Journal of Zhejiang University SCIENCE A
ISSN 1009-3095 (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Numerical solution of geodesic through two given points on a simple surface

WU Ming-hua[†1], MO Guo-liang[†‡1], YU Yi-yue[2]

([1]School of Computational Science, Zhejiang University City College, Hangzhou 310015, China)

([2]Department of Mathematics, Zhejiang University, Hangzhou 310027, China)

[†]E-mail: wmhua@zju.edu.cn; mogl@zucc.edu.cn

Received Jan. 10, 2006; revision accepted Apr. 21, 2006

Abstract: The algorithm for the approximate solution of a geodesic connecting two given points on a simple surface is discussed in this paper. It arises from practical demands of the filament winding technique. Geodesic is the shortest path connecting two given points on a surface and it can also be regarded as the extremal curve of the arc length functional. The nonlinear equation system of the geodesic on some discrete points by means of the direct variation method is explored. By employing Newton's iterative method, this nonlinear system is transformed into a linear one. And the approximate solution to the geodesic is obtained by solving the resultant linear system. This paper also proves that the iteration is convergent under certain circumstance. Moreover, the result is illustrated with three examples and an appropriate comparison between the analytical solution and the approximate solution to the geodesic is described on the cone surface.

Key words: Geodesic, Filament winding, Functional of arc length
doi:10.1631/jzus.2006.AS0187 **Document code:** A **CLC number:** TP39

INTRODUCTION

Fiber composite of resin matrix has the advantages of corrosion-resistant, insulating, high-strength and lightweight. They are widely used in aerospace, national defense, petroleum and chemical industry. Such materials can be manufactured by using the filament winding technique in which the continuous fibers are wound around a given mandrel along some certain paths (Polini and Sorrentino, 2005; Ghasemi Nejhad *et al*., 2001). When this path is a geodesic, the fiber wound on a mandrel would not slide and the desirable effect can be achieved. So the geodesic is of much importance in this technique (Mazumdar and Hoa, 1995).

A geodesic can be represented by the solution of a second-order ordinary differential equation. Traditionally, with a given point and its tangent direction being the initial condition, the numerical solution of a geodesic can be obtained (Wu *et al*., 2001). But this method cannot ensure the requirement that the obtained geodesic pass through another fixed terminal point with this requirement being essential in practical winding.

The classical conclusion of differential geometry is that exists a unique geodesic passing through two given points under certain circumstance and that this geodesic problem is actually a boundary value one of the corresponding second-order ordinary differential equation (Hsiung, 1981). Except for a few surfaces, there does not exist an analytical solution to such boundary value problem, so the numerical methods, such as the methods of "shooting" or "difference", have to be applied (Heath, 2002).

Different from the traditional methods, the algorithm, presented in this paper, first converts the boundary value problem into an extremal problem of the arc length functional, then obtains the nonlinear equation system on some discrete points by means of the difference method of variation (Hu and Hu, 1987),

[‡] Corresponding author

then linearizes the resultant equation by using Newton's iterative method, and finally finds the numerical solution to the linear equation system, which is the approximate solution to the geodesic. This paper proves that when the coefficients of the first fundamental form of the surface are only dependent on a single parameter, the presented algorithm is convergent. Moreover the effectiveness of the presented algorithm is illustrated with three examples.

BASIC ALGORITHM

Given a parametric surface $\boldsymbol{r}(u,v)$, its first fundamental form is

$$\mathrm{d}s^2=E(u,v)\mathrm{d}u^2+2F(u,v)\mathrm{d}u\mathrm{d}v+G(u,v)\mathrm{d}v^2.$$

The arc length between two given points $(u_\mathrm{b},v_\mathrm{b})$, $(u_\mathrm{e},v_\mathrm{e})$ is given by the following formula

$$L(u,v)=\int_{(u_\mathrm{b},v_\mathrm{b})}^{(u_\mathrm{e},v_\mathrm{e})}\sqrt{E(u,v)\mathrm{d}u^2+2F(u,v)\mathrm{d}u\mathrm{d}v+G(u,v)\mathrm{d}v^2}. \quad (1)$$

The geodesic is the extremal curve of this arc functional. Let the first variation of this arc functional be equal to zero, i.e., $\delta L(u,v)=0$, we can get the corresponding differential equation of the geodesic, from which the geodesic we want can be found. But in general, the analytical expression of the geodesic cannot be obtained by integration. So we present here a numerical method by means of the direct method of variation.

For convenience of computation, we evenly divide the parameter v into

$$v_i=v_\mathrm{b}+i\Delta v,\ \ i=0,\ 1,\ \ldots,\ n,$$

where $\Delta v=(v_\mathrm{e}-v_\mathrm{b})/n$, and $v_0=v_\mathrm{b}$, $v_n=v_\mathrm{e}$. The corresponding points on the geodesic are

$$(u_i,v_i),\ u_0=u_\mathrm{b},\ u_n=u_\mathrm{e},\ i=0,\ 1,\ \ldots,\ n,$$

and

$$\left.\frac{\mathrm{d}u}{\mathrm{d}v}\right|_{(u_i,v_i)}\approx\frac{1}{\Delta v}(u_{i+1}-u_i).$$

We get

$$L(u_1,u_2,\ldots,u_{n-1})$$
$$\approx\sum_{i=0}^{n-1}\sqrt{E_i(u_{i+1}-u_i)^2+2F_i(u_{i+1}-u_i)\Delta v+G_i\Delta v^2}, \quad (2)$$

where

$$E_i=E(u_i,v_i),\ \ F_i=F(u_i,v_i),\ \ G_i=G(u_i,v_i).$$

It follows from $\partial L/\partial u_i=0$ (i=1, 2, …, $n-1$) that

$$\frac{E_{i-1}(u_i-u_{i-1})+F_{i-1}\Delta v}{\sqrt{H_{i-1}}}+\frac{H_i'}{2\sqrt{H_i}}=0,\ \ i=1,\ 2,\ \ldots,\ n-1, \quad (3)$$

where

$$H_i=E_i(u_{i+1}-u_i)^2+2F_i(u_{i+1}-u_i)\Delta v+G_i\Delta v^2,$$

$$H_i'=\frac{\partial H_i}{\partial u_i}=-2[E_i(u_{i+1}-u_i)+F_i\Delta v]+[E_i'(u_{i+1}-u_i)^2+2F_i'(u_{i+1}-u_i)\Delta v+G_i'\Delta v^2],$$

$$E_i'=\frac{\partial E_i}{\partial u_i},\quad E_i''=\frac{\partial^2 E_i}{\partial u_i^2}.$$

Each equation in Eq.(3) is dependent on at most three adjacent variables. We can rewrite Eq.(3) as

$$\begin{cases}f_1(u_1,u_2)=0,\\ f_i(u_{i-1},u_i,u_{i+1})=0, \qquad i=2,\ 3,\ \ldots,\ n-2,\\ f_{n-1}(u_{n-2},u_{n-1})=0.\end{cases} \quad (3')$$

The system Eq.(3) is linearized by Newton's iteration. We thus obtain a linear system

$$\sum_{j=1}^{n-1}\frac{\partial f_i}{\partial u_j}\delta u_j=-f_i,\quad i=1,\ 2,\ \ldots,\ n-1, \quad (4)$$

where $\delta u_j=u_j^{(l)}-u_j^{(l-1)}$, and $u_j^{(l)}$ is the lth iterative value of u_j.

Let

$$a_{i,\,i-1}=\frac{\partial f_i}{\partial u_{i-1}}=\frac{-(E_{i-1}G_{i-1}-F_{i-1}^2)\Delta v^2}{H_{i-1}^{3/2}}+\frac{1}{2H_{i-1}^{3/2}}\{2[E_{i-1}'(u_i-u_{i-1})+F_{i-1}'\Delta v]\cdot[E_{i-1}(u_i-u_{i-1})^2+2F_{i-1}(u_i-u_{i-1})\Delta v+G_{i-1}\Delta v^2]-[E_{i-1}(u_i-u_{i-1})+F_{i-1}\Delta v]\cdot[E_{i-1}'(u_i-u_{i-1})^2+2F_{i-1}'(u_i-u_{i-1})\Delta v+G_{i-1}'\Delta v^2]\},$$

$$a_{ii}=\frac{\partial f_i}{\partial u_i}=\frac{(E_{i-1}G_{i-1}-F_{i-1}^2)\Delta v^2}{H_{i-1}^{3/2}}+\frac{(E_iG_i-F_i^2)\Delta v^2}{H_i^{3/2}}$$
$$+\frac{1}{H_i^{3/2}}\{[E_i(u_{i+1}-u_i)+F_i\Delta v][E_i'(u_{i+1}-u_i)^2$$
$$+2F_i'(u_{i+1}-u_i)\Delta v+G_i'\Delta v^2]-2[E_i'(u_{i+1}-u_i)$$
$$+F_i'\Delta v]H_i\}+\frac{1}{4H_i^{3/2}}\{2H_i[E_i''(u_{i+1}-u_i)^2$$
$$+2F_i''(u_{i+1}-u_i)\Delta v+G_i''\Delta v^2]-[E_i'(u_{i+1}-u_i)^2$$
$$+2F_i'(u_{i+1}-u_i)\Delta v+G_i'\Delta v^2]^2\},$$
$$a_{i,i+1}=\frac{\partial f_i}{\partial u_{i+1}}=-\frac{(E_iG_i-F_i^2)\Delta v^2}{H_i^{3/2}}+\frac{1}{2H_i^{3/2}}\{2[E_i'(u_{i+1}$$
$$-u_i)+F_i'\Delta v]H_i-[E_i(u_{i+1}-u_i)+F_i\Delta v]$$
$$\cdot[E_i'(u_{i+1}-u_i)^2+2F_i'(u_{i+1}-u_i)\Delta v$$
$$+G_i'\Delta v^2]\},$$

and the other entries $a_{ij}=0$, the matrix $\boldsymbol{A}=(a_{ij})_{(n-1)\times(n-1)}$ is then a tridiagonal matrix. Further, put $\boldsymbol{b}=(-f_1, -f_2, \ldots, -f_{n-1})^{\mathrm{T}}$, $\boldsymbol{\delta u}=(\delta u_1, \delta u_2, \ldots, \delta u_{n-1})^{\mathrm{T}}$.

Then the linear system Eq.(4) can be written in matrix form

$$\boldsymbol{A}\boldsymbol{\delta u}=\boldsymbol{b}, \tag{4′}$$

and the above system can be easily solved numerically.

With parameters selected properly, the coefficients of the first fundamental form of surfaces, such as rotating surfaces, are dependent on only one parameter of the surface, i.e.,

$$E(u,v)=E(v),\ F(u,v)=F(v),\ G(u,v)=G(v),$$

we have

$$\frac{\partial E}{\partial u}=\frac{\partial F}{\partial u}=\frac{\partial G}{\partial u}\equiv 0,$$

and the system Eq.(3) can be reduced to

$$f_i(u_{i-1},u_i,u_{i+1})=\frac{E_{i-1}(u_i-u_{i-1})+F_{i-1}\Delta v}{\sqrt{H_{i-1}}}-\frac{E_i(u_{i+1}-u_i)+F_i\Delta v}{\sqrt{H_i}}=0. \tag{5}$$

and the corresponding matrix $\boldsymbol{A}$ has entries

$$a_{i,i-1}=-\frac{(E_{i-1}G_{i-1}-F_{i-1}^2)\Delta v^2}{H_{i-1}^{3/2}}<0,$$
$$a_{ii}=\frac{(E_{i-1}G_{i-1}-F_{i-1}^2)\Delta v^2}{H_{i-1}^{3/2}}+\frac{(E_iG_i-F_i^2)\Delta v^2}{H_i^{3/2}}>0,$$
$$a_{i,i+1}=-\frac{(E_iG_i-F_i^2)\Delta v^2}{H_i^{3/2}}<0,$$
$$a_{ii}=-(a_{i,i-1}+a_{i,i+1}),\ i=2,\ 3,\ \ldots,\ n-2,$$

and

$$a_{11}>|a_{12}|,\ a_{n-1,\ n-1}>|a_{n-1,\ n-2}|.$$

Therefore, the matrix $\boldsymbol{A}$ is irreducibly diagonally dominant. According to the weak row sum criterion (Stoer and Bulirsch, 1980), this iteration is convergent.

EXAMPLES

We apply the above-mentioned algorithm to three examples. In each case, the iterative initial value $u_i^{(0)}$ is taken to be linear interpolation

$$u_i^{(0)}=u_{\mathrm{b}}+\frac{i}{n}(u_{\mathrm{e}}-u_{\mathrm{b}}),\ i=0,\ 1,\ \ldots,\ n.$$

Example 1 Cone $\boldsymbol{r}(u,v)=\{v\cos u,\ v\sin u,\ kv\}$ with $\mathrm{d}s^2=v^2\mathrm{d}u^2+(1+k^2)\mathrm{d}v^2$.

By integrating its differential equation its geodesic has the analytic expression as follows:

$$\sin\left(\frac{u}{\sqrt{1+k^2}}-\varphi\right)=\frac{v_{\mathrm{b}}\sin\frac{u_{\mathrm{b}}}{\sqrt{1+k^2}}-v_{\mathrm{b}}k_1\cos\frac{u_{\mathrm{b}}}{\sqrt{1+k^2}}}{v\sqrt{1+k_1^2}}, \tag{6}$$

where

$$k_1=\frac{v_{\mathrm{e}}\sin\frac{u_{\mathrm{e}}}{\sqrt{1+k^2}}-v_{\mathrm{b}}\sin\frac{u_{\mathrm{b}}}{\sqrt{1+k^2}}}{v_{\mathrm{e}}\cos\frac{u_{\mathrm{e}}}{\sqrt{1+k^2}}-v_{\mathrm{b}}\cos\frac{u_{\mathrm{b}}}{\sqrt{1+k^2}}},$$
$$\varphi=\arctan k_1.$$

We take (π/6,2.0) as the starting point and (π/2,5.0) the terminal point. The exact values of some of the points in the geodesic can be calculated from

Eq.(6). They are listed in the first column in Table 1.

Then we subdivide the interval of parameter v for n=90, and n=9 to obtain the approximate geodesic values by means of the above algorithm. These values are listed in the second and third columns of Table 1. The prescribed accuracy ε of the iteration is

$$|\delta u_j| = |u_j^{(l)} - u_j^{(l-1)}| < \varepsilon = 10^{-5}\,.$$

From the above cases, we see that the desired accuracy is attained through four iterations, which shows that the convergence is quite fast. Compared with the corresponding exact analytic values in the first column, we see that the larger the number of the subdivision is, the closer is the iterative solutions u_i to the exact value. When n=90 and 9, the maximal error is less than 1.25×10^{-3} and 1.23×10^{-2} respectively.

Example 2 Torus $\boldsymbol{r}(u,v)=\{(a+b\cos v)\cos u, (a+b\cos v)\cdot\sin u, -b\sin v\}$ with $ds^2=(a+b\cos v)^2du^2+b^2dv^2$.

Take a=12.0, b=4.0, $\varepsilon=10^{-5}$, n=9, and take (0.0,π/8) as the starting point, and (π/4,π/3) the terminal point. The iterative results u_i are convergent after 7 times, as shown in Table 2.

From the above two examples, we conclude that the linear iteration converges very fast when the iterative accuracy is less than 10^{-5}. When the subdivision gets finer, the numerical solution of the variation will converge to the geodesic. Such accuracy is enough for filament winding.

Example 3 We now consider the geodesic of a B-spline surface whose starting point and terminal point are on different surface patches.

The B-spline surface is given by

$$\boldsymbol{r}(u,v) = \sum_{i=-3}^{n-1}\sum_{j=-3}^{m-1} B_i(u)\,B_j(v)\boldsymbol{V}_{i+2,j+2}\,,$$

and the surface patches are given by

$$\boldsymbol{r}_{ij}(u,v) = \sum_{k=i-3}^{0}\sum_{l=j-3}^{0} B_k(u)\,B_l(v)\boldsymbol{V}_{k+2,l+2}\,,$$

where $\boldsymbol{V}_{ij}$ are the controlling points of the B-spline surface, and

Table 1 Comparison between iterated algorithm and analytic formula of geodesic on cone

n=9 (From Eq.(6))	n=90 (From algorithm)	n=9 (From algorithm)
(u_0,v_0)=(0.52360, 2.00000)	(u_0,v_0)=(0.5236, 2.0000)	(u_0,v_0)=(0.5236, 2.0000)
(u_1,v_1)=(0.77804, 2.33333)	(u_{10},v_{10})=(0.7790, 2.3333)	(u_1,v_1)=(0.7873, 2.3333)
(u_2,v_2)=(0.9664, 2.6667)	(u_{20},v_{20})=(0.9676, 2.6667)	(u_2,v_2)=(0.9787, 2.6667)
(u_3,v_3)=(1.1116, 3.0000)	(u_{30},v_{30})=(1.1129, 3.0000)	(u_3,v_3)=(1.1240, 3.0000)
(u_4,v_4)=(1.2272, 3.3333)	(u_{40},v_{40})=(1.2283, 3.3333)	(u_4,v_4)=(1.2382, 3.3333)
(u_5,v_5)=(1.3213, 3.6667)	(u_{50},v_{50})=(1.3222, 3.6667)	(u_5,v_5)=(1.3304, 3.6667)
(u_6,v_6)=(1.3995, 4.0000)	(u_{60},v_{60})=(1.4002, 4.0000)	(u_6,v_6)=(1.4063, 4.0000)
(u_7,v_7)=(1.4655, 4.3333)	(u_{70},v_{70})=(1.4659, 4.3333)	(u_7,v_7)=(1.4700, 4.3333)
(u_8,v_8)=(1.5219, 4.6667)	(u_{80},v_{80})=(1.5222, 4.6667)	(u_8,v_8)=(1.5242, 4.6667)
(u_9,v_9)=(1.5708, 5.0000)	(u_{90},v_{90})=(1.5708, 5.0000)	(u_9,v_9)=(1.5708, 5.0000)

Table 2 Iterated results of geodesic on torus

No.	u (From algorithm)	v	(x, y, z) (From algorithm)	δu
0	0.0000	0.3927	(15.6955, 0.0000, −1.5307)	0.0
1	0.0396	0.4654	(15.5623, 0.6169, −1.7952)	0.56×10^{-5}
2	0.0814	0.5381	(15.3836, 1.2545, −2.0502)	0.56×10^{-5}
3	0.1260	0.6109	(15.1556, 1.9192, −2.2943)	0.55×10^{-5}
4	0.1744	0.6836	(14.8722, 2.6201, −2.5263)	0.55×10^{-5}
5	0.2282	0.7563	(14.5231, 3.3724, −2.7450)	0.54×10^{-5}
6	0.2899	0.8290	(14.0888, 4.2030, −2.9491)	0.97×10^{-5}
7	0.3650	0.9018	(13.5270, 5.1688, −3.1377)	0.98×10^{-5}
8	0.4687	0.9745	(12.7098, 6.4335, −3.3096)	0.93×10^{-5}
9	0.7854	1.0472	(9.8995, 9.8995, −3.4641)	0.0

$$E(u,v)=\boldsymbol{r}_u\cdot\boldsymbol{r}_u,\ F(u,v)=\boldsymbol{r}_u\cdot\boldsymbol{r}_v,\ \ G(u,v)=\boldsymbol{r}_v\cdot\boldsymbol{r}_v,$$
$$E_u'(u,v)=2\boldsymbol{r}_u\cdot\boldsymbol{r}_{uu},\ F_u'(u,v)=\boldsymbol{r}_{uu}\cdot\boldsymbol{r}_v+\boldsymbol{r}_u\boldsymbol{r}_{uv},$$
$$G_u'(u,v)=2\boldsymbol{r}_v\cdot\boldsymbol{r}_{uv},\ E_{uu}''(u,v)=2\boldsymbol{r}_{uu}\cdot\boldsymbol{r}_{uu}+2\boldsymbol{r}_u\cdot\boldsymbol{r}_{uuu},$$
$$F_{uu}''(u,v)=\boldsymbol{r}_{uuu}\cdot\boldsymbol{r}_v+2\boldsymbol{r}_{uu}\cdot\boldsymbol{r}_{uv}+\boldsymbol{r}_u\cdot\boldsymbol{r}_{uuv},$$
$$G_{uu}''(u,v)=2\boldsymbol{r}_{uv}\cdot\boldsymbol{r}_{uv}+2\boldsymbol{r}_v\cdot\boldsymbol{r}_{uuv}.$$

On the surface of a torus, we use the equally-spaced mesh of parameters (u,v) as the group of offsets to evaluate the controlling points and then to get the corresponding B-spline surface. Here two cases are taken into consideration and the corresponding results are presented:

(1) When the group of offsets of the torus is so intensive that the resultant B-spline surface is very close to the original torus and the parameters of the starting and terminal points are taken as the same ones in Example 3, the iteration converges very fast and the maximal error between the numerical solution to the geodesic of the B-spline surface and the one to the geodesic of the original torus in Example 2 is less than 0.0052 [see (x,y,z) in Table 2 and Table 3].

Take a=12.0, b=4.0, ε=10^{-5}, n=9, and take (0.0,π/8) as the starting point, (π/4,π/3) as the terminal point. The iterative results u_i are convergent after 7 times, as shown in Table 3.

(2) When the group of offsets is taken from a tubiform surface, which has variable sectional radii, the iteration also converges very fast.

Now the equation of tubiform surface is given by

$$\boldsymbol{r}(u,v)=\{(a+(b_0+h\cdot u)\cos v)\cos u,(a+(b_0+h\cdot u)\cos v)\sin u,(b_0+h\cdot u)\sin v\}.$$

Take a=10.0, b_0=5.0, h=2/π, ε=10^{-5}, n=9, and take (2π/9,2π/9) as the starting point, (π/3,π/3) as the terminal point. The iterative results u_i are convergent after 7 times and the resultant numerical solutions to the geodesic of the B-spline converge very fast (see Table 3 and Table 4). The results are shown in Table 4.

Table 3 Iterated results of geodesic on B-spline surface generated by points on torus

No.	u (From algorithm)	v	(x, y, z) (From algorithm)	δu
0	0.0000	0.3927	(15.6955, 0.0000, −1.5307)	0.0
1	0.0393	0.4654	(15.5622, 0.6118, −1.7952)	-0.052×10^{-5}
2	0.0812	0.5381	(15.3839, 1.2516, −2.0502)	0.002×10^{-5}
3	0.1257	0.6109	(15.1560, 1.9154, −2.2943)	-0.028×10^{-5}
4	0.1742	0.6836	(14.8727, 2.6171, −2.5263)	-0.018×10^{-5}
5	0.2279	0.7563	(14.5238, 3.3693, −2.7450)	-0.027×10^{-5}
6	0.2897	0.8290	(14.0897, 4.2001, −2.9491)	0.351×10^{-5}
7	0.3648	0.9018	(13.5281, 5.1661, −3.1377)	0.361×10^{-5}
8	0.4685	0.9745	(12.7110, 6.4335, −3.3096)	0.311×10^{-5}
9	0.7854	1.0472	(9.8995, 9.8995, −3.4641)	0.0

Table 4 Iterated results of geodesic on B-spline surface generated by points on tubiform surface of variable sectional radii

No.	u (From algorithm)	v	δu
0	0.6981	0.6981	0.0
1	0.7247	0.7369	0.007×10^{-5}
2	0.7529	0.7757	0.012×10^{-5}
3	0.7831	0.8145	0.017×10^{-5}
4	0.8155	0.8533	0.021×10^{-5}
5	0.8509	0.8921	0.022×10^{-5}
6	0.8899	0.9308	0.022×10^{-5}
7	0.9339	0.9696	0.018×10^{-5}
8	0.9849	1.0084	0.013×10^{-5}
9	1.0472	1.0472	0.0

CONCLUSION

From the practical problem of filament winding, we need to find the geodesic passing through two given fixed points. We obtain the numerical solution to the geodesic through the difference method of the arc length functional. When the coefficients of the first fundamental form of the surface is dependent on a single parameter, such as rotating surfaces, we prove that the iteration is convergent. We also illustrate the effectiveness of the presented algorithm with three examples, i.e., the cone surface, torus surface and B-spline surface.

ACKNOWLEDGEMENT

We sincerely thank Professor Huang Zhaozheng for suggestions during the writing process.

References

Heath, M.T., 2002. Scientific Computing: An Introductory Survey (2nd Ed.). McGraw-Hill Co., p.430-432.

Hsiung, C.C., 1981. A First Course of Differential Geometry. John Wiley & Son, p.299-300.

Hu, H.C., Hu, R.M., 1987. Variation. Architecture Publishing Company China, Beijing, p.**7** (in Chinese).

Leek, C., 1998. Filament winding breathes life into gas bottles. *Reinforce Plastics*, **9**:52-53.

Mazumdar, S.K., Hoa, S.V., 1995. Analytical models for low cost manufacturing of composite components by filament winding, Part I: Direct kinematics. *Journal of Composite Materials*, **29**(11):1515-1541.

Ghasemi Nejhad, M.N., Chandramouli, M.V., Yousefpour, A., 2001. Processing and Performance of continuous fiber ceramic composites by preceramic polymer pyrolysis: I—filament winding. *Journal of Composite Materials*, **35**(24):2207-2237. [doi:10.1106/X6MK-AJBB-UHUH-KTBB]

Polini, W., Sorrentino, L., 2005. Winding trajectory and winding time in robotized filament winding of asymmetric shape parts. *Journal of Composite Materials*, **39**(15): 1391-1411. [doi:10.1177/0021998305050431]

Stoer, J., Bulirsch, R., 1980. Introduction to Numerical Analysis. Springer-Verlag, **247**:542-544.

Wu, M.H., Liang, Y.D., Yu, Y.Y., 2001. Stabilities of geodesics on torus. *Appl. Math. J. Chinese Univ. Ser A*, **16**(4):481-485.