Journal of Zhejiang University SCIENCE A
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Nonlinear dynamics analysis of a new autonomous chaotic system*

CHU Yan-dong[1], LI Xian-feng[†‡1], ZHANG Jian-gang[1,2], CHANG Ying-xiang[1,2]
([1]School of Mathematics, Physics and Software Engineering, Lanzhou Jiaotong University, Lanzhou 730070, China)
([2]Nonlinear Science Research Center, Lanzhou Jiaotong University, Lanzhou 730070, China)
[†]E-mail: lixf1979@126.com

Received Feb. 2, 2007; revision accepted Apr. 1, 2007

Abstract: In this paper, a new nonlinear autonomous system introduced by Chlouverakis and Sprott is studied further, to present very rich and complex nonlinear dynamical behaviors. Some basic dynamical properties are studied either analytically or numerically, such as Poincaré map, Lyapunov exponents and Lyapunov dimension. Based on this flow, a new almost-Hamilton chaotic system with very high Lyapunov dimensions is constructed and investigated. Two new nonlinear autonomous systems can be changed into one another by adding or omitting some constant coefficients.

Key words: Lyapunov exponents, Bifurcation, Chaos, Phase space, Poincaré sections
doi:10.1631/jzus.2007.A1408 **Document code:** A **CLC number:** O322; TB535

INTRODUCTION

A new nonlinear autonomous system named B was found by modifying a chaotic system named A (Chlouverakis and Sprott, 2005). The chaotic attractor obtained from the system by detailed numerical as well as theoretical analysis is similar to system A, also exhibiting complex chaotic dynamics. Chlouverakis and Sprott used it for investigating and expatiating the comparison of correlation and Lyapunov dimensions as one of three examples that they adduced. System A is used to model semiconductor lasers optically driven by a monochromatic light beam (Wieczorek *et al*., 1999), the dynamics of which have been well studied in (Wieczorek *et al*., 1999; Chlouverakis and Adams, 2003; Chlouverakis and Sprott, 2005). Both 3D nonlinear systems are capable of filling most of their phase-space (Chlouverakis and Sprott, 2006), and producing chaotic attractors which with Lyapunov dimensions almost anywhere between 2 and 3, even some attractors can reach a dimension very close to D_{KY}=3 with some specific values sets of parameters in these two chaotic systems.

The new simpler chaotic system is given by

$$\begin{cases} \mathrm{d}x/\mathrm{d}t=K+z(x-\alpha y), \\ \mathrm{d}y/\mathrm{d}t=z(\alpha x-\varepsilon y), \\ \mathrm{d}z/\mathrm{d}t=1-x^2-y^2. \end{cases} \quad (1)$$

It is a 3D autonomous system which with six nonlinearities, including two quadratic terms, to introduce the nonlinearity necessary for folding trajectories. The advantage of system Eq.(1) is that it can produce much higher Lyapunov dimensions than some other famous autonomous chaotic systems, e.g. (Lorenz, 1963; Rössler, 1976; Chua *et al*., 1986; Chen and Ueta, 1999; Lü and Chen, 2002), which are all low-dimensional.

A typical chaotic attractor of system Eq.(1) was simulated and several chaotic attractors' Lyapunov dimensions were analyzed with some given parameters, which are presented in Table 1 in (Chlouverakis and Sprott, 2005), although the stability of its fixed points had not been studied before, so more chaotic dynamics of this system should be revealed and investigated further. In this paper, the stability of the fixed points is first, and the numerical analysis is given

‡ Corresponding author
* Project supported by the National Natural Science Foundation of China (No. 50475109) and the Natural Science Foundation of Gansu Province (No. 3ZS-042-B25-049), China

later. Because of the simulation, the results support brief theoretical derivations. Simulation results are given by presenting bifurcation diagrams, chaotic attractors on Poincaré maps, Lyapunov exponents, fractal dimension, chaotic attractor, limit circles.

FIXED POINTS AND THEIR LINEAR STABILITY

First note that the system has symmetry S because the transformation

$$S: (x, y, z, t)\to(-x, -y, -z, -t) \quad (2)$$

permits the system Eq.(1) to be invariant for all parameters K, α, ε. If an orbit P:[x,y,z] is invariant under S, it is called a symmetric orbit. Otherwise, it is called an asymmetric orbit and has its conjugate orbit (Kim and Kim, 2000).

System Eq.(1) has two fixed points, which are respectively described as follows:

$$f_1(\varepsilon/\mathfrak{A}, \alpha/\mathfrak{A}, -k\mathfrak{A}/(\varepsilon-\alpha^2)),$$
$$f_2(-\varepsilon/\mathfrak{A}, -\alpha/\mathfrak{A}, k\mathfrak{A}/(\varepsilon-\alpha^2)),$$

where $\mathfrak{A}=\sqrt{\varepsilon^2+\alpha^2}$.

At the fixed points f_1 and f_2, system Eq.(1) is linearized, the Jacobian matrix is defined as:

$$\boldsymbol{J}_i=\begin{bmatrix} z & -\alpha z & x-\alpha y \\ \alpha z & -\varepsilon z & 0 \\ -2x & -2y & 0 \end{bmatrix}_{f_i}, \quad i=1,2. \quad (3)$$

Obviously one has $\boldsymbol{J}_1=-\boldsymbol{J}_2$, and the eigenvalues corresponding to the fixed points f_1 and f_2 will be additive inverses (including complex eigenvalues). Furthermore, the origin (0,0,0) is not a fixed point of the system for all parameters K, α and ε obviously, which permits that at least one of the fixed points should be unstable. The characteristic polynomial is as follows:

$$\begin{cases} P(\lambda)=\det(\lambda\boldsymbol{E}-\boldsymbol{J}_i)=|\lambda\boldsymbol{E}-\boldsymbol{J}_i|=0, \\ P(\lambda)=\lambda^3-\lambda^2(1-\varepsilon)z+2(x-\alpha y)z(\alpha y-\varepsilon x) \\ \qquad -\lambda(\varepsilon z^2+2x^2-2\alpha xy+\alpha^2 z^2)=0, \end{cases} \quad (4)$$

where $\boldsymbol{E}$ is a third-order identity matrix.

The system also can be a dissipative system, because the divergence of the vector field (the trace of the Jacobian matrix)

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t}=\mathrm{div}\boldsymbol{V}=\frac{\partial\dot{x}}{\partial x}+\frac{\partial\dot{y}}{\partial y}+\frac{\partial\dot{z}}{\partial z}=z(1-\varepsilon) \quad (5)$$

is negative if and only if $z(1-\varepsilon)<0$. That is, a volume element V_0 is contract by the flow into a volume element $V_0\mathrm{e}^{z(1-\varepsilon)}$ in time t. Hence, 3D volumes in the phase space contract at a uniform exponential rate $z(1-\varepsilon)$. This suggests that the dynamics may tend to an "attractor" as $t\to\infty$.

Let K=0.4, α=3, ε=0.73. These eigenvalues corresponding to f_1 are respectively obtained as follows:

$\lambda_1=-0.5756+1.2334\mathrm{i}$, $\lambda_2=-0.5756-1.2334\mathrm{i}$,
$\lambda_3=1.2962$,

where λ_3 is a positive real number, λ_1 and λ_2 are a couple of complex conjugate eigenvalues with negative real parts. So the fixed point f_1 is a saddle-focus point in 3D phase space. So equilibrium f_1 is unstable. For the fixed point f_2, the eigenvalues corresponding to it are

$\lambda_1=0.5756-1.2334\mathrm{i}$, $\lambda_2=0.5756+1.2334\mathrm{i}$,
$\lambda_3=-1.2962$,

where λ_3 is a negative real number, λ_2 and λ_3 become a pair of complex conjugate eigenvalues with positive real parts. Therefore, this fixed point f_2 is a saddle-focus point, also unstable.

Different values of the parameters in it will produce different behaviors of the system. For example, the flow converges to a chaotic attractor; the numerical simulations can be observed in Fig.1 with parameters K=0.4, α=3, ε=0.73 and the initial condition $x(0)$=0.1, $y(0)$=0.1, $z(0)$=0.1.

NUMERICAL SIMULATIONS

To distinguish a chaotic response from a regular one, the Poincaré mapping technique proves, as is well known, very informative. A Poincaré section is

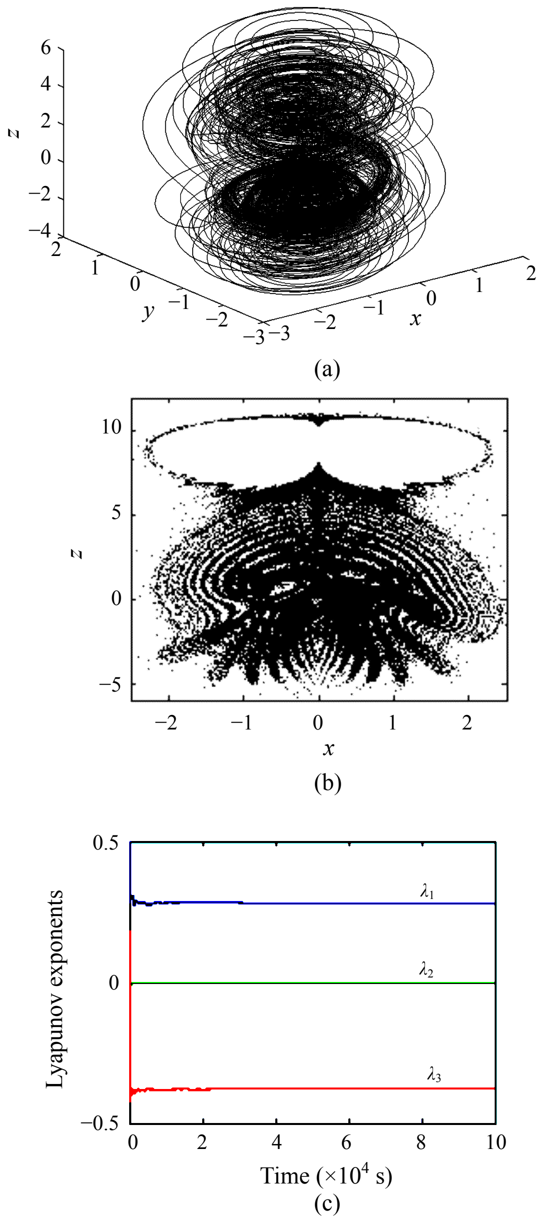

Fig.1 (a) Phase trajectory in 3D space; (b) Poincaré map in *x*-*z* plane {*y*=0}; (c) Lyapunov-exponent spectrum for ε=0.73 with $\lambda\approx$(0.29, 0, −0.38) and $D_{KY}\approx$2.76

often used to reduce a 3D (or higher) continuous system to a discrete map of dimension one or two. The strength behind this tool is that these sections have the same topological properties as their continuous counterparts. Often, the local maxima of a variable are used as a 1D Poincaré map. For more complicated systems, the distance between the maxima is more descriptive of the system's characteristics. Thus, plotting points of the section after many iterations of the dynamical system can generate the bifurcation diagram. Once again, this procedure is applied to a range of values of a given parameter.

The bifurcation diagram provides a summary of essential dynamics and is therefore a useful method for acquiring this overview. In the present study, the nonlinear equation of system Eq.(1) is integrated numerically in order to obtain the various dynamic behaviors of the three parameters K, α, ε. In this letter, the chaotic dynamics of the system versus the control parameter K will be studied chiefly and especially, which is considered as the bifurcation parameter only on the close interval [−4,4] in this paper. For this system, bifurcation can easily be detected by examining graphs of y versus control parameter K. The bifurcation diagram that will be obtained by the fourth order Runge-Kutta numerical integration algorithm is presented in Fig.2. The period-doubling bifurcation phenomena can easily be observed. The Poincaré map is also used to examine the behavior of the system. It is a 3D problem with x, y and z as independent variables. The Poincaré surface of the section, just on the x-z plane {y=0} or the x-y plane {z=0}, is plotted with the critical bifurcation parameter K increasing.

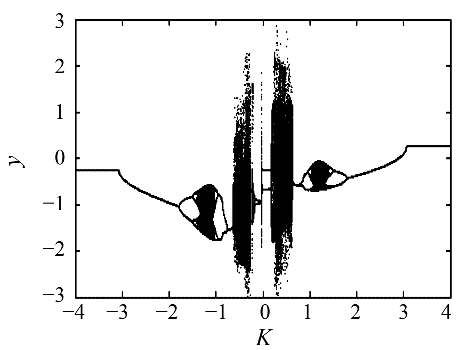

Fig.2 Bifurcation diagram for specific values set (ε=−0.8, α=3) vs *K*

However, when the bifurcation diagram loses continuity, it means that the system attractor is either in periodic motion, quasi-periodic motion or chaotic motion. As for ε=0.99, it is very hard for us to declare that the attractor of the system Eq.(1) is chaotic or periodic directly from the phase portraits in 3D space and the corresponding Poincaré section in the x-z plane {y=0} as shown in Figs.3a~3d, although there are many points in the Poincaré map. Therefore, further tests for the attractor are required to classify the dynamics.

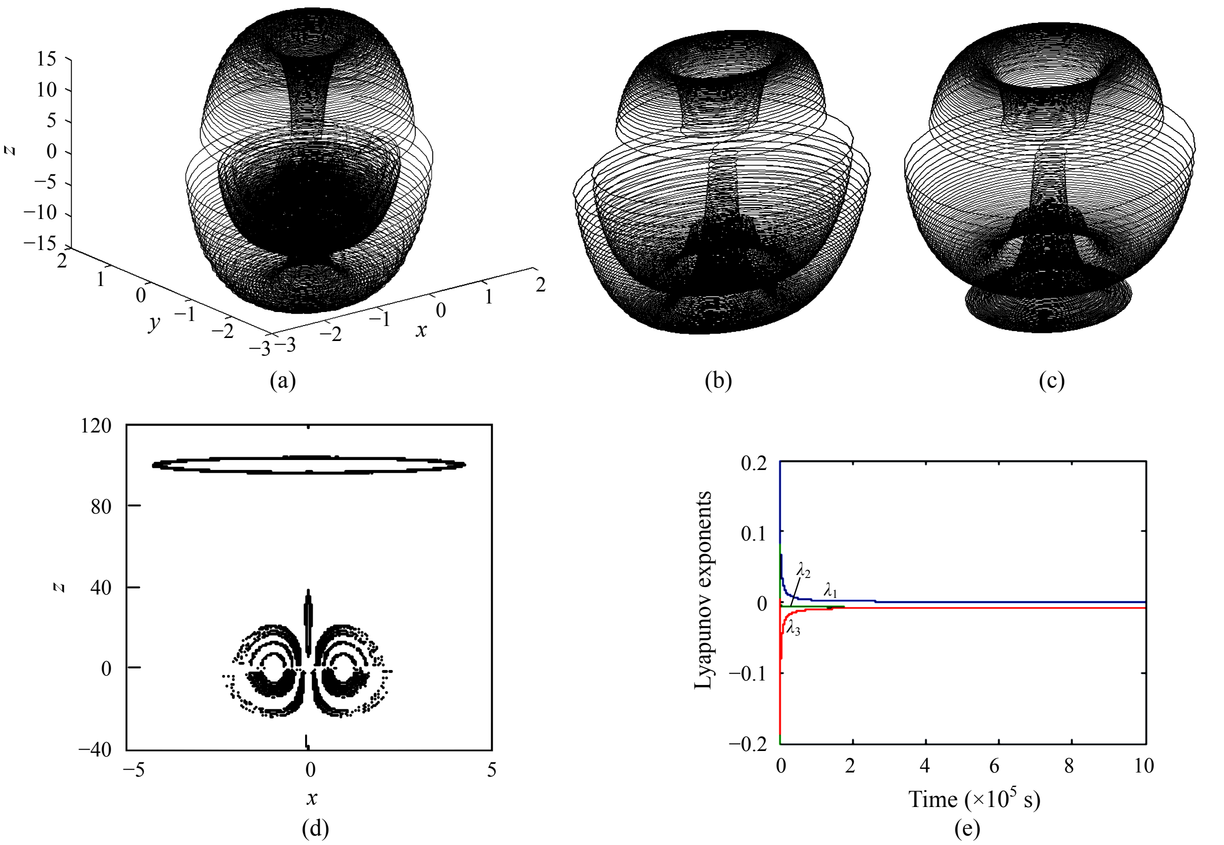

Fig.3 (a) Phase trajectory in 3D space; (b), (c) Interior phase trajectory of (a); (d) Poincaré map in *x*-*z* plane {*y*=0}; (e) Lyapunov-exponent spectrum for ε=0.99 with $\lambda\approx(0, -0.007, -0.007)$ and $D_{KY}\approx1$

LYAPUNOV EXPONENTS AND LYAPUNOV DIMENSION FROM A TIME SERIES

According to the previous section, quantifying chaos has become an important problem. Lyapunov exponents can provide qualitative and quantitative tests for dynamic behavior, which give the means of attractors and other invariant sets. The algorithm developed by Wolf *et al*.(1985) has been widely used to determine the Lyapunov exponents from a time series.

Lyapunov exponents measure the exponential rates of divergence or convergence of nearby trajectories in phase space, which can also be used to measure the sensitive dependence of the initial conditions. Lyapunov exponents are generalizations of eigenvalues of linear systems, and for linear systems they are exactly the eigenvalues. They are calculated using longtime averages of local behavior, leading to a description that reflects the global behavior of the system response. Different solutions of a dynamic system, such as fixed point, periodic motion, quasi-periodic motion and chaotic motion can be distinguished from it.

The dimension of a strange attractor also is a measure of its geometric scaling properties or its "complexity" and has been considered as the most basic property of an attractor. Numerous methods have been proposed for characterizing the fractal dimension of the strange attractors produced by chaotic flows. The Kaplan-Yorke dimension, also called the Lyapunov dimension, has been conjectured and proposed based on the Lyapunov exponents by Frederickson *et al*.(1983), which is described as

$$D_{KY} = k + \frac{1}{-\lambda_{k+1}}\sum_{i=1}^{k}\lambda_i, \tag{6}$$

where integer k satisfies $\sum_{i=1}^{k}\lambda_i > 0$ and $\sum_{i=1}^{k+1}\lambda_i < 0$.

The attractors of a 3D dissipative system and corresponding movement state are classified using the sign of Lyapunov exponents and value of Lyapunov dimensions, as shown in Table 1.

Table 1 Categories of attractors in 3D space (Liu and Chen, 2001)

λ_1	λ_2	λ_3	D_{KY}	State
−	−	−	0	Fixed points
0	−	−	1	Periodic state
0	0	−	2	Quasi-periodic state
+	0	−	$2<D_{KY}<3$	Chaos

The first step in order to calculate the exponent is to start iterating the system until the orbit is on the attractor. In order to achieve this and avoid errors it is necessary to iterate a few thousand times. The Lyapunov exponents *LEs* and the Lyapunov dimension D_{KY} resulted after 10^5 iterations with a normalized step-size 0.05 for every attractor, which has been plotted and discussed above, are shown in Figs.4 and 5 respectively. It should not be a chaotic attractor with one Lyapunov exponent $\lambda_1\to0$, λ_2, $\lambda_3\approx-0.007$ for $\varepsilon=0.99$, which is demonstrated in Fig.3e.

Figs.4 and 5 show the Lyapunov-exponent spectrum and the Lyapunov dimensions for specific values set ($\varepsilon=-0.8$, $\alpha=3$) vs the bifurcation parameter K on the close interval [−4,4], respectively.

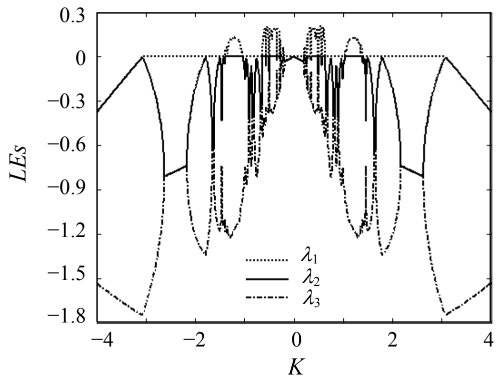

Fig.4 Lyapunov-exponent spectrum against *K* corresponding to the bifurcation diagram of Fig.2

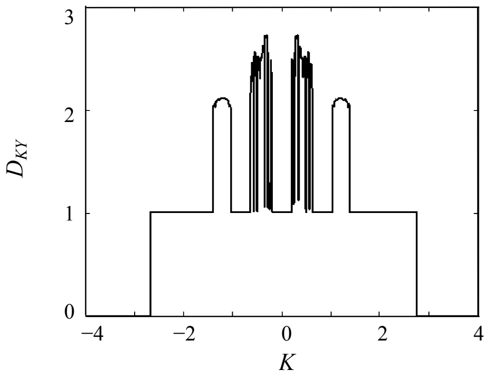

Fig.5 Lyapunov dimension vs the control parameter *K* corresponding to the bifurcation diagram of Fig.2

ALMOST-HAMILTONIAN CHAOS

System Eq.(1) also could be modified directly to another new chaotic flow, by adding the constant coefficient 0.5 to the first and the second equation. The new chaotic flow is described as (Chlouverakis, 2005):

$$\begin{cases} \mathrm{d}x/\mathrm{d}t=K+0.5z(x-\alpha y), \\ \mathrm{d}y/\mathrm{d}t=0.5z(\alpha x-\varepsilon y), \\ \mathrm{d}z/\mathrm{d}t=1-x^2-y^2. \end{cases} \tag{7}$$

The trace of the Jacobian matrix is equal to $0.5(1-\varepsilon)z$; especially, when $\varepsilon=-1$, the trace is equal to z, and averaging along the trajectory we get $z\approx0$, but still negative for ensuring it is dissipative, so system Eq.(7) maybe result in almost-Hamiltonian chaos from the idea above we have discussed about system Eq.(1) when $\varepsilon=1$. In this case (Fig.6), the parameters are selected as $\alpha=20$, $K=0.2$, $\varepsilon=-1$; the chaotic attractor is demonstrated in the x-y $\{z=0\}$ and x-z $\{y=0\}$ plane, respectively; the most important criterion for this case is validated by the three Lyapunov exponents with Kaplan-Yorke dimensions being almost equal to 3 after 10^6 iterations with a normalized step-size 0.01.

CONCLUSION AND DISCUSSION

In this paper, a new chaotic system Eq.(1) which was simplified and modified by Chlouverakis and Sprott is studied in detail by varying three control parameters, chiefly the control parameter K. The dynamical behaviors of the chaotic system are analyzed, both theoretically and numerically only on the interval [−4,4]. And a new chaotic flow is constructed based on the new chaotic system Eq.(1), which can produce an almost-Hamiltonian chaotic attractor reaching a dimension almost close to $D_{KY}\approx3$ with given parameters values.

Abundant and complex dynamical behaviors, produced by the new autonomous systems, are investigated and expatiated in this paper. Many nonlinear dynamical behaviors are investigated and presented, but the attractors and their forming mechanism need to be further explored, and their topological structure should be completely and thoroughly investigated. Therefore, further research into these two chaotic systems is still important and insightful.

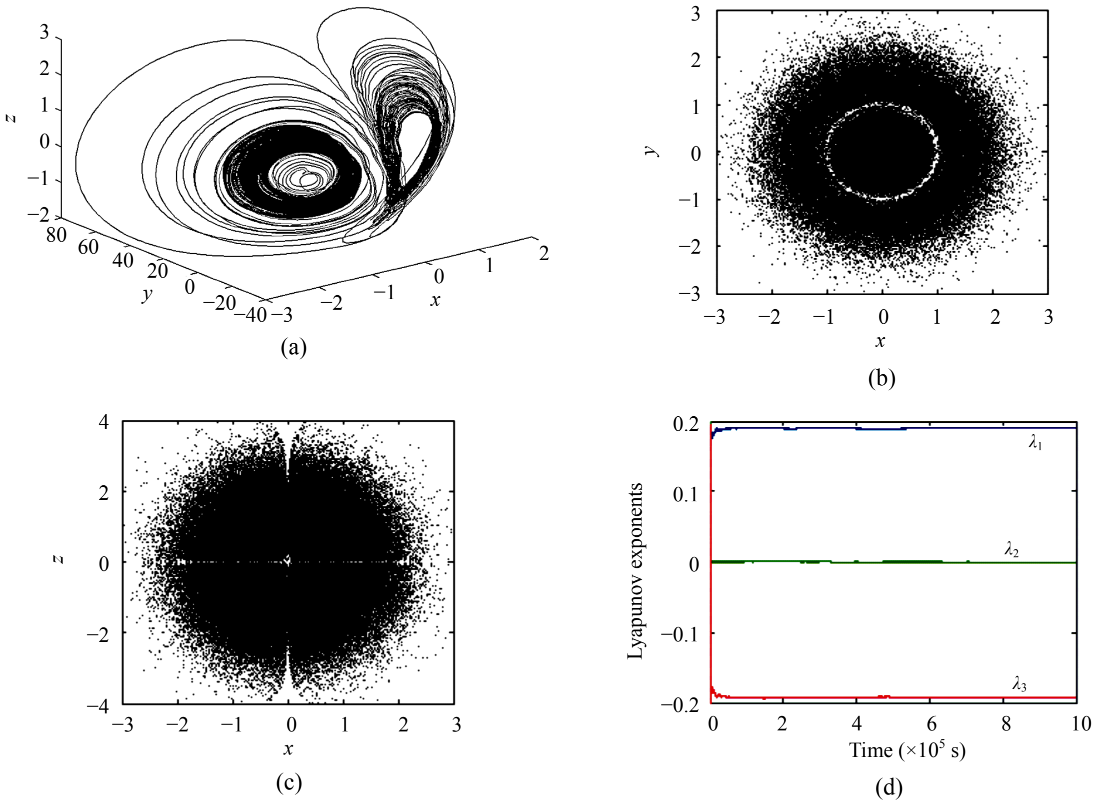

Fig.6 (a) Phase trajectory in 3D space; (b) Poincaré map in *x*-*y* plane {*z*=0}; (c) Poincaré map in *x*-*z* plane {*y*=0}; (d) Lyapunov-exponent spectrum for α=20, *K*=0.2, ε=−1 with λ≈(0.19,0,−0.192) and D_{KY}≈3

References

Chen, G.R., Ueta, T., 1999. Yet another chaotic attractor. *Int. J. Bifurcat. & Chaos*, **9**:1465-1466. [doi:10.1142/S0218127499001024]

Chlouverakis, K.E., 2005. Color maps of the Kaplan-Yorke dimension in optically driven lasers: maximizing the dimension and almost-Hamiltonian chaos. *Int. J. Bifurcat. & Chaos*, **15**:3011-3021. [doi:10.1142/S0218127405013848]

Chlouverakis, K.E., Adams, M.J., 2003. Stability maps of injection-locked laser diodes using the largest Lyapunov exponent. *Opt. Commun.*, **216**:405-412. [doi:10.1016/S0030-4018(02)02357-X]

Chlouverakis, K.E., Sprott, J.C., 2005. A comparison of correlation and Lyapunov dimensions. *Physica D*, **200**:156-164. [doi:10.1016/j.physd.2004.10.006]

Chlouverakis, K.E., Sprott, J.C., 2006. Chaotic hyperjerk systems. *Chaos, Solitons and Fractals*, **28**:739-746. [doi:10.1016/j.chaos.2005.08.019]

Chua, L.O., Komuro, M., Matsum, T., 1986. The double scroll family. Part I: Rigorous proof of chaos. *IEEE Trans. Circuits. Syst.*, **33**:1072-1096. [doi:10.1109/TCS.1986.1085869]

Frederickson, P., Kaplan, J.L., Yorke, E.D., 1983. The Lyapunov dimension of strange attractors. *J. Differential Equations*, **49**:185-207. [doi:10.1016/0022-0396(83)90011-6]

Kim, S.Y., Kim, Y., 2000. Dynamic stabilization in the double-well Duffing oscillator. *Phys. Rev. E*, **61**:6517-6520. [doi:10.1103/PhysRevE.61.6517]

Liu, Y.Z., Chen, L.Q., 2001. Nonlinear Vibrations. Higher Education Press, Beijing (in Chinese).

Lorenz, E.N., 1963. Deterministic nonperiodic flow. *J. Atmos. Sci.*, **20**:130-141. [doi:10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2]

Lü, J.H., Chen, G.R., 2002. A new chaotic attractor coined. *Int. J. Bifurcat. & Chaos*, **12**(3):659-661. [doi:10.1142/S0218127402004620]

Rössler, O.E., 1976. An equation for continuous chaos. *Phys. Lett. A*, **57**:397-398. [doi:10.1016/0375-9601(76)90101-8]

Wieczorek, S., Krauskopf, B., Lenstra, D., 1999. A unifying view of bifurcations in a semiconductor laser subject to optical injection. *Opt. Commun.*, **172**:279-295. [doi:10.1016/S0030-4018(99)00603-3]

Wolf, A., Swift, J.B., Swinney, H.L., Vastano, J.A., 1985. Determining Lyapunov exponents from a time series. *Physica D: Nonlinear Phenomena*, **16**:285-317. [doi:10.1016/0167-2789(85)90011-9]