction of the curve is in the first quadrant, then $\Delta x=dx>0$, $\Delta y=dy>0$. If $F(x_0+\Delta x, y_0)<0$, then $F(x_0, y_0+\Delta y)>0$ according to the property that the curve partitions a plane to positive and negative domains. If the starting direction of the curve is in other quadrant, we get the same conclusion. Therefore $F(x_0, y_0+\Delta y)>0$.

After forming the initial state and making the function $F(x, y)$, we begin to trace the implicit curve according to the two rules which are presented below.

2. Two rules for plotting a planar implicit curve

As soon as the initial state is formed, we begin to trace the planar implicit curve. As shown in Fig.1, given the ranges of arguments x and y: $x_l\le x\le x_r$, $y_b\le y\le y_t$, we explain our method of tracing the planar implicit curve $f(x, y)=0$ using a large step. First, we formulate $F(x, y)$ according to Eq.(2), where $f(x, y)=0$ and $F(x, y)=0$ define the same curve, so that $F(x_0+\Delta x, y_0)<0$. Suppose that the starting point $\boldsymbol{P}_0=(x_0, y_0)$ is on the curve $F(x, y)=0$. That is $\boldsymbol{P}_0\in F^0$, where F^0 means $F(x, y)=0$. First, we move one step of Δx, and get to $\boldsymbol{P}_1=(x_0+\Delta x, y_0)$. According to Eq.(2), we have $F(\boldsymbol{P}_1)=F(x_0+\Delta x, y_0)<0$. As soon as we have entered into F^-: $F(x, y)<0$, we move to the domain of F^+:$F(x, y)>0$ in an appropriate direction (positive or negative) along the y-axis. In Fig.1, we apparently must move forward by $|\Delta y|$, until we enter into the domain of F^+, or get to F^0. Suppose the point that we get to is $\boldsymbol{P}_2$. Then we continue moving forward to F^- by selecting an appropriate direction. The rest to get to ($\boldsymbol{P}_3$, $\boldsymbol{P}_4$, …) may be deduced by analogy. In Fig.1, before getting to $\boldsymbol{P}_7$, the selected directions of Δx and Δy are positive. As soon as we stride over the turning point (x_1^*, y_1^*) of the y direction on the curve, where the turning point means the point (x, y) where $F_x(x, y)=0$ or $F_y(x, y)=0$, Δy is changed by $-|\Delta y|$. In the same way, as soon as we stride over the turn point (x_2^*, y_2^*) of the x direction on the curve, Δx is changed by $-|\Delta x|$. If we go on, we will get a ladder-shape polyline ($\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_3\boldsymbol{P}_4$…) approaching the curve $F(x, y)=0$. During the tracing procedure, the two rules given in the following are obeyed.

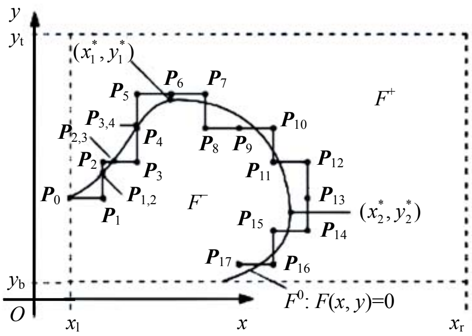


Fig.1 Plotting curve $F(x, y)=0$ using our method

Rule 1 for plotting a monotony planar implicit curve At first, we discuss how to plot a monotony planar implicit curve. Theorem 1 tells us that if Δx and Δy are determined and we obey Eq.(3), we must move close to the curve at the monotony-segment of the curve.

Theorem 1 If Δx and Δy are determined, and we obey Eq.(3), then we must move close to the curve $F(x, y)=0$ at the monotony-segment of the curve. Set $\boldsymbol{P}_0\boldsymbol{P}_1\ldots\boldsymbol{P}_n$ as the moving path, and d_i (i=0,1,…,n) as the distance between $\boldsymbol{P}_i$ and the curve. Set the error $\varepsilon_d=\max(d_0, d_1, \ldots, d_n)$, then we have Eq.(4).

$$\begin{cases}\text{if } (x,y)\in F^+\cup F_0,\ \text{namely, } F(x,y)\ge 0, & \\ \qquad \text{then move one step } \Delta x; & \text{(3a)}\\ \text{if } (x,y)\in F^-,\ \text{namely, } F(x,y)<0, & \\ \qquad \text{then move one step } \Delta y, & \text{(3b)}\end{cases}$$

$$\varepsilon_d\le\max(dx, dy). \tag{4}$$

Here Eq.(3) is called Rule 1 for plotting a monotony planar implicit curve.

Proof As in Fig.2, since we prescribe that the starting point (x_0, y_0) is at the curve, and $F(x_0, y_0)=0$ satisfies Eq.(3a), we move forward in the x direction at the first step. In Fig.2, the solid lines with arrow represent the direction of movement. After moving one step Δx, we get to $(x_0+\Delta x, y_0)$. Since $F(x_0+\Delta x, y_0)<0$, we move one step Δy for the second step according to Eq.(3b). For the third step, whether we move Δx or Δy in the same way is decided by $F(x_0+\Delta x, y_0+\Delta y)$ which satisfies Eq.(3a) or Eq.(3b). The rest after the third step may be deduced by analogy.

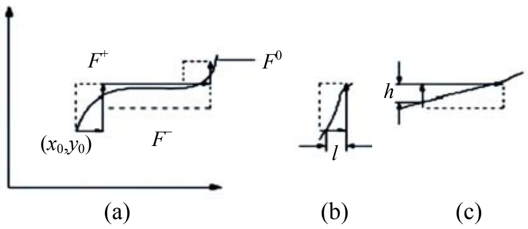


Fig.2 Drawing a monotony-segment of planar implicit curve.

Since the segment from the starting point to the first turning point (if the curve has turning points) must be monotony, if we have not reached the first turning point, we move towards the curve, and the moving path must intersect with the curve if we continue on. To prove Eq.(4), we suppose that the curve keeps monotony-adding in the first quadrant, as in Fig.2a. The curve can be surrounded by some rectangles which only contain the two types as shown in Fig.2b and Fig.2c. As shown in Fig.2b, in the rectangle, the distance from a point on the curve to the edge of the rectangle is less than or equal to l. Since the error $\varepsilon_d=\max(d_0, d_1, \ldots, d_n)$, where d_i $(i=0,1,\ldots,n)$ is the distance between $\boldsymbol{P}_i$ (the vertex of moving path) and the curve, we have $\varepsilon_d\leq l$. As shown in Fig.2c, in the rectangle, the distance from a point on the curve to the edge of the rectangle is less than or equal to h. Therefore, the error ε_d satisfies $\varepsilon_d\leq h$. $l, h\leq\max(dx, dy)$, and $\varepsilon_d\leq\max(dx, dy)$.

Rule 2 for plotting a non-monotony planar implicit curve From Fig.1, we know that we need to automatically control the direction if we want to pass the turning points on the planar implicit curve. However, generally, it is not easy to compute the coordinates of the turning points. The following describes how we can automatically change directions without computing the coordinates of the turning points.

Definition 1 In an xy-plane, for $g(x, y)=c_1$ and $g(x, y)=c_2$, if $|c_1|>|c_2|$, then we state that $g(x, y)=c_1$ is farther than $g(x, y)=c_2$ off from $g(x, y)=0$.

Definition 2 In an xy-plane, if $g(\boldsymbol{A})$ and $g(\boldsymbol{B})$ (where $\boldsymbol{A}$ and $\boldsymbol{B}$ are points on the xy-plane) have the same signs, and $|g(\boldsymbol{B})|>|g(\boldsymbol{A})|$, then we state that point $\boldsymbol{A}$ to point $\boldsymbol{B}$ is off the curve. If $g(\boldsymbol{A})$ and $g(\boldsymbol{B})$ have the different signs, or if $g(\boldsymbol{A})$ and $g(\boldsymbol{B})$ have the same signs but $|g(\boldsymbol{B})|\leq|g(\boldsymbol{A})|$, then we state that point $\boldsymbol{A}$ to $\boldsymbol{B}$ is towards the curve.

Theorem 2 In an xy-plane, from point $\boldsymbol{A}$ to point $\boldsymbol{B}$ is considered off the curve $g(x, y)=0$ if and only if

$$g(\boldsymbol{A})(g(\boldsymbol{B})-g(\boldsymbol{A}))>0. \tag{5}$$

Proof Sufficiency condition:

Since $g(\boldsymbol{A})(g(\boldsymbol{B})-g(\boldsymbol{A}))>0$, and if $g(\boldsymbol{A})>0$, then $g(\boldsymbol{B})>g(\boldsymbol{A})>0$. According to Definition 2, point $\boldsymbol{A}$ to point $\boldsymbol{B}$ are off the curve $g(x, y)=0$. If $g(\boldsymbol{A})<0$, then $g(\boldsymbol{B})<g(\boldsymbol{A})<0$. According to Definition 2, point $\boldsymbol{A}$ to point $\boldsymbol{B}$ are off the curve $g(x, y)=0$.

Necessary condition:

Point $\boldsymbol{A}$ to point $\boldsymbol{B}$ are off the curve $g(x, y)=0$, according to Definition 2, if $g(\boldsymbol{A})$ and $g(\boldsymbol{B})$ have the same signs, and $|g(\boldsymbol{B})|>|g(\boldsymbol{A})|$, then if $g(\boldsymbol{A})<0$, then $g(\boldsymbol{B})<g(\boldsymbol{A})<0$; else if $g(\boldsymbol{A})>0$, then $g(\boldsymbol{B})>g(\boldsymbol{A})>0$. Therefore $g(\boldsymbol{A})(g(\boldsymbol{B})-g(\boldsymbol{A}))>0$.

Theorem 2 tell us that if Eq.(5) is satisfied when we move from $\boldsymbol{A}$ to $\boldsymbol{B}$ for plotting the planar implicit curve, we need to change direction. According to Theorem 2, we suppose that $\boldsymbol{A}$ is (x, y) and $\boldsymbol{B}$ is $(x+\Delta x, y)$ for the planar implicit curve $F(x, y)=0$, where $F(x, y)$ is defined by Eq.(2), we have Eq.(6a); if we suppose that $\boldsymbol{A}$ is (x, y) and $\boldsymbol{B}$ is $(x, y+\Delta y)$ for the planar implicit curve $F(x, y)=0$, where $F(x, y)$ is defined by Eq.(2), we have Eq.(6b).

$$\begin{cases} \text{if } F(x,y)(F(x+\Delta x,y)-F(x,y)) \\ \quad =\sigma f(x,y)(\sigma f(x+\Delta x,y)-\sigma f(x,y)) \qquad \text{(6a)} \\ \quad =f(x,y)(f(x+\Delta x,y)-f(x,y))>0, \\ \Delta y \text{ and } \sigma \text{ change their signs;} \\ \text{if } F(x,y)(F(x,y+\Delta y)-F(x,y)) \\ \quad =\sigma f(x,y)(\sigma f(x,y+\Delta y)-\sigma f(x,y)) \qquad \text{(6b)} \\ \quad =f(x,y)(f(x,y+\Delta y)-f(x,y))>0, \\ \Delta x \text{ and } \sigma \text{ change their signs.} \end{cases}$$

We assume the following: if $f(x+\Delta x, y)-f(x, y)=0$, Δy and σ change their signs, and if $f(x, y+\Delta y)-f(x, y)=0$, Δx and σ change their signs.

If $f(x, y)>0$ and move one step Δx, then before Δy and σ change their signs according to Eq.(3a), $\sigma=1$. According to Eq.(6a) and the assumption, before Δy and σ change their signs, means $f(x, y)(f(x+\Delta x, y)-f(x, y))<0$. Therefore when $\sigma(f(x+\Delta x, y)-f(x, y))<0$, Δy and σ do not change their signs. That is, when $\sigma(f(x+\Delta x, y)-f(x, y))\geq 0$, Δy and σ change their signs.

If $f(x, y)<0$ and move one step Δx, then before Δy and σ change their signs, according to Eq.(3b), $\sigma=-1$. According to Eq.(6a) and the assumption, before Δy and σ change their signs means $f(x, y)(f(x+\Delta x, y)-f(x, y))<0$. Therefore when $\sigma(f(x+\Delta x, y)-f(x, y))<0$, Δy and σ do not change their signs. That is, when $\sigma(f(x+\Delta x, y)-f(x, y))\geq 0$, Δy and σ change their signs.

If $f(x, y)=0$ and $f(x+\Delta x, y)=0$ according to the assumption, Δy and σ change their signs. If $f(x, y)=0$ and $f(x+\Delta x, y)\neq 0$, according to Eq.(2), $\sigma(f(x+\Delta x, y)-f(x, y))=\sigma f(x+\Delta x, y)<0$, Δy and σ do not change their signs.

In general, if $\sigma(f(x+\Delta x, y)-f(x, y))\geq 0$, Δy and σ change their signs. Similarly, if $\sigma(f(x, y+\Delta y)-f(x,y))<0$, Δx and σ change their signs.

From Eq.(6a), if $f(x, y)(f(x+\Delta x, y)-f(x, y))>0$, move one step Δx. According to Eq.(3a), if we move one step Δx, then $F(x, y)\geq 0$. Therefore $\sigma^{-1}F(x, y)(f(x+\Delta x, y)-f(x, y))>0$. Since $F(x, y)\geq 0$, we have $\sigma(f(x+\Delta x, y)-f(x, y))>0$. Similarly, if $f(x, y)(f(x, y+\Delta y)-f(x, y))<0$, then $\sigma(f(x, y+\Delta y)-f(x, y))<0$.

Therefore we have Eq.(7)

$$\begin{cases} \text{if } \sigma(f(x+\Delta x, y)-f(x,y))\geq 0, \\ \quad \Delta y \text{ and } \sigma \text{ change their signs;} & \text{(7a)} \\ \text{if } \sigma(f(x, y+\Delta y)-f(x,y))\leq 0, \\ \quad \Delta x \text{ and } \sigma \text{ change their signs.} & \text{(7b)} \end{cases}$$

Eq.(7) is called Rule 2 for plotting a non-monotony planar implicit curve.

3. Tracing a planar implicit curve

Before we give the algorithm for tracing a planar implicit curve by equal step size or adaptive step size, we discuss how to obtain an exact polyline by intersecting the ladder polyline with the curve using recursive binary subdivision.

As shown in Fig.1, we trace the curve by using two rules to obtain a ladder polyline ($\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{P}_3\boldsymbol{P}_5\boldsymbol{P}_7\boldsymbol{P}_8\boldsymbol{P}_{10}\ldots$) approaching the curve. Then, we get the intersection point $\boldsymbol{P}_{i,j}$ between the line segment $\boldsymbol{P}_i\boldsymbol{P}_j$ and the curve using recursive binary subdivision. We get an exact polyline ($\boldsymbol{P}_0\boldsymbol{P}_{1,2}\boldsymbol{P}_{2,3}\boldsymbol{P}_{3,4}\ldots$) that approaches the curve $F(x, y)=0$. However, the line segment whose next tracing direction is changed, such as line segment $\boldsymbol{P}_5\boldsymbol{P}_7$, does not intersect the curve, therefore, we make the middle perpendicular line of the line segment. Compute the intersection point between the perpendicular line and the curve by the recursive binary subdivision, so that we get a vertex of the exact polyline, namely the intersection point.

When we render a planar implicit curve $f(x, y)=0$ by adaptive step size, how do we decide dx and dy by giving the value d ($d=dx+dy$)? As shown in Fig.1, $\boldsymbol{P}_0$ is the starting point on the curve. Suppose that the tangent vector at $\boldsymbol{P}_0$ on the curve $T\equiv(tx,ty)=(-f_y, f_x)|_{\boldsymbol{P}_0}$, then $dx=d|tx|/(|tx|+|ty|)$, $dy=d|ty|/(|tx|+|ty|)$. Using the size of dx and dy, we can move two steps to get to $\boldsymbol{P}_2$. Let l_x represent the length of the last moving poly-line segment in the x direction, l_y the length of the last moving polyline segment in the y direction. When we get to $\boldsymbol{P}_2$, $\boldsymbol{P}_5$, $\boldsymbol{P}_8$, $\boldsymbol{P}_{11}$ and $\boldsymbol{P}_{14}$, we can determine dx and dy again according to Eq.(8).

$$\begin{cases} dx = d\cdot l_x/(l_x+l_y), \\ dy = d\cdot l_y/(l_x+l_y). \end{cases} \tag{8}$$

Now we give the following algorithm.

Step 1 (Form the initial state): Input the following data: the ranges of arguments x and y ($x_l\leq x\leq x_r$ and $y_b\leq y\leq y_t$), step size dx and dy for equal step size ($d\equiv dx+dy$ for adaptive step size), starting point (x_0, y_0) and end point (x_e, y_e) on the curve, the quadrant number G of the starting direction, compute the starting direction (Δx, Δy) and σ in $F(x, y)\equiv\sigma f(x, y)$.
$f(x_0, y_0)\Rightarrow f$

Step 2: $f\Rightarrow f_0$

Step 3 (According to Rule 1, move a step in Δx direction or Δy direction): If $\sigma f\geq 0$, then {if the curve traced by adaptive step, compute dx and dy according to Eq.(8), Δx and Δy; else if the curve traced by equal step, do not change dx and dy, Δx and Δy; Do Step 3.1} else do Step 3.2

Step 3.1: $x+\Delta x\Rightarrow x, f(x, y)\Rightarrow f$. Do Step 3.1.1.

Step 3.1.1 (According to Rule 2, decide if the signs of Δy and σ chage): If $\sigma(f-f_0)\geq 0$, then $\{-\Delta y\Rightarrow\Delta y, -\sigma\Rightarrow\sigma\}$

Step 3.2: $y+\Delta y\Rightarrow y, f(x, y)\Rightarrow f$. Do Step 3.2.1.

Step 3.2.1 (According to Rule 2, decide if the signs of Δx and σ change): If $\sigma(f-f_0)\geq 0$, then $\{-\Delta x \Rightarrow \Delta x, -\sigma \Rightarrow \sigma\}$.
Step 4: Join current path point (x, y) with the last path point
Step 5: If we get to the end point, then output the ladder polyline; else goto Step 2
Step 6: Get an exact polyline approaching a planar implicit curve by recursive subdivision.

If the implicit curve is monotony, the algorithm becomes simpler because only Rule 1 is adapted. The algorithm for plotting the monotony curve need not be listed any more.

Plotting planar implicit curves with multiple points

How to find the multiple point $\boldsymbol{P}_m$ on the curve? Make a circle whose center is $\boldsymbol{P}_m$, radius is small, such as about step size. Divide the circle into n equal parts to get the points $(\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2\ldots\boldsymbol{P}_{n-1})$ on the circle. Compute the signs of $f(\boldsymbol{P}_i)$ (i=0, 1, …, $n-1$). As shown in Fig.3, here n=8, mark $\boldsymbol{P}_i(+)$ (i=0, 1, …, $n-1$) if $f(\boldsymbol{P}_i)\geq 0$; mark $\boldsymbol{P}_i(-)$ (i=0, 1, …, $n-1$) if $f(\boldsymbol{P}_i)<0$. If the marks of $\boldsymbol{P}_i\boldsymbol{P}_{i+1}$ (i=0,1,…,$n-2$) are not the same, there is a segment of the implicit curve passing through the small arc $\boldsymbol{P}_i\boldsymbol{P}_{i+1}$; If the marks of $\boldsymbol{P}_0\boldsymbol{P}_{n-1}$ are not the same, there is a segment of the implicit curve passing through the small arc $\boldsymbol{P}_0\boldsymbol{P}_{n-1}$. If the number of these segments passing through the curve is greater than 3, then the center of the circle is the multiple point.

Now we discuss how to find the starting point (or end point) of a segment around the multiple point. As shown in Fig.3, compute the intersection point between line $\boldsymbol{P}_1\boldsymbol{P}_2$ and the curve by recursive subdivision. Regard the intersection point as the starting point and end point of the segment of the curve.

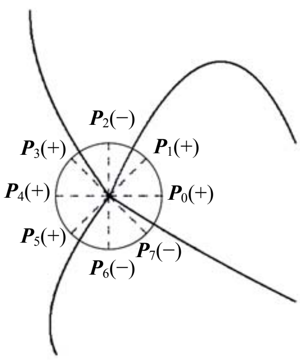


Fig.3 Multiple point on the implicit curve

EXAMPLES

Example 1

$$f_1(x,y)\equiv 25-((x-3)^2+(y+4)^2),$$
$$f_2(x,y)\equiv 25-((x+3)^2+(y+4)^2),$$
$$f_3(x,y)=f_1(x,y)+f_2(x,y)-\sqrt{f_1^2(x,y)+f_2^2(x,y)},$$
$$f_4(x,y)\equiv x^2+(y+4)^2-1,$$
$$f(x,y)=f_3(x,y)+f_4(x,y)-\sqrt{f_3^2(x,y)+f_4^2(x,y)}.$$

Fig.4 is the example of plotting the curve $f(x, y)$=0 where $-5\leq x\leq 5$, $-10\leq y\leq 5$.

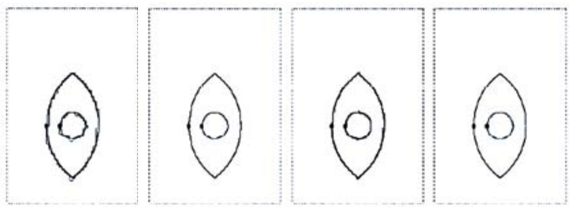


Fig.4 (a) Getting the ladder polyline when plotting the planar implicit curve by equal step size (*dx*=*dy*=0.2); (b) The result of plotting the curve by equal step size (*dx*=*dy*=0.2); (c) Getting the ladder polyline when plotting the planar implicit curve by equal step size (*dx*=*dy*=0.1); (d) The result of plotting the curve by equal step size (*dx*=*dy*=0.1)

Fig.4a yields the ladder polyline when plotting the planar implicit curve by equal step size (dx=dy=0.2). Fig.4b is the result of plotting the curve by equal step size (dx=dy=0.2). Fig.4c yields the ladder polyline when plotting the planar implicit curve by equal step size (dx=dy=0.1). Fig.4d is the result of plotting the curve by equal step size (dx=dy=0.1).

The black points in Fig.4 are the starting point or end point for plotting the curve.

Example 2 A planar implicit curve

$$f(x, y)\equiv y-\sin(1/x)=0,$$

where $0.1\leq x\leq 2.7$, $-1.5\leq y\leq 1.5$.

Fig.5a is the original planar implicit curve generated by the following method: Dividing interval [0.1, 2.7] in the x axis by 13000, we get x_i=0.1+

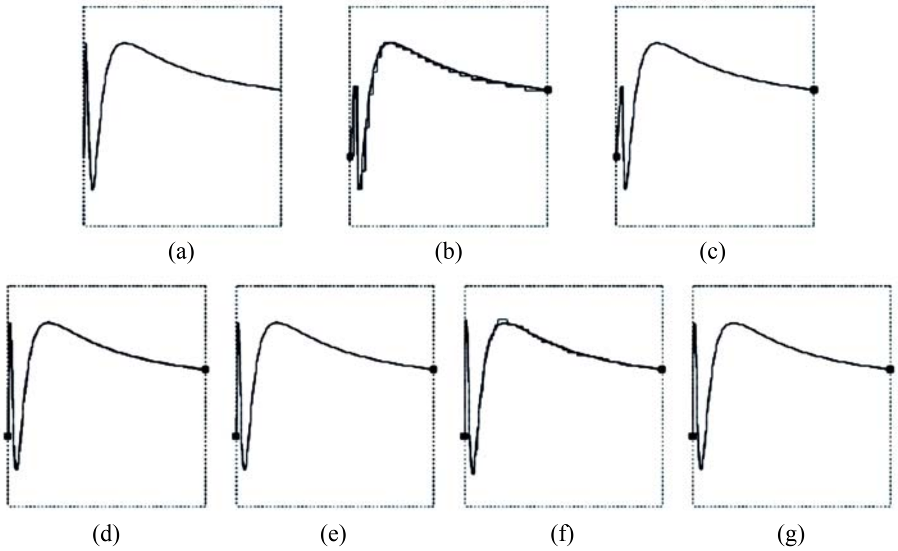


Fig.5 (a) Original planar implicit curve; (b) Getting the ladder polyline when plotting the planar implicit curve by equal step size (*dx*=*dy*=0.05); (c) The result of plotting the curve by equal step size (*dx*=*dy*=0.05); (d) Getting the ladder polyline when plotting the planar implicit curve by equal step size (*dx*=*dy*=0.01); (e) The result of plotting the curve by equal step size (*dx*=*dy*=0.01); (f) Getting the ladder polyline when plotting the planar implicit curve by adaptive step size (*dx*+*dy*=0.1); (g) The result of plotting the curve by adaptive step size(*dx*+*dy*=0.1)

$i\times(2.7-0.1)/13000$ (i=0,1,…,13000). Then mark points $(x_i,\sin(1/x_i))$ (i=0,1,…,13000), so as to plot the curve. Fig.5b yields the ladder polyline when plotting the planar implicit curve by equal step size ($dx=dy=0.05$), Fig.5c is the result of plotting the curve by equal step size ($dx=dy=0.05$). Fig.5d yields the ladder polyline when plotting the planar implicit curve by equal step size ($dx=dy=0.01$). Fig.5e is the result of plotting the curve by equal step size ($dx=dy=0.01$). Fig.5f yields the ladder polyline when plotting the planar implicit curve by the adaptive step size ($dx+dy=0.1$). Fig.5g is the result of plotting the curve by the adaptive step size ($dx+dy=0.1$). The black point in Fig.5 is the starting point or end point for plotting the curve.

Example 3 A planar implicit curve is:

$$f_1(x, y)\equiv x^4-x^2y+y^3=0,$$

where $-0.5\le x\le 0.5$, $-0.5\le y\le 0.5$.

First give the starting point $\boldsymbol{P}_0$, end point $\boldsymbol{P}_7$, and the starting direction of the curve. From the starting point, according to the starting direction, we move forward until get to the multiple point of the curve. Then find the starting points or end points of the curve around the multiple point. As shown in Fig.6a, the starting points of the curve around the multiple point are $\boldsymbol{P}_3$, $\boldsymbol{P}_5$; the end points of the curve around the multiple point are $\boldsymbol{P}_4$, $\boldsymbol{P}_6$. From the starting point $\boldsymbol{P}_3$, according to the starting direction, we move forward until we get to $\boldsymbol{P}_4$. From the starting point $\boldsymbol{P}_5$, according to the starting direction, we move forward until we get to $\boldsymbol{P}_6$. From the starting point $\boldsymbol{P}_2$, according to the starting direction, we move forward until get to $\boldsymbol{P}_7$. Fig.6a yields the ladder polyline when plotting the planar implicit curve by equal step size ($dx=dy=0.01$). Fig.6b is the result of plotting the curve by equal step size ($dx=dy=0.01$).

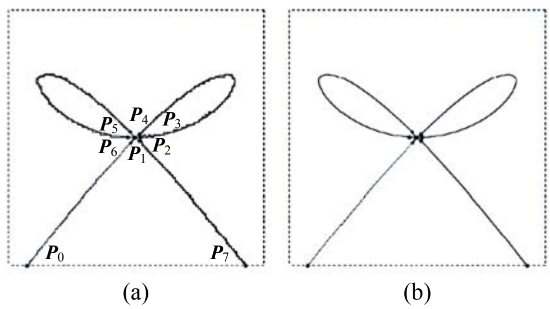


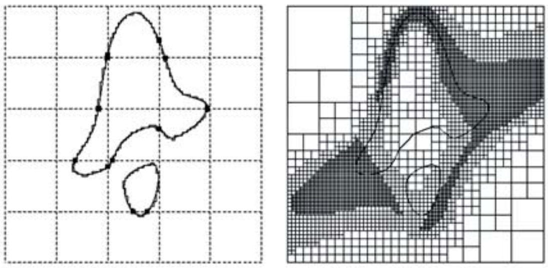
Fig.6 (a) Getting the ladder polyline when plotting the planar implicit curve by equal step size (*dx*=*dy*=0.01); (b) The result of plotting the curve by equal step size (*dx*=*dy*=0.01)

Example 4 A planar implicit curve is

$$f(x, y)\equiv 0.004+0.110x-0.177y-0.174x^2+0.224xy -0.303y^2-0.1168x^3+0.327x^2y-0.087xy^2 -0.013y^3+0.235x^4-0.667x^3y+0.745x^2y^2 -0.029xy^3+0.072y^4=0,$$

where $-2.5\le x\le 2.5$, $-2.5\le y\le 2.5$.

Fig.7a is the rendered planar implicit curve using our method by equal step size ($dx=dy=0.05$). Fig.7b is the rendered planar implicit curve by geometric adaptive polygonal approximation from the paper (Lopes *et al*., 2002), where the spatial tolerance was $0.05\sqrt{2}$, which means that the bound length of the smallest cell is 0.05, equal to the step size in our method, and the tolerance for gradient estimates was $\sigma=0.99$. The run time of Fig.7a is 0.024 s. The run time of Fig.7b is 0.124 s.



(a) (b)

Fig.7 (a) Rendering the planar implicit curve using our method by equal step size (*dx*=*dy*=0.05); (b) Rendering the planar implicit curve by geometric adaptive polygonal approximation from (Lopes *et al*., 2002)

The black points in Fig.7a are the starting points or end points for plotting the curve, obtained by the lattice method (Barnhill *et al*., 1987).

Example 1 is the planar implicit curve which maintains C^0 continuity. For Example 2, if we plot the planar implicit curves by an equal step size, the step size is small enough to plot it without losing its truth. However, if we plot it by adaptive step size, the efficiency is higher and the result is more satisfactory. Example 3 is the application of our method in which the planar implicit curve with multiple points can be plotted. Example 4 shows our method is efficient.

SUMMARY AND FUTURE WORK

In this paper, we present two rules for plotting a planar implicit curve. Rule 1 is for plotting a monotony planar implicit curve. Rule 2 is for plotting a non-monotony planar implicit curve. We present the method for plotting the planar implicit curve by using equal step size and adaptive step size. For an irregular planar implicit curve, it is better to use adaptive step size. When we use the equal step size to plot the curve, the result of drawing may be lose its truth if the step size is too large and the computation efficiency is slower if the step size is too small. Our method can plot C^0 implicit curves, or curves with multiple points.

There are several directions for future work. How to give $dx(dy)$ according to the shape of a planar implicit curve if we use equal step size for plotting the curve? How to give d (where $d=dx+dy$) according to the shape of a planar implicit curve if we use adaptive step size for plotting the curve? How to use more adaptive step size according to curvatures of a planar implicit curve for plotting the curve? How to extend our method to plot an implicit surface?

ACKNOWLEDGEMENT

This work was completed with the support of the APEC post-doc scholarship by Korea Science and Engineering Foundation (KOSEF).

References

Barnhill, R.E., Farin, G., Jordan, M., Piper, B.R., 1987. Surface/surface intersection. *Computer Aided Geometric Design*, **4**(1-2):3-16. [doi:10.1016/0167-8396(87)90020-3]

Bresenham, J.E., 1965. Algorithm for computer control of a digital plotter. *IBM Systems Journal*, **4**(1):25-30.

Bresenham, J.E., 1977. A linear algorithm for incremental digital display of circular arcs. *Comm. ACM*, **20**(2): 100-106. [doi:10.1145/359423.359432]

Cai, Y.Z., 1990. Numerical Control Rendering Using Positive-negative Method. Zhejiang University Press, Hangzhou.

Chandler, R.E., 1988. A tracking algorithm for implicitly defined curves. *IEEE Computer Graphics and Applications*, **8**(2):83-89. [doi:10.1109/38.506]

Cohen, E., 1976. A method for plotting curves defined by implicit equation. *Computer Graphics (SIGGRAPH'76)*,

10(2):263-265. [doi:10.1145/965143.563321]

Lennon, W.J., Jordan, B.W., Holm, B.C., 1973. An improved algorithm for the generation of nonparametric curves. *IEEE Transactions on Computers*, **C-22**:1052-1060.

Lopes, H., Oliverira, J.B., de Figueiredo, L.H., 2002. Robust adaptive polygonal approximation of implicit curves. *Computer & Graphics*, **26**(6):841-852. [doi:10.1016/S0097-8493(02)00173-5]

Martin, R., Shou, H., Voiculescu, I., Bowyer, A., Wang, G., 2002. Comparision of interval methods for plotting algebraic curves. *Computer Aided Geometric Design*, **19**(7):553-587. [doi:10.1016/S0167-8396(02)00146-2]

Shou, H.H., Martin, R.R., Wang, G.J., Bowyer, A., Voiculescu, I., 2005. A Recursive Taylor Method for Algebraic Curves and Surfaces. *In*: Dokken, T., Jüttler, B. (Eds.), Computational Methods for Algebraic Spline Surface (COMPASS). Springer, p.135-155.

van Aken, J., 1984. An efficient ellipse-drawing algorithm. *IEEE Computer Graphics and Applications*, **3**:24-35.

van Aken, J., Novak, M., 1985. Curve-drawing algorithms for raster displays. *ACM Transactions on Graphics*, **4**(2): 147-169. [doi:10.1145/282918.282943]

Editors-in-Chief: Pan Yun-he

ISSN 1009-3095 (Print); ISSN 1862-1775 (Online), monthly

Journal of Zhejiang University

SCIENCE A

www.zju.edu.cn/jzus; www.springerlink.com

jzus@zju.edu.cn

JZUS-A **focuses on "Applied Physics & Engineering"**

➢ Welcome Your Contributions to *JZUS-A*

Journal of Zhejiang University SCIENCE A warmly and sincerely welcomes scientists all over the world to contribute Reviews, Articles and Science Letters focused on **Applied Physics & Engineering**. Especially, Science Letters (3−4 pages) would be published as soon as about 30 days (Note: detailed research articles can still be published in the professional journals in the future after Science Letters is published by *JZUS-A*).