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Stability analysis of neutral-type nonlinear delayed systems: An LMI approach*

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Abstract: The problems of determining the global asymptotic stability and global exponential stability for a class of norm-bounded nonlinear neutral differential systems with constant or time-varying delays are investigated in this work. Lyapunov method was used to derive some useful criteria of the systems' global asymptotic stability and global exponential stability. The stability conditions are formulated as linear matrix inequalities (LMIs) which can be easily solved by various convex optimization algorithms. Moreover, for the exponentially stable system, the exponential convergence rates of the system's states can be estimated by some parameters of the LMIs. Numerical examples are given to illustrate the application of the proposed method.

Key words: Convergence rate, Generalized eigenvalue problem, Linear matrix inequality (LMI), Nonlinear neutral system, Stability, Time delay
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INTRODUCTION

The theory of neutral delay-differential systems is of theoretical and practical interest. For example, neutral-type functional differential equations are natural models of voltage and current fluctuations in problems arising in transmission lines. Also, neutral systems often appear in the study of automatic control, population dynamics, and vibrating masses attached to an elastic bar, etc. (Park, 2003). So far, the stability analysis of various neutral delay-differential systems has received considerable attention. For the stability analysis of the linear neutral system, specially, with constant delays, many results have been published (Park, 2003; Fridman, 2001; Hu *et al*., 2004; Bellen *et al*., 1999; Cao and He, 2004; Agarwal and Grace, 2000; Park and Won, 1999; 2000; Park, 2002). Several researches have also been conducted for the same problem of neutral-type nonlinear delay differential equations (Si and Ma, 1995; Ma and Takeuchi, 1998; Amemiya and Ma, 2003). However, these papers only deal with the asymptotic stability of such systems, without providing any conditions for exponential stability and any information about the decay rates (i.e. exponential convergence rates) of the system's states. Although the linear matrix inequality (LMI) approach is widely applied for stability analysis of linear neutral system because the stability conditions represented as LMIs are easily verified, and make the stability criteria less conservative (Park and Won, 2000; Park, 2002), there does not seem to be much (if any) study on the stability analysis of nonlinear neutral system via the LMI approach.

In this work, we study the global asymptotic stability and global exponential stability of nonlinear neutral differential systems with constant or time-varying delays. The nonlinearity is assumed to be norm-bounded. Such nonlinear system is common in many practical or industrial processes. Combing Lyapunov-Krasovskii functionals with the LMI ap-

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proach, we derive some criteria on global asymptotic stability and global exponential stability, and estimate the exponential convergence rates. These stability conditions can be transformed into LMIs which can be easily solved by various effective optimization algorithms (Boyd *et al*., 1994) or computing software [e.g. MATLAB LMI Control Toolbox (Gahinet *et al*., 1995)]. The main advantages of our approach include: (1) The criteria are relatively less conservative, and can be applied under more relaxed assumptions; (2) It can be efficiently verified via numerically solving the LMIs. Moreover, our approach can be also applied to stability analysis of linear neutral systems.

NOTATIONS AND PRELIMINARIES

Let $\mathbb{R}^n$ denote the n-dimensional real space and $\mathbb{R}^{n\times n}$ denote the set of all real n by n matrices. If $\boldsymbol{X}\in\mathbb{R}^p$ and $Y\in\mathbb{R}^q$, $\boldsymbol{C}(\boldsymbol{X};\boldsymbol{Y})$ denotes the space of all continuous functions mapping $\mathbb{R}^p\to\mathbb{R}^q$. $\boldsymbol{I}$ denotes the unit matrix of appropriate order. $\lambda_M(\boldsymbol{A})$ and $\lambda_m(\boldsymbol{A})$ denote the maximal and minimal eigenvalue of a square matrix $\boldsymbol{A}$, respectively. $\|\boldsymbol{x}\|$ denotes the Euclid norm of the vector $\boldsymbol{x}$, and $\|\boldsymbol{A}\|$ denotes the induced norm of the matrix $\boldsymbol{A}$, that is, $\|\boldsymbol{A}\|=\sqrt{\lambda_M(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})}$. $|a|$ denote the absolute value of the scalar a.

Consider a specific class of nonlinear neutral differential system with time-varying delays having the form

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{f}(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau(t)))+\boldsymbol{g}(t,\dot{\boldsymbol{x}}(t-h(t))), \tag{1}$$

with the initial condition function

$$\boldsymbol{x}(t_0+\theta)=\boldsymbol{\phi}(\theta),\ \forall\theta\in[-\rho,0], \tag{2}$$

where $\boldsymbol{x}(t)\in\mathbb{R}^n$ is the state vector, $\boldsymbol{A}\in\mathbb{R}^{n\times n}$ is constant real system matrix, $\boldsymbol{f}\in\boldsymbol{C}(\mathbb{R}^{2n+1};\mathbb{R}^n)$ and $\boldsymbol{g}\in\boldsymbol{C}(\mathbb{R}^{n+1};\mathbb{R}^n)$ are time-varying nonlinear functions satisfying $\boldsymbol{f}(t,0,0)=0$ and $\boldsymbol{g}(t,0)=0$, respectively, and the norm-bounded condition, i.e., there exist nonnegative constants α_1, α_2 and α_3 such that

$$\|f(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau(t)))\|\le\alpha_1\|\boldsymbol{x}(t)\|+\alpha_2\|\boldsymbol{x}(t-\tau(t))\|, \tag{3}$$

$$\|\boldsymbol{g}(t,\dot{\boldsymbol{x}}(t-\tau(t)))\|\le\alpha_3\|\dot{\boldsymbol{x}}(t-h(t))\|, \tag{4}$$

$\tau(t)$ and $h(t)$ are positive time-varying differentiable bounded delays satisfying

$$\begin{cases}0<\tau(t)\le\overline{\tau}<\infty,\ \dot{\tau}(t)\le 1,\\ 0<h(t)\le\overline{h}<\infty,\ \dot{h}(t)\le 1,\end{cases} \tag{5}$$

$\overline{\tau}=\max\tau(t)$, $\overline{h}=\max h(t)$, $\rho=\max\{\overline{\tau},\overline{h}\}$, and $\boldsymbol{\phi}(\cdot)$ is the given continuously differentiable function on $[-\rho,0]$. In this paper, the system matrix $\boldsymbol{A}$ is assumed to be a Hurwitz matrix with all the eigenvalues of $\boldsymbol{A}$ having negative real parts. So there exists at least one equilibrium point in the system Eq.(1).

Definition 1 (Liao *et al*., 2002) If there exist $\gamma>0$ and $f(\gamma)>0$ such that

$$\|\boldsymbol{x}(t)\|\le f(\gamma)\mathrm{e}^{-\gamma t},\ \forall t>0, \tag{6}$$

the system Eq.(1) is said to be exponentially stable at the equilibrium point, where γ is called the degree of exponential stability.

Before proceeding further, we present well-known lemmas.

Lemma 1 (Khargonekar *et al*., 1990) For any real vector $\boldsymbol{D}$ and $\boldsymbol{E}$ with appropriate dimension and any positive scalar δ, we have

$$\boldsymbol{D}\boldsymbol{E}+\boldsymbol{E}^{\mathrm{T}}\boldsymbol{D}^{\mathrm{T}}\le\delta\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}}+\delta^{-1}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}. \tag{7}$$

Lemma 2 (Schur complement) (Boyd *et al*., 1994) The following LMI,

$$\begin{bmatrix}\boldsymbol{Q}(x) & \boldsymbol{S}(x)\\ \boldsymbol{S}(x)^{\mathrm{T}} & \boldsymbol{R}(x)\end{bmatrix}<0, \tag{8}$$

where $\boldsymbol{Q}(x)=\boldsymbol{Q}(x)^{\mathrm{T}}$, $\boldsymbol{R}(x)=\boldsymbol{R}(x)^{\mathrm{T}}$, and $\boldsymbol{S}(x)$ depend affinely on x, is equivalent to

$$\boldsymbol{R}(x)<0,\ \boldsymbol{Q}(x)-\boldsymbol{S}(x)\boldsymbol{R}(x)^{-1}\boldsymbol{S}(x)^{\mathrm{T}}<0. \tag{9}$$

GLOBAL STABILITY FOR NEUTRAL SYSTEMS WITH CONSTANT DELAYS

First, we assume that the time-delay $\tau(t)$ and $h(t)$

are constant, i.e. $\tau(t)=\tau>0$, $h(t)=h>0$. Then the following criteria are established by using Lyapunov method in terms of LMIs.

Theorem 1 Suppose that there exist positive definite matrices $\boldsymbol{X}$ and $\boldsymbol{Q}$, and positive scalars β_1 and β_2, such that

$$\begin{bmatrix} \boldsymbol{AX}+\boldsymbol{XA}^{\mathrm{T}}+\beta_1\boldsymbol{I}+\beta_2\boldsymbol{I} & \sqrt{2}\alpha_1\boldsymbol{X} & \boldsymbol{XQ} & \sqrt{2\alpha_1^2+\|\boldsymbol{A}\|^2}\boldsymbol{X} \\ \sqrt{2}\alpha_1\boldsymbol{X} & -\beta_1\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{QX} & \boldsymbol{0} & -\boldsymbol{Q} & \boldsymbol{0} \\ \sqrt{2\alpha_1^2+\|\boldsymbol{A}\|^2}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix}<0, \tag{10}$$

$$\begin{bmatrix} 2\alpha_2^2\boldsymbol{I}-\boldsymbol{Q} & \sqrt{2}\alpha_2\boldsymbol{I} \\ \sqrt{2}\alpha_2\boldsymbol{I} & -\beta_1\boldsymbol{I} \end{bmatrix}<0, \tag{11}$$

$$\begin{bmatrix} \alpha_3^2\boldsymbol{I}-\boldsymbol{I} & \alpha_3\boldsymbol{I} \\ \alpha_3\boldsymbol{I} & -\beta_2\boldsymbol{I} \end{bmatrix}<0, \tag{12}$$

the origin of system Eq.(1) is globally asymptotically stable.

Proof For simplicity, we denote $\boldsymbol{x}(t)$ as $\boldsymbol{x}$, $\boldsymbol{x}(t-\tau)$ as $\boldsymbol{x}_\tau$, $\boldsymbol{f}(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau))$ as $\boldsymbol{f}$, $\dot{\boldsymbol{x}}(t-h)$ as $\dot{\boldsymbol{x}}_h$, $\boldsymbol{g}(t,\dot{\boldsymbol{x}}(t-h))$ as $\boldsymbol{g}$. We construct the following positive definite Lyapunov-Krasovskii functional:

$$\begin{aligned} V(\boldsymbol{x}) &= \boldsymbol{x}^{\mathrm{T}}\boldsymbol{Px}+\int_{-\tau}^{0}\boldsymbol{x}^{\mathrm{T}}(t+s)\boldsymbol{Qx}(t+s)\mathrm{d}s \\ &\quad +\int_{-h}^{0}\dot{\boldsymbol{x}}^{\mathrm{T}}(t+s)\dot{\boldsymbol{x}}(t+s)\mathrm{d}s, \end{aligned} \tag{13}$$

where $\boldsymbol{P}$, $\boldsymbol{Q}$ are positive definite matrices. The time derivative of $V(\boldsymbol{x}(t))$ along the trajectories of system Eq.(1) is

$$\begin{aligned} \dot{V}(\boldsymbol{x}) &= 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\dot{\boldsymbol{x}}+\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Qx}-\boldsymbol{x}_\tau^{\mathrm{T}}\boldsymbol{Qx}_\tau+\dot{\boldsymbol{x}}^{\mathrm{T}}\dot{\boldsymbol{x}}-\dot{\boldsymbol{x}}_h^{\mathrm{T}}\dot{\boldsymbol{x}}_h \\ &= \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P})\boldsymbol{x}+2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Pf}+2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Pg}+\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Qx} \\ &\quad -\boldsymbol{x}_\tau^{\mathrm{T}}\boldsymbol{Qx}_\tau-\dot{\boldsymbol{x}}_h^{\mathrm{T}}\dot{\boldsymbol{x}}_h+(\boldsymbol{Ax}+\boldsymbol{f}+\boldsymbol{g})^{\mathrm{T}}(\boldsymbol{Ax}+\boldsymbol{f}+\boldsymbol{g}). \end{aligned} \tag{14}$$

Using Eq.(3), the term $\boldsymbol{f}^{\mathrm{T}}\boldsymbol{f}$ satisfies the following inequality:

$$\begin{aligned} \boldsymbol{f}^{\mathrm{T}}\boldsymbol{f} &= \|\boldsymbol{f}\|^2 \le \alpha_1^2\|\boldsymbol{x}\|^2+\alpha_2^2\|\boldsymbol{x}_\tau\|^2+2\alpha_1\alpha_2\|\boldsymbol{x}\|\cdot\|\boldsymbol{x}_\tau\| \\ &\le \alpha_1^2\|\boldsymbol{x}\|^2+\alpha_2^2\|\boldsymbol{x}_\tau\|^2+(\alpha_1^2\|\boldsymbol{x}\|^2+\alpha_2^2\|\boldsymbol{x}_\tau\|^2) \\ &= 2\alpha_1^2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}+2\alpha_2^2\boldsymbol{x}_\tau^{\mathrm{T}}\boldsymbol{x}_\tau. \end{aligned} \tag{15}$$

Again, Using Eqs.(4), (15) and Lemma 1, the terms on the right-hand side of Eq.(14) satisfy

$$\begin{aligned} &(\boldsymbol{Ax}+\boldsymbol{f}+\boldsymbol{g})^{\mathrm{T}}(\boldsymbol{Ax}+\boldsymbol{f}+\boldsymbol{g}) = \|\boldsymbol{Ax}+\boldsymbol{f}+\boldsymbol{g}\|^2 \\ &\le \|\boldsymbol{A}\|^2\|\boldsymbol{x}\|^2+\|\boldsymbol{f}\|^2+\|\boldsymbol{g}\|^2 \\ &\le (2\alpha_1^2+\|\boldsymbol{A}\|^2)\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}+2\alpha_2^2\boldsymbol{x}_\tau^{\mathrm{T}}\boldsymbol{x}_\tau+\alpha_3^2\dot{\boldsymbol{x}}_h^{\mathrm{T}}\dot{\boldsymbol{x}}_h, \end{aligned} \tag{16}$$

$$\begin{aligned} 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Pf} &\le \beta_1\boldsymbol{x}^{\mathrm{T}}\boldsymbol{PPx}+\beta_1^{-1}\boldsymbol{f}^{\mathrm{T}}\boldsymbol{f} \\ &\le \beta_1\boldsymbol{x}^{\mathrm{T}}\boldsymbol{PPx}+2\alpha_1^2\beta_1^{-1}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}+2\alpha_2^2\beta_1^{-1}\boldsymbol{x}_\tau^{\mathrm{T}}\boldsymbol{x}_\tau, \end{aligned} \tag{17}$$

$$\begin{aligned} 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Pg} &\le \beta_2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{PPx}+\beta_2^{-1}\boldsymbol{g}^{\mathrm{T}}\boldsymbol{g} \\ &\le \beta_2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{PPx}+\alpha_3^2\beta_2^{-1}\dot{\boldsymbol{x}}_h^{\mathrm{T}}\dot{\boldsymbol{x}}_h, \end{aligned} \tag{18}$$

where β_1 and β_2 are positive scalars to be chosen.

Then, we obtain

$$\begin{aligned} \dot{V}(\boldsymbol{x}) &\le \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\beta_1\boldsymbol{PP}+\beta_2\boldsymbol{PP}+2\alpha_1^2\beta_1^{-1}\boldsymbol{I} \\ &\quad +\boldsymbol{Q}+2\alpha_1^2\boldsymbol{I}+\|\boldsymbol{A}\|^2\boldsymbol{I})\boldsymbol{x}+\boldsymbol{x}_\tau^{\mathrm{T}}(2\alpha_2^2\beta_1^{-1}\boldsymbol{I}-\boldsymbol{Q} \\ &\quad +2\alpha_2^2\boldsymbol{I})\boldsymbol{x}_\tau+\dot{\boldsymbol{x}}_h^{\mathrm{T}}(\alpha_3^2\beta_2^{-1}\boldsymbol{I}-\boldsymbol{I}+\alpha_3^2\boldsymbol{I})\dot{\boldsymbol{x}}_h \\ &= \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{x}_\tau\\ \dot{\boldsymbol{x}}_h\end{bmatrix}^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{P},\boldsymbol{Q},\beta_1,\beta_2) \begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{x}_\tau\\ \dot{\boldsymbol{x}}_h\end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{G}(\boldsymbol{P},\boldsymbol{Q},\beta_1,\beta_2) &= \mathrm{diag}[\boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\beta_1\boldsymbol{PP}+\beta_2\boldsymbol{PP} \\ &\quad +2\alpha_1^2\beta_1^{-1}\boldsymbol{I}+\boldsymbol{Q}+2\alpha_1^2\boldsymbol{I}+\|\boldsymbol{A}\|^2\boldsymbol{I},\ 2\alpha_2^2\beta_1^{-1}\boldsymbol{I} \\ &\quad -\boldsymbol{Q}+2\alpha_2^2\boldsymbol{I},\ \alpha_3^2\beta_2^{-1}\boldsymbol{I}-\boldsymbol{I}+\alpha_3^2\boldsymbol{I}]. \end{aligned}$$

If $\boldsymbol{G}(\boldsymbol{P},\boldsymbol{Q},\beta_1,\beta_2)<0$ holds, then $\dot{V}(\boldsymbol{x})\le 0$. Therefore, the origin of system Eq.(1) is globally asymptotically stable. Pre- and post-multiplying the matrix $\boldsymbol{G}(\boldsymbol{P},\boldsymbol{Q},\beta_1,\beta_2)$ by $\mathrm{diag}(\boldsymbol{P}^{-1},\boldsymbol{I},\boldsymbol{I})$ reveals the fact that $\boldsymbol{G}(\boldsymbol{P},\boldsymbol{Q},\beta_1,\beta_2)<0$ is equivalent to

$$\begin{aligned} &\boldsymbol{AX}+\boldsymbol{XA}^{\mathrm{T}}+\beta_1\boldsymbol{I}+\beta_2\boldsymbol{I}+2\alpha_1^2\beta_1^{-1}\boldsymbol{XX} \\ &\quad +\boldsymbol{XQX}+(2\alpha_1^2+\|\boldsymbol{A}\|^2)\boldsymbol{XX}<0, \end{aligned} \tag{19}$$

$$2\alpha_2^2\beta_1^{-1}\boldsymbol{I}-\boldsymbol{Q}+2\alpha_2^2\boldsymbol{I}<0, \tag{20}$$

$$\alpha_3^2\beta_2^{-1}\boldsymbol{I}-\boldsymbol{I}+\alpha_3^2\boldsymbol{I}<0, \tag{21}$$

where $\boldsymbol{X}=\boldsymbol{P}^{-1}$. By Lemma 2, Eqs.(19)~(21) are equivalent to Eqs.(10)~(12). This completes the proof.

Next, we will derive some criteria of exponential stability for system Eq.(1).

Theorem 2 If there exist positive definite matrices $\boldsymbol{X}$, $\boldsymbol{Q}$, $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, and $\boldsymbol{K}_3$, and positive scalars β_1, β_2, and $0<\alpha<1$, satisfying the following generalized eigenvalue problem (GEVP):

$$\text{minimize} \quad \alpha, \tag{22}$$

subject to

$$\begin{bmatrix} \boldsymbol{AX}+\boldsymbol{XA}^{\mathrm{T}}+\beta_1\boldsymbol{I}+\beta_2\boldsymbol{I}+\boldsymbol{K}_1 & \sqrt{2}\alpha_1\boldsymbol{X} & \boldsymbol{XQ} & \sqrt{2\alpha_1^2+\|\boldsymbol{A}\|^2}\boldsymbol{X} \\ \sqrt{2}\alpha_1\boldsymbol{X} & -\beta_1\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{QX} & \boldsymbol{0} & -\boldsymbol{Q} & \boldsymbol{0} \\ \sqrt{2\alpha_1^2+\|\boldsymbol{A}\|^2}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix}<0, \tag{23}$$

$$\begin{bmatrix} 2\alpha_2^2\boldsymbol{I}-\boldsymbol{K}_2 & \sqrt{2}\alpha_2\boldsymbol{I} \\ \sqrt{2}\alpha_2\boldsymbol{I} & -\beta_1\boldsymbol{I} \end{bmatrix}<0, \tag{24}$$

$$\begin{bmatrix} \alpha_3^2\boldsymbol{I}-\boldsymbol{K}_3 & \alpha_3\boldsymbol{I} \\ \alpha_3\boldsymbol{I} & -\beta_2\boldsymbol{I} \end{bmatrix}<0, \tag{25}$$

$$\rho^{-1}\boldsymbol{X}<\alpha(\boldsymbol{K}_1+\rho^{-1}\boldsymbol{X}), \tag{26}$$

$$\boldsymbol{K}_2<\alpha\boldsymbol{Q}, \tag{27}$$

$$\boldsymbol{K}_3<\alpha\boldsymbol{I}, \tag{28}$$

the origin of system Eq.(1) is globally exponentially stable. Moreover,

$$\|\boldsymbol{x}(t)\|\le \sqrt{\frac{\left[\lambda_M(\boldsymbol{P})+\lambda_M(\boldsymbol{Q})\dfrac{1-\mathrm{e}^{-2\gamma\tau}}{2\gamma}\right]\|\boldsymbol{\Omega}_1\|^2+\dfrac{1-\mathrm{e}^{-2\gamma h}}{2\gamma}\|\boldsymbol{\Omega}_2\|^2}{\lambda_m(\boldsymbol{P})}}\mathrm{e}^{-\gamma t}, \tag{29}$$

where $\rho=\max\{\tau,h\}$, $0<\alpha=\mathrm{e}^{-2\gamma\rho}<1$, $\boldsymbol{X}=\boldsymbol{P}^{-1}$, $\|\boldsymbol{\Omega}_1\|=\sup\limits_{-\tau\le s\le 0}\|\boldsymbol{x}(s)\|$, $\|\boldsymbol{\Omega}_2\|=\sup\limits_{-h\le s\le 0}\|\dot{\boldsymbol{x}}(s)\|$.

Proof By virtue of Eqs.(26)~(28), we have

$$\begin{aligned} 0<(1+2\gamma\rho)\rho^{-1}\boldsymbol{X}<\mathrm{e}^{2\gamma\rho}\rho^{-1}\boldsymbol{X}=\alpha^{-1}\rho^{-1}\boldsymbol{X}<\boldsymbol{K}_1+\rho^{-1}\boldsymbol{X}, \\ \Leftrightarrow 0<2\gamma\boldsymbol{X}<\boldsymbol{K}_1, \\ \Leftrightarrow 0<2\gamma\boldsymbol{I}<\boldsymbol{K}_1\boldsymbol{P}, \end{aligned} \tag{30}$$

$$0<\boldsymbol{K}_2<\alpha\boldsymbol{Q}=\mathrm{e}^{-2\gamma\rho}\boldsymbol{Q}\le\mathrm{e}^{-2\gamma\tau}\boldsymbol{Q}, \tag{31}$$

$$0<\boldsymbol{K}_3<\alpha\boldsymbol{I}=\mathrm{e}^{-2\gamma\rho}\boldsymbol{I}\le\mathrm{e}^{-2\gamma h}\boldsymbol{I}. \tag{32}$$

We define the following positive definite Lyapunov-Krasovskii functional:

$$\begin{aligned} V(\boldsymbol{x})=\mathrm{e}^{2\gamma t}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Px}+\int_{-\tau}^{0}\mathrm{e}^{2\gamma(t+s)}\boldsymbol{x}^{\mathrm{T}}(t+s)\boldsymbol{Qx}(t+s)\mathrm{d}s \\ +\int_{-h}^{0}\mathrm{e}^{2\gamma(t+s)}\dot{\boldsymbol{x}}^{\mathrm{T}}(t+s)\dot{\boldsymbol{x}}(t+s)\mathrm{d}s, \end{aligned} \tag{33}$$

where $\boldsymbol{P}$, $\boldsymbol{Q}$ are positive definite matrices, γ is positive scalar. Using Eqs.(16)~(18) and Eqs.(30)~(32), the time derivative of $V(\boldsymbol{x}(t))$ along the trajectories of system Eq.(1) is

$$\begin{aligned} \dot{V}(\boldsymbol{x})&=2\gamma\mathrm{e}^{2\gamma t}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Px}+2\mathrm{e}^{2\gamma t}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{Ax}+\boldsymbol{f}+\boldsymbol{g})+\mathrm{e}^{2\gamma t}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Qx} \\ &\quad-\mathrm{e}^{2\gamma(t-\tau)}\boldsymbol{x}_\tau^{\mathrm{T}}\boldsymbol{Qx}_\tau+\mathrm{e}^{2\gamma t}\dot{\boldsymbol{x}}^{\mathrm{T}}\dot{\boldsymbol{x}}-\mathrm{e}^{2\gamma(t-h)}\dot{\boldsymbol{x}}_h^{\mathrm{T}}\dot{\boldsymbol{x}}_h \\ &\le\mathrm{e}^{2\gamma t}[\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\beta_1\boldsymbol{PP}+\beta_2\boldsymbol{PP} \\ &\quad+2\alpha_1^2\beta_1^{-1}\boldsymbol{I}+\boldsymbol{Q}+2\alpha_1^2\boldsymbol{I}+\|\boldsymbol{A}\|^2\boldsymbol{I}+2\gamma\boldsymbol{P})\boldsymbol{x} \\ &\quad+\boldsymbol{x}_\tau^{\mathrm{T}}(2\alpha_2^2\beta_1^{-1}\boldsymbol{I}-\mathrm{e}^{-2\gamma\tau}\boldsymbol{Q}+2\alpha_2^2\boldsymbol{I})\boldsymbol{x}_\tau \\ &\quad+\dot{\boldsymbol{x}}_h^{\mathrm{T}}(\alpha_3^2\beta_2^{-1}\boldsymbol{I}-\mathrm{e}^{-2\gamma h}\boldsymbol{I}+\alpha_3^2\boldsymbol{I})\dot{\boldsymbol{x}}_h] \\ &<\mathrm{e}^{2\gamma t}[\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\beta_1\boldsymbol{PP}+\beta_2\boldsymbol{PP} \\ &\quad+2\alpha_1^2\beta_1^{-1}\boldsymbol{I}+\boldsymbol{Q}+2\alpha_1^2\boldsymbol{I}+\|\boldsymbol{A}\|^2\boldsymbol{I}+\boldsymbol{PK}_1\boldsymbol{P})\boldsymbol{x} \\ &\quad+\boldsymbol{x}_\tau^{\mathrm{T}}(2\alpha_2^2\beta_1^{-1}\boldsymbol{I}-\boldsymbol{K}_2+2\alpha_2^2\boldsymbol{I})\boldsymbol{x}_\tau \\ &\quad+\dot{\boldsymbol{x}}_h^{\mathrm{T}}(\alpha_3^2\beta_2^{-1}\boldsymbol{I}-\boldsymbol{K}_3+\alpha_3^2\boldsymbol{I})\dot{\boldsymbol{x}}_h] \\ &=\mathrm{e}^{2\gamma t}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{x}_\tau\\ \dot{\boldsymbol{x}}_h\end{bmatrix}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{P},\boldsymbol{Q},\beta_1,\beta_2)\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{x}_\tau\\ \dot{\boldsymbol{x}}_h\end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{P},\boldsymbol{Q},\beta_1,\beta_2)=\mathrm{diag}[&\boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\beta_1\boldsymbol{PP}+\beta_2\boldsymbol{PP} \\ &+2\alpha_1^2\beta_1^{-1}\boldsymbol{I}+\boldsymbol{Q}+2\alpha_1^2\boldsymbol{I}+\|\boldsymbol{A}\|^2\boldsymbol{I}+\boldsymbol{PK}_1\boldsymbol{P}, \\ &2\alpha_2^2\beta_1^{-1}\boldsymbol{I}-\boldsymbol{K}_2+2\alpha_2^2\boldsymbol{I},\alpha_3^2\beta_2^{-1}\boldsymbol{I}-\boldsymbol{K}_3+\alpha_3^2\boldsymbol{I}]. \end{aligned}$$

If $\boldsymbol{M}(\boldsymbol{P},\boldsymbol{Q},\beta_1,\beta_2)<0$ holds, then $\dot{V}(\boldsymbol{x})\le 0$. Pre- and post-multiplying the matrix $\boldsymbol{M}(\boldsymbol{P},\boldsymbol{Q},\beta_1,\beta_2)$ by $\mathrm{diag}(\boldsymbol{P}^{-1},\boldsymbol{I},\boldsymbol{I})$, and using Lemma 2, we have Eqs.(26)~(28), where $\boldsymbol{X}=\boldsymbol{P}^{-1}$. Since $\dot{V}(\boldsymbol{x})\le 0$, we get

$$V(\boldsymbol{x}(t)) \leq V(\boldsymbol{x}(0)). \tag{34}$$

However,

$$\begin{aligned} V(\boldsymbol{x}(0)) &= \boldsymbol{x}(0)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(0) + \int_{-\tau}^{0} \mathrm{e}^{2\gamma s}\boldsymbol{x}^{\mathrm{T}}(s)\boldsymbol{Q}\boldsymbol{x}(s)\mathrm{d}s \\ &\quad + \int_{-h}^{0} \mathrm{e}^{2\gamma s}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\dot{\boldsymbol{x}}(s)\mathrm{d}s \\ &\leq \left[\lambda_M(\boldsymbol{P}) + \lambda_M(\boldsymbol{Q})\int_{-\tau}^{0}\mathrm{e}^{2\gamma s}\mathrm{d}s\right]\|\boldsymbol{\Omega}_1\|^2 \\ &\quad + \|\boldsymbol{\Omega}_2\|^2\int_{-h}^{0}\mathrm{e}^{2\gamma s}\mathrm{d}s \\ &= \left[\lambda_M(\boldsymbol{X}^{-1}) + \lambda_M(\boldsymbol{Q})\frac{1-\mathrm{e}^{-2\gamma\tau}}{2\gamma}\right]\|\boldsymbol{\Omega}_1\|^2 \\ &\quad + \frac{1-\mathrm{e}^{-2\gamma h}}{2\gamma}\|\boldsymbol{\Omega}_2\|^2, \end{aligned} \tag{35}$$

and

$$\begin{aligned} V(\boldsymbol{x}(t)) &\geq \mathrm{e}^{2\gamma t}\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) \geq \mathrm{e}^{2\gamma t}\lambda_m(\boldsymbol{P})\|\boldsymbol{x}(t)\|^2 \\ &\geq \mathrm{e}^{2\gamma t}\lambda_m(\boldsymbol{X}^{-1})\|\boldsymbol{x}(t)\|^2, \end{aligned}$$

therefore, we can get the convergence rates of the system's states, i.e. Eq.(29). From Definition 1, we conclude that the equilibrium point is globally exponentially stable. We hope that the degree of exponential stability γ is maximal (or α is minimal) such that the system Eq.(1) converges to the equilibrium point as fast as possible. It requires solving the generalized eigenvalue minimization problem Eqs.(22)~(28), which is a quasi-convex optimization problem and can be solved by using the MATLAB LMI Control Toolbox (Gahinet *et al*., 1995). Theorem 2 provides a simple method to determine the exponential stability of system Eq.(1) and get the upper bound of the exponential convergence rate, and can be widely applied to stability analysis. We thus complete the proof.

GLOBAL STABILITY FOR NEUTRAL SYSTEMS WITH TIME-VARYING DELAYS

Time-varying delay is commonly encountered in the field of automatic control or population dynamics, and its existence is frequently a source of oscillation and instability. Now we consider the case where the delays in system Eq.(1) are time-varying (nonconstant). Assuming that $\tau(t)$ and $h(t)$ satisfy Eq.(5), we have the following results.

Theorem 3 Suppose that $\tau(t)$ and $h(t)$ satisfy $\dot{\tau}(t) \leq 1$ and $\dot{h}(t) \leq 1$, respectively. If there exist positive definite matrices $\boldsymbol{X}$ and $\boldsymbol{Q}$, and positive scalars β_1 and β_2, such that

$$\begin{bmatrix} \boldsymbol{AX}+\boldsymbol{XA}^{\mathrm{T}}+\beta_1\boldsymbol{I}+\beta_2\boldsymbol{I} & \sqrt{2}\alpha_1\boldsymbol{X} & \boldsymbol{XQ} & \sqrt{2\alpha_1^2+\|\boldsymbol{A}\|^2}\boldsymbol{X} \\ \sqrt{2}\alpha_1\boldsymbol{X} & -\beta_1\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{QX} & \boldsymbol{0} & -\boldsymbol{Q} & \boldsymbol{0} \\ \sqrt{2\alpha_1^2+\|\boldsymbol{A}\|^2}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix} < 0, \tag{36}$$

$$\begin{bmatrix} 2\alpha_2^2\boldsymbol{I}-\sigma_1\boldsymbol{Q} & \sqrt{2}\alpha_2\boldsymbol{I} \\ \sqrt{2}\alpha_2\boldsymbol{I} & -\beta_1\boldsymbol{I} \end{bmatrix} < 0, \tag{37}$$

$$\begin{bmatrix} \alpha_3^2\boldsymbol{I}-\sigma_2\boldsymbol{I} & \alpha_3\boldsymbol{I} \\ \alpha_3\boldsymbol{I} & -\beta_2\boldsymbol{I} \end{bmatrix} < 0, \tag{38}$$

where $\sigma_1 = \inf_{t\geq 0}(1-\dot{\tau}(t))$, $\sigma_2 = \inf_{t\geq 0}(1-\dot{h}(t))$, the origin of system Eq.(1) is globally asymptotically stable.

Proof For simplicity, we denote $\boldsymbol{x}(t)$ as $\boldsymbol{x}$, $\boldsymbol{x}(t-\tau(t))$ as $\boldsymbol{x}_\tau$, $\dot{\boldsymbol{x}}(t-h(t))$ as $\dot{\boldsymbol{x}}_h$. Define a positive definite Lyapunov-Krasovskii functional as follows:

$$\begin{aligned} V(\boldsymbol{x}) &= \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x} + \int_{-\tau(t)}^{0}\boldsymbol{x}^{\mathrm{T}}(t+s)\boldsymbol{Q}\boldsymbol{x}(t+s)\mathrm{d}s \\ &\quad + \int_{-h(t)}^{0}\dot{\boldsymbol{x}}^{\mathrm{T}}(t+s)\dot{\boldsymbol{x}}(t+s)\mathrm{d}s. \end{aligned} \tag{39}$$

Using Eqs.(16)~(18), the time derivative of $V(\boldsymbol{x}(t))$ along the trajectories of system Eq.(1) is

$$\begin{aligned} \dot{V}(\boldsymbol{x}) &= 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\dot{\boldsymbol{x}} + \boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} - (1-\dot{\tau}(t))\boldsymbol{x}_\tau^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}_\tau \\ &\quad + \dot{\boldsymbol{x}}^{\mathrm{T}}\dot{\boldsymbol{x}} - (1-\dot{h}(t))\dot{\boldsymbol{x}}_h^{\mathrm{T}}\dot{\boldsymbol{x}}_h \\ &\leq \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\beta_1\boldsymbol{PP}+\beta_2\boldsymbol{PP} \\ &\quad + 2\alpha_1^2\beta_1^{-1}\boldsymbol{I}+\boldsymbol{Q}+2\alpha_1^2\boldsymbol{I}+\|\boldsymbol{A}\|^2\boldsymbol{I})\boldsymbol{x} \\ &\quad + \boldsymbol{x}_\tau^{\mathrm{T}}(2\alpha_2^2\beta_1^{-1}\boldsymbol{I}-\sigma_1\boldsymbol{Q}+2\alpha_2^2\boldsymbol{I})\boldsymbol{x}_\tau \\ &\quad + \dot{\boldsymbol{x}}_h^{\mathrm{T}}(\alpha_3^2\beta_2^{-1}\boldsymbol{I}-\sigma_2\boldsymbol{I}+\alpha_3^2\boldsymbol{I})\dot{\boldsymbol{x}}_h. \end{aligned} \tag{40}$$

The remaining part of the proof is similar to that of Theorem 1. According to the conditions Eqs.(36)~(38), the origin of system Eq.(1) is globally asymptotically stable. The proof of Theorem 3 is thus completed.

Following the same ideas as the proof of Theorem 2, we have the following theorem on exponential stability of neutral system Eq.(1).

Theorem 4 Suppose that $\tau(t)$ and $h(t)$ satisfy $\dot{\tau}(t)\le 1$ and $\dot{h}(t)\le 1$, respectively. If there exist positive definite matrices $\boldsymbol{X}$, $\boldsymbol{Q}$, $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, and $\boldsymbol{K}_3$, and positive scalars β_1, β_2, and $0<\alpha<1$, satisfying the following GEVP:

$$\text{minimize } \alpha, \tag{41}$$

subject to

$$\begin{bmatrix} \boldsymbol{AX}+\boldsymbol{XA}^{\mathrm{T}}+\beta_1\boldsymbol{I}+\beta_2\boldsymbol{I}+\boldsymbol{K}_1 & \sqrt{2}\alpha_1\boldsymbol{X} & \boldsymbol{XQ} & \sqrt{2\alpha_1^2+\|\boldsymbol{A}\|^2}\boldsymbol{X} \\ \sqrt{2}\alpha_1\boldsymbol{X} & -\beta_1\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{QX} & \boldsymbol{0} & -\boldsymbol{Q} & \boldsymbol{0} \\ \sqrt{2\alpha_1^2+\|\boldsymbol{A}\|^2}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix}<0, \tag{42}$$

$$\begin{bmatrix} 2\alpha_2^2\boldsymbol{I}-\sigma_1\boldsymbol{K}_2 & \sqrt{2}\alpha_2\boldsymbol{I} \\ \sqrt{2}\alpha_2\boldsymbol{I} & -\beta_1\boldsymbol{I} \end{bmatrix}<0, \tag{43}$$

$$\begin{bmatrix} \alpha_3^2\boldsymbol{I}-\sigma_2\boldsymbol{K}_3 & \alpha_3\boldsymbol{I} \\ \alpha_3\boldsymbol{I} & -\beta_2\boldsymbol{I} \end{bmatrix}<0, \tag{44}$$

$$\bar{\rho}^{-1}\boldsymbol{X}<\alpha(\boldsymbol{K}_1+\bar{\rho}^{-1}\boldsymbol{X}), \tag{45}$$

$$\boldsymbol{K}_2<\alpha\boldsymbol{Q}, \tag{46}$$

$$\boldsymbol{K}_3<\alpha\boldsymbol{I}, \tag{47}$$

where $\sigma_1=\inf\limits_{t\ge 0}(1-\dot{\tau}(t))$, $\sigma_2=\inf\limits_{t\ge 0}(1-\dot{h}(t))$, $\bar{\rho}=\max\{\bar{\tau},\bar{h}\}$, the origin of system Eq.(1) is globally exponentially stable. Moreover,

$$\|\boldsymbol{x}(t)\| \le \sqrt{\frac{\left[\lambda_M(\boldsymbol{X}^{-1})+\lambda_M(\boldsymbol{Q})\frac{1-\mathrm{e}^{-2\gamma\bar{\tau}}}{2\gamma}\right]\|\boldsymbol{\Omega}_1\|^2+\frac{1-\mathrm{e}^{-2\gamma\bar{h}}}{2\gamma}\|\boldsymbol{\Omega}_2\|^2}{\lambda_m(\boldsymbol{X}^{-1})}}\mathrm{e}^{-\gamma t}, \tag{48}$$

where $0<\alpha=\mathrm{e}^{-2\gamma\bar{\rho}}<1$, $\|\boldsymbol{\Omega}_1\|=\sup\limits_{-\bar{\tau}\le s\le 0}\|\boldsymbol{x}(s)\|$, $\|\boldsymbol{\Omega}_2\|=\sup\limits_{-\bar{h}\le s\le 0}\|\dot{\boldsymbol{x}}(s)\|$.

Here, it is worth noting that, in Theorem 3 and Theorem 4, we require for the derivative of the time-varying delay $\tau(t)$ and $h(t)$ to be equal to or less than 1, i.e. $\dot{\tau}(t)\le 1$ and $\dot{h}(t)\le 1$. Such an assumption is often needed in many papers dealing with the stability problem of various neutral differential systems with time-varying delays (Park, 2002).

ILLUSTRATIVE EXAMPLES

To illustrate the usefulness of the proposed approach, we present the following three examples. Examples 1 and 2 show the application of the criteria in the case of constant delay, and Example 3 for the time-varying delay.

Example 1 Consider the following linear neutral system with constant delays (Park and Won, 2000):

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{Ax}(t)+\boldsymbol{Bx}(t-\tau)+\boldsymbol{C}\dot{\boldsymbol{x}}(t-h), \tag{49}$$

where

$$\boldsymbol{A}=\begin{bmatrix}-3 & -2\\ 1 & 0\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}0 & -0.1\\ -0.1 & 0\end{bmatrix},\ \boldsymbol{C}=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix},$$

$$\tau=2,\ h=1,\ \rho=2.$$

We can get $\alpha_1=0$, $\alpha_2=\|\boldsymbol{B}\|=0.1$, and $\alpha_3=\|\boldsymbol{C}\|=0.1$. Example 1 is Example 1 in the paper by Park and Won (2000). Although many approaches in the literature (Cao and He, 2004; Agarwal and Grace, 2000; Park and Won, 1999; 2000) are used to judge the global asymptotical stability of the system Eq.(49), our approach not only determines whether the system Eq.(49) is asymptotically stable or not by solving Eqs.(10)~(12), but also judges the global exponential stability of the system Eq.(49) by solving the GEVP Eqs.(22)~(28). Since Eqs.(10)~(12) have the solutions as

$$\boldsymbol{X}=\begin{bmatrix}0.1752 & -0.0585\\ -0.0585 & 0.0416\end{bmatrix},$$

$$\boldsymbol{Q}=\begin{bmatrix}3.2645 & 0.0691\\ 0.0691 & 3.4827\end{bmatrix},$$

$$\beta_1=0.0064,\ \beta_2=0.0105,$$

then we can conclude that the system Eq.(49) is globally asymptotically stable by Theorem 1. Solving the GEVP Eqs.(22)~(28), we can obtain the solutions as

$$\alpha=0.9483,\ \boldsymbol{X}=\begin{bmatrix}0.1394 & -0.0489\\ -0.0489 & 0.0364\end{bmatrix},$$
$$\boldsymbol{Q}=\begin{bmatrix}4.8967 & 0.1036\\ 0.1036 & 5.2241\end{bmatrix},$$
$$\boldsymbol{K}_1=\begin{bmatrix}0.1457 & -0.0354\\ -0.0354 & 0.0092\end{bmatrix},$$
$$\boldsymbol{K}_2=\begin{bmatrix}4.6292 & 0.0494\\ 0.0494 & 4.7851\end{bmatrix},\ \boldsymbol{K}_3=\begin{bmatrix}0.9842 & 0\\ 0 & 0.9842\end{bmatrix},$$
$$\beta_1=0.0044,\ \beta_2=0.0107,$$

so the system Eq.(49) is also globally exponentially stable by Theorem 2.

Example 2 Consider the following nonlinear neutral system with constant delays:

$$\begin{aligned}\dot{\boldsymbol{x}}(t)&=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{A}_1\tanh(\boldsymbol{x}(t))\\ &\quad+\boldsymbol{B}\tanh(\boldsymbol{x}(t-\tau))+\boldsymbol{C}\tanh(\dot{\boldsymbol{x}}(t-h)),\end{aligned} \tag{50}$$

where $\tanh(\cdot)$ is hyperbolic tangent, $\tanh(\boldsymbol{x}(t))=[\tanh(\boldsymbol{x}_1(t))\ \ \tanh(\boldsymbol{x}_1(t))]^{\mathrm{T}}$, $\tau=3$, $h=4$, $\rho=4$, $\boldsymbol{A}=\begin{bmatrix}-0.5 & 2\\ -3 & -1\end{bmatrix}$, $\boldsymbol{A}_1=\begin{bmatrix}-0.15 & -0.05\\ 0.12 & 0\end{bmatrix}$, $\boldsymbol{B}=\begin{bmatrix}-0.05 & 0.05\\ 0.05 & -0.05\end{bmatrix}$, $\boldsymbol{C}=\begin{bmatrix}0.1 & 0\\ 0 & -0.1\end{bmatrix}$.

Since $\tanh(s)/s\in[0,\ 1]$, we can get $\alpha_1=\|\boldsymbol{A}_1\|=0.1961$, $\alpha_2=\|\boldsymbol{B}\|=0.1$, $\alpha_3=\|\boldsymbol{C}\|=0.1$. We solve Eqs.(10)~(12), and have the solutions as

$$\boldsymbol{X}=\begin{bmatrix}0.0378 & -0.0046\\ -0.0046 & 0.0552\end{bmatrix},$$
$$\boldsymbol{Q}=\begin{bmatrix}2.2545 & 1.4678\\ 1.4678 & 2.5827\end{bmatrix},$$
$$\beta_1=0.0222,\ \beta_2=0.0108.$$

Then we can state that the system Eq.(50) is globally asymptotically stable according to Theorem 1. We solve the GEVP Eqs.(22)~(28), and obtain the solutions as

$$\alpha=0.8553,\ \boldsymbol{X}=\begin{bmatrix}0.0287 & -0.0034\\ -0.0034 & 0.0413\end{bmatrix},$$
$$\boldsymbol{Q}=\begin{bmatrix}4.5090 & 2.9356\\ 2.9356 & 5.1654\end{bmatrix},$$
$$\boldsymbol{K}_1=\begin{bmatrix}0.0012 & -0.0001\\ -0.0001 & 0.0017\end{bmatrix},$$
$$\boldsymbol{K}_2=\begin{bmatrix}3.0125 & 1.5673\\ 1.5673 & 3.3629\end{bmatrix},\ \boldsymbol{K}_3=\begin{bmatrix}0.8552 & 0\\ 0 & 0.8552\end{bmatrix},$$
$$\beta_1=0.0126,\ \beta_2=0.0118.$$

So the system Eq.(50) is also globally exponentially stable according to Theorem 2.

Example 3 Consider the following nonlinear neutral system with time-varying delays:

$$\begin{aligned}\dot{x}_1(t)&=-3x_1(t)+0.1x_2(t)\\ &\quad+0.4\sqrt{x_1(t)x_1(t-0.4|\sin t|)}\\ &\quad-0.2\sqrt{x_1(t)x_2(t-0.4|\sin t|)}\\ &\quad+0.06x_2(t-0.4|\sin t|)\\ &\quad+0.25\tanh[x_1(t-0.2|\cos t|)]\\ &\quad-0.15\tanh[(t-0.2|\cos t|)],\\ \dot{x}_2(t)&=-0.2x_1(t)-3x_2(t)\\ &\quad+0.6\sqrt{x_2(t)x_2(t-0.4|\sin t|)}\\ &\quad+0.6\sqrt{x_1(t)x_2(t)}\\ &\quad-0.4\sqrt{x_1(t-0.4|\sin t|)x_2(t-0.4|\sin t|)}\\ &\quad-0.2\tanh[x_1(t-0.2|\cos t|)]\\ &\quad+0.1\tanh[x_2(t-0.2|\cos t|)].\end{aligned} \tag{51}$$

From Eq.(51), we can get $\boldsymbol{A}=\begin{bmatrix}-3 & 0.1\\ -0.2 & -3\end{bmatrix}$, $\alpha_1=0.6723$, $\alpha_2=0.3$, $\alpha_3=0.3672$. The time-varying delays $\tau(t)=0.4|\sin t|$ and $h(t)=0.2|\cos t|$ are bounded as $\bar{\tau}=0.4$ and $\bar{h}=0.2$, respectively. The maximum of $\bar{\tau}$ and $\bar{h}$ is 0.4, i.e. $\bar{\rho}=0.4$. The infimums of $1-\dot{\tau}(t)$ and $1-\dot{h}(t)$ are 0.6 and 0.8, respectively. According to Theorem 3, by solving Eqs.(36)~(38), we can obtain the solutions as

$$\boldsymbol{X}=\begin{bmatrix}0.2177 & 0.0030\\ 0.0030 & 0.2160\end{bmatrix},\ \boldsymbol{Q}=\begin{bmatrix}1.2645 & 0.2691\\ 0.2691 & 1.5827\end{bmatrix},$$
$$\beta_1=0.3762,\ \beta_2=0.2314.$$

This implies that the system Eq.(51) is globally asymptotically stable. According to Theorem 4, by solving the GEVP Eqs.(41)~(47), we can obtain the solutions as

$$\alpha\text{=}0.8652,\ \boldsymbol{X}=\begin{bmatrix}0.1575 & 0.0061\\ 0.0061 & 0.1611\end{bmatrix},$$
$$\boldsymbol{Q}=\begin{bmatrix}2.4025 & 0.5113\\ 0.5113 & 3.0071\end{bmatrix},\ \boldsymbol{K}_1=\begin{bmatrix}0.0678 & 0.0038\\ 0.0038 & 0.0635\end{bmatrix},$$
$$\boldsymbol{K}_2=\begin{bmatrix}1.9559 & 0.2453\\ 0.2453 & 2.2459\end{bmatrix},\ \boldsymbol{K}_3=\begin{bmatrix}0.8617 & 0\\ 0 & 0.8617\end{bmatrix},$$
$$\beta_1=0.1990,\ \beta_2=0.2432.$$

Then the system Eq.(51) is also globally exponentially stable.

CONCLUSION

In this work, we study the problems of global asymptotical stability and global exponential stability for norm-bounded nonlinear neutral differential systems with constant or time-varying delays. We derive some stability criteria by means of the Lyapunov-Krasovskii functionals and the LMI approach. The criteria are expressed in terms of LMIs, which are less conservative and less restrictive, and can be easily solved by using the MATLAB LMI Control Toolbox (Gahinet *et al*., 1995). While the system is exponentially stable, we can also estimate the exponential convergence rates by the solutions of the LMIs. Moreover, our approach can be applied to the stability analysis of linear neutral systems. However, we need some assumptions which restrict the nonlinearity of the neutral system, that is, the nonlinearity is norm-bonded. It means that the proposed results cannot be applied to all nonlinear neutral systems. We will attempt to develop these results to more general nonlinear neutral systems, removing such restrictions in future.

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