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Restrained torsion of open thin-walled beams including shear deformation effects*

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Abstract: A first-order torsion theory based on Vlasov theory has been developed to investigate the restrained torsion of open thin-walled beams. The total rotation of the cross section is divided into a free warping rotation and a restrained shear rotation. In first-order torsion theory, St. Venant torque is only related to the free warping rotation and the expression of St. Venant torque is derived by using a semi-inverse method. The relationship between the warping torque and the restrained shear rotation is established by using an energy method. The torsion shear coefficient is then obtained. On the basis of the torsion equilibrium, the governing differential equation of the restrained torsion is derived and the corresponding initial method is given to solve the equation. The relationship between total rotation and free warping rotation is obtained. A parameter λ, which is associated with the stiffness property of a cross section and the beam length, is introduced to determine the condition, under which the St. Venant constant is negligible. Consequently a simplified theory is derived. Numerical examples are illustrated to validate the current approach and the results of the current theory are compared with those of some other available methods. The results of comparison show that the current theory provides more accurate results. In the example of a channel-shaped cantilever beam, the applicability of the simplified theory is determined by the parameter study of λ.

Key words: Thin-walled beam, Restrained torsion, Shear deformation, Warping, Shear coefficient
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1 Introduction

Thin-walled structures are widely used in the fields of civil, mechanical and naval constructions, aeronautical/aerospace, automotive and helicopter rotor blades, and comprise an important and growing proportion of engineering structures (Librescu and Song, 2006). The most well-known advantages of thin-walled members are light weight and high strength. There has been a growing interest in the foundation of the theory of thin-walled members.

The classical theory of thin-walled beams was developed by Vlasov (1961) and Gjelsvik (1981). The theory assumed that the contour of a thin wall does not deform in its own plane and the shear strain of the middle surface is negligible. Mohareb and Nowzartash (2003) developed a finite beam element formulation based on St. Venant and Vlasov theories. As Vlasov theory does not consider the shear deformation effects, it is only applicable for a slender beam. When the beam is short or unsymmetrical open-typed thin-walled section, the transverse shear effect becomes very important because of the flexural-torsional behavior of the beam (Park *et al*., 1997). There are a number of flexure theories of beams that consider shear deformation, including the first-, second-, and higher-order beam theories (Wang, 1995; Reddy *et al*., 1997).

The widely used torsion theory which included shear deformation is Benscoter theory, and in this theory the twist rate function of thin-walled beams was replaced by another function which had no

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obvious physical meaning (Shakourzadeh *et al*., 1994; Chen and Hu, 1998). Several beam finite elements were developed to account for the shear deformation due to restrained torsion. Shakourzadeh *et al*. (1994) developed a finite element based on Benscoter thin-walled beam theory. Back and Will (1998) presented a finite element for the analysis of open thin-walled beams with arbitrary cross sections, which included both transverse shear deformation and warping deformation. Kim and Kim (2005) developed an improved thin-walled beam theory considering the shear deformation due to the shear force and restrained warping rotation and the coupled effect between these two shear deformations by introducing Vlasov's assumption and applying Hellinger-Reissner principle. Lee (2005) developed an analytical model to study the flexural behavior of thin-walled composite beams with doubly symmetric I-section subjected to uniformly distributed vertical load. Yang and Wang (2010) developed a new geometrical and physical nonlinear beam element model. Back and Will (2008) developed a shear flexible finite element for the flexural and buckling analysis of thin-walled composite I-beams with both doubly and mono-symmetrical cross sections. Kim (2011) derived a shear deformable beam element for the coupled flexural and torsional analyses of thin-walled composite I-beams and evaluated the stiffness matrix of thin-walled composite I-beams.

Many researchers have obtained analytical solutions for torsional problems of thin-walled beams. Roberts and Al-Ubaidi (2001) presented an approximate theory considering the influence of shear deformation on the restrained torsional warping, which has been validated by a series of bending and torsion experiments on pultruded fiber reinforced plastic (FRP) I-beams. Pavazza (2005) assumed that the shear stress was constant along the length of an open thin-walled beam, and developed an approximate analytical approach to the torsion of open thin-walled beams with effect of shear deformation. Erkmen and Mohareb (2006) developed a theory based on postulated stress field for the torsional analysis of open steel thin-walled beams of general cross sections which accounted for shear deformation effects. Based on the Prokić's work, Saadé *et al*. (2004) analyzed the behavior of thin-walled beams by using a beam theory with a single warping function valid for arbitrary form of cross sections. El Fatmi (2007a; 2007b) presented a beam theory with a non-uniform warping including the effects of torsion and shear forces, and the theory is valid for any homogeneous cross sections made of isotropic elastic material. Mokos and Sapountzakis (2011) proposed a non-uniform torsion theory of doubly symmetrical arbitrary constant cross section including secondary torsional moment deformation effect and developed the corresponding boundary element method.

In this paper, a first-order torsion theory for open thin-walled structures is developed on the basis of Vlasov theory. The current theory considers the effects of shear deformations on total rotation and St. Venant torque. The torsion shear coefficient is derived using an energy method. An initial method is employed to solve the governing equation efficiently. The simplified style of the first-order theory is derived. Numerical examples are presented to validate the first-order torsion theory and the simplified approach. The first-order theory provided more accurate results than other theories. The parameter λ, which is proposed to determine the applicability of the simplified theory, is evaluated in the example of a channel-shaped cantilever beam.

2 Displacement field and resultants of thin-walled beams

The basic assumptions of the first-order torsion theory of thin-walled beams are as follows:

(1) The contour of a cross section does not deform in its own plane;

(2) Transverse shear strains γ_{zx} and γ_{zy} are not negligible and they are constant over the middle surface of the cross section;

(3) The total rotation of a cross section is divided into the free warping rotation and the restrained shear rotation. The free warping rotation θ_ω generates the warping deformation without the shear strain. The restrained shear rotation θ_s generates only the shear deformation of the cross section without the warping deformation;

(4) The deformations are small with respect to the dimensions of the cross section.

Fig. 1 shows a typical open cross section of a thin-walled beam. The description of displacement field needs three coordinate systems: a global right-handed Cartesian coordinate system (x, y, z), a local right-handed Cartesian coordinate system (n, s, z), and a curvilinear coordinate s along the contour of the cross section with its origin at point O on the contour. The three coordinate systems are as shown in Fig. 1. The global coordinate system is fixed, and the local coordinate follows point P on the contour. The n axis is normal to the contour and the s axis is tangent to the contour. The wall thickness is assumed to be constant in the longitudinal direction, and the thickness $t(s)$ is a function of s only.

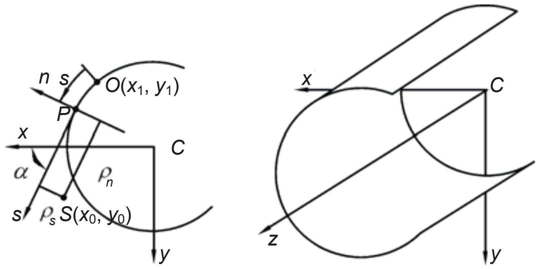

Fig. 1 Cross section of a thin-walled beam

As shown in Fig. 1, the shear center S is an arbitrary point and x_0 and y_0 are its coordinates, s is the curvilinear coordinate of the point P from the origin point O. The displacements of the pole S of the cross section in x and y directions are $u=u(z)$ and $v=v(z)$, respectively, and $\theta=\theta(z)$ is the rotation angle of the cross section about the pole S.

According to assumption 1, the beam is deformed in such a way that the shape of the cross section remains unchanged. The tangential displacement v_s of the arbitrary point P on the contour can be expressed in terms of the displacements u and v, and the rotation angle θ of the pole S:

$$v_s(s,z)=u(z)\frac{\mathrm{d}x}{\mathrm{d}s}+v(z)\frac{\mathrm{d}y}{\mathrm{d}s}+\theta(z)\rho_s, \tag{1}$$

where

$$\frac{\mathrm{d}x}{\mathrm{d}s}=\cos\alpha,\quad \frac{\mathrm{d}y}{\mathrm{d}s}=\sin\alpha, \tag{2}$$

and $\alpha=\alpha(s)$ is the angle between the tangent of point P and the axis x, ρ_s is the distance from S to the tangent of an arbitrary point P on the contour.

According to assumption 2, the shear strain induced by transverse shear effects is given as follows by the transformation rule:

$$\gamma_{zs}^{\mathrm{T}}=\gamma_{xz}\frac{\mathrm{d}x}{\mathrm{d}s}+\gamma_{yz}\frac{\mathrm{d}y}{\mathrm{d}s}. \tag{3}$$

And according to assumption 3, which indicates that the cross section is subjected to a rigid body rotation, as in the case of circular cylinder, and therefore the shear strain induced by the restrained shear rotation can be written as

$$\gamma_{zs}^{\mathrm{R}}=\theta_s'\rho_s, \tag{4}$$

where θ_s' is the twist rate of the thin-walled beam induced by the restrained shear rotation.

According to the principle of superposition, the total shear strain is given by

$$\gamma_{zs}=\gamma_{xz}\frac{\mathrm{d}x}{\mathrm{d}s}+\gamma_{yz}\frac{\mathrm{d}y}{\mathrm{d}s}+\theta_s'\rho_s. \tag{5}$$

When the beam is subjected to pure torsion, Eqs. (1) and (5) respectively become

$$v_s(s,z)=\theta(z)\rho_s, \tag{6}$$

$$\gamma_{zs}=\theta_s'\rho_s. \tag{7}$$

From the definition of shear strain, γ_{zs} can also be given by

$$\gamma_{zs}=\frac{\partial w}{\partial s}+\frac{\partial v_s}{\partial z}. \tag{8}$$

In view of Eqs. (1), (5), and (8), the longitudinal displacement $w(s, z)$ can be obtained as

$$\frac{\partial w}{\partial s}=-\theta_\omega'(z)\rho_s(s), \tag{9}$$

where

$$\theta_\omega'(z)=\theta'(z)-\theta_s'(z) \tag{10}$$

and θ_ω' is the twist rate of the thin-walled beam induced by the free warping rotation, while the prime (′) indicates derivatives with respect to z. Integration of

Eq. (9) with respect to s from the origin point O to the arbitrary point P on the contour yields

$$w(s,z)=w_0(z)-\theta'_\omega(z)\omega(s), \tag{11}$$

where $w_0(z)$ can be interpreted as the average displacement of cross section, and $\omega=\omega(s)$ is the sectorial coordinate with respect to the pole S and the origin point O:

$$\omega=\int_0^s \rho_s \mathrm{d}s. \tag{12}$$

It is assumed that in-plane stresses of a beam cross section are negligible, and the normal stress in the longitudinal direction from Hooke's law is given by

$$\sigma_z=E\varepsilon_z=E\frac{\mathrm{d}w}{\mathrm{d}z}, \tag{13}$$

where $\sigma_z=\sigma_z(s, z)$ is the normal stress in the longitudinal direction, $\varepsilon_z=\varepsilon_z(s, z)$ is the corresponding normal strain, and E is the modulus of elasticity.

By substituting Eq. (11) into Eq. (13), the normal stress can be written as

$$\sigma_z=E\left[\frac{\mathrm{d}w_0}{\mathrm{d}z}-\frac{\mathrm{d}\theta'_\omega}{\mathrm{d}z}\omega(s)\right]. \tag{14}$$

When the thin-walled beam is only subjected to torsion, the axial force N and bending moment M_x and M_y about the x and y axes defined over the cross section are zero. Thus,

$$\begin{cases} N=\int_s \sigma_z t\mathrm{d}s=0, \\ M_x=\int_s \sigma_z ty\mathrm{d}s=0, \\ M_y=\int_s \sigma_z tx\mathrm{d}s=0. \end{cases} \tag{15}$$

A sectorial origin point is chosen which makes

$$\frac{\mathrm{d}w_0}{\mathrm{d}z}=0. \tag{16}$$

Substituting Eq. (16) into Eq. (14) yields

$$\sigma_\omega=-E\omega\theta''_\omega. \tag{17}$$

A differential element P of the thin wall is taken out to evaluate the shear stress (Fig. 2). The dimensions of the differential element are dz along the length of the beam and ds along the contour. The positive direction of coordinate s is as shown in Fig. 2.

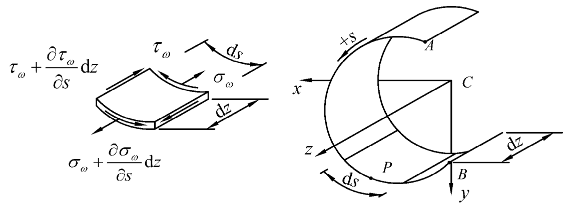

Fig. 2 Equilibrium of a differential element

The equilibrium of the differential element in the longitudinal direction can be written as

$$\frac{\partial(\tau_\omega t)}{\partial s}+\frac{\partial(\sigma_\omega t)}{\partial z}=0, \tag{18}$$

where $t=t(s)$ is the wall thickness.

In view of Eq. (17) and with the opening edge as initial integral point, it yields

$$\tau_\omega t=E\theta'''_\omega S_\omega, \tag{19}$$

where

$$S_\omega=\int_0^P \omega t\mathrm{d}s \tag{20}$$

is the sectorial area moment of the cross section.

Bimoment B and warping torque M_ω are defined as

$$B=\int_s \sigma_z t\omega\mathrm{d}s, \quad M_\omega=\int_s \tau_\omega t\rho_s\mathrm{d}s. \tag{21}$$

By substituting Eqs. (17) and (19) into Eq. (21), one obtains

$$B=-EI_\omega\theta''_\omega, \quad M_\omega=-EI_\omega\theta'''_\omega, \tag{22}$$

where $I_\omega=\int_A \omega^2\mathrm{d}A$, and

$$\sigma_\omega=B\omega/I_\omega,\ \tau_\omega t=-M_\omega S_\omega/I_\omega. \tag{23}$$

Eq. (22) yields

$$M_\omega=\frac{\mathrm{d}B_\omega}{\mathrm{d}z}. \tag{24}$$

Compared with the Vlasov theory, the bimoment and warping torque of cross section given in Eq. (22) are related with the free warping rotation rather than the total rotation.

3 St. Venant torque in restrained torsion

When an open thin-walled beam is subjected to a restrained rotation, the total rotation of cross section is divided into free warping rotation and restrained shear rotation:

$$\theta(z)=\theta_{\omega}(z)+\theta_{s}(z), \tag{25}$$

where $\theta_\omega(z)$ and $\theta_s(z)$ are free warping rotation and restrained shear rotation, respectively.

An infinitesimal element of the beam of length dz is as shown in Fig. 3. According to assumption 1, the in-plane displacement of an arbitrary point P on the right cross section corresponding to rotation of cross section is given by

$$\begin{Bmatrix} \mathrm{d}u(s,z) \\ \mathrm{d}v(s,z) \end{Bmatrix}=\theta'(z)\begin{Bmatrix} -y\mathrm{d}z \\ x\mathrm{d}z \end{Bmatrix}. \tag{26}$$

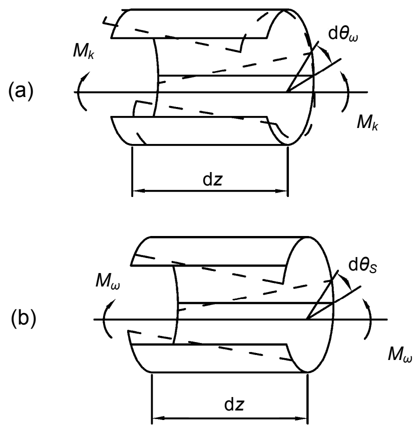

Fig. 3 Free warping rotation (a) and restrained shear rotation (b) of middle surface of an open thin-walled beam

In view of Eq. (25), Eq. (26) becomes

$$\begin{Bmatrix} \mathrm{d}u(s,z) \\ \mathrm{d}v(s,z) \end{Bmatrix}=\theta'_{\omega}(z)\begin{Bmatrix} -y\mathrm{d}z \\ x\mathrm{d}z \end{Bmatrix}+\theta'_{s}(z)\begin{Bmatrix} -y\mathrm{d}z \\ x\mathrm{d}z \end{Bmatrix}, \tag{27}$$

where θ'_ω is the twist rate of the thin-walled beam induced by the free warping rotation, and $\omega(s)$ is the warping function. The in-plane displacements d$u(s, z)$ and d$v(s, z)$ are due to the rigid body rotation of the cross section, as in the case of the circular cylinder.

Considering Eqs. (11) and (16), the out-of-plane displacement of an arbitrary point P on the right cross section corresponding to rotation of the cross section is written as

$$\mathrm{d}w(s,z)=-\theta'_{\omega}(z)\omega(s), \tag{28}$$

where the out-of-plane displacement $w(s, z)$ is proportional to the twist rate of the free warping rotation and has an arbitrary variation over the cross section described by the warping function $\omega(s)$.

Hence, the displacement of an arbitrary point P on the right cross section can be described by

$$\begin{Bmatrix} \mathrm{d}u(s,z) \\ \mathrm{d}v(s,z) \\ \mathrm{d}w(s,z) \end{Bmatrix}=\theta'_{\omega}(z)\begin{Bmatrix} -y\mathrm{d}z \\ x\mathrm{d}z \\ -\omega(s) \end{Bmatrix}+\theta'_{s}(z)\begin{Bmatrix} -y\mathrm{d}z \\ x\mathrm{d}z \\ 0 \end{Bmatrix}. \tag{29}$$

According to assumption 3, the free warping rotation generates the warping deformation without shear strain. However, the restrained shear rotation generates shear strain without the warping deformation, as shown in Fig. 3. Hence, it is postulated that in the restrained torsion, St. Venant torque is only related to the free warping rotation and the warping torque is only associated with the restrained shear rotation.

St. Venant torque and stresses of an open thin-walled beam in the restrained torsion will be investigated by using the semi-inverse method. Since St. Venant torque is only related to the free warping rotation of thin-walled beams, the displacement of an arbitrary point P on the contour associated to St. Venant torque can be described by

$$\begin{Bmatrix} \mathrm{d}u(s,z) \\ \mathrm{d}v(s,z) \\ \mathrm{d}w(s,z) \end{Bmatrix}=\theta'_{\omega}(z)\begin{Bmatrix} -y\mathrm{d}z \\ x\mathrm{d}z \\ -\omega(s) \end{Bmatrix}. \tag{30}$$

Prandtl stress function Φ must satisfy the differential equation which is given in the coordinate system (s, n) by

$$\frac{\partial^2 \Phi}{\partial s^2}+\frac{\partial^2 \Phi}{\partial n^2}=\nabla^2 \Phi=-2G\theta'_\omega. \tag{31}$$

For the cross section of a thin-walled beam as shown in Fig. 1, there is little change of Φ in the s direction, except regions near the far edges of the cross section. This means that Eq. (31) can be simplified to

$$\frac{\partial^2 \Phi}{\partial n^2}=-2G\theta'_\omega. \tag{32}$$

Integrating and enforcing Φ=0 at the edges of the cross section, i.e., at $n=\pm t/2$, it gives

$$\Phi=G\theta'_\omega\left(\frac{t^2}{4}-n^2\right). \tag{33}$$

The St. Venant stress distribution now is given by

$$\tau_{zn}=-\frac{\partial \Phi}{\partial s}=0,\quad \tau_{zs}=\frac{\partial \Phi}{\partial n}=2G\theta'_\omega n. \tag{34}$$

And the torque associated with the St. Venant stress distribution is evaluated with the help of the membrane analogy:

$$\begin{aligned} M_k &=2\iint_A \Phi \mathrm{d}x\mathrm{d}y=2G\theta'_\omega\int_s\left[\int_{-\frac{t}{2}}^{\frac{t}{2}}\left(\frac{t^2}{4}-n^2\right)\mathrm{d}n\right]\mathrm{d}s \\ &=\frac{1}{3}G\theta'_\omega\int_s t^3\mathrm{d}s. \end{aligned} \tag{35}$$

Here again a linear relationship is found between the St. Venant torque and the resulting free twist rate. The torsion constant is taken as

$$I_k=\frac{1}{3}\int_s t^3\mathrm{d}s. \tag{36}$$

If the cross section consists of several portions of different plates, the St. Venant torsion constant can be assumed to be

$$I_k=\frac{1}{3}\sum_i b_i t_i^3, \tag{37}$$

where b_i and t_i are the width and thickness of the ith plate, respectively.

Thus, the St. Venant torque of the open thin-walled beam in the restrained torsion can be written as

$$M_k=GI_k\theta'_\omega. \tag{38}$$

The stress of an arbitrary point P as the result of the St. Venant torque is

$$\tau_{zs}=2M_k n/I_k, \tag{39}$$

where n is the normal coordinate of point P.

Unlike the Vlasov theory or Benscoter theory or other first-order theory, Eq. (38) shows that in the current theory, the St. Venant torque is related with the twist rate of free warping rotation rather than the twist rate of total rotation.

4 Restrained shear rotation and torsion shear coefficient

As mentioned previously, in the restrained torsion, the restrained shear rotation generates the shear strain without the warping deformation, which looks like a rigid body rotation, and is only concerned with the warping torque of the cross section. Thus, the restrained shear rotation θ_s is seen as the generalized displacement of warping torque M_ω.

To determine the relationship between the warping shear rotation and the warping torque, an element of length dz of a thin-walled beam is considered, as depicted in Fig. 3b.

As shown in Fig. 3b, only shear strain is generated and the cross section remains planar. The rotation angle per unit length can be calculated by

$$\frac{\mathrm{d}\theta_s}{\mathrm{d}z}=\theta'_s, \tag{40}$$

where θ'_s can be considered as the twist rate of the restrained shear rotation.

If the warping torque of the cross section is indicated by M_ω, the work done by the warping torque is written as

$$\Delta W=\frac{1}{2}M_\omega\theta'_s\mathrm{d}z. \tag{41}$$

Under the warping shear stress, each differential element of the thin wall deforms and a shear strain γ_ω is created (Fig. 4). The strain energy stored in this differential element is

$$\mathrm{d}U = \frac{1}{2}\gamma_\omega \tau_\omega t \mathrm{d}s\mathrm{d}z. \quad (42)$$

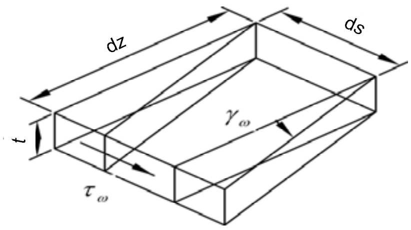

Fig. 4 Deformation of a differential element

The total strain energy for the entire element is then

$$\Delta U = \int_s \mathrm{d}U = \frac{1}{2}\int_s \gamma_\omega \tau_\omega t \mathrm{d}s\mathrm{d}z. \quad (43)$$

Considering Hooke's law, $\tau_\omega = G\gamma_\omega$, Eq. (43) yields

$$\Delta U = \mathrm{d}z \int_s \frac{1}{2}\tau_\omega t \frac{\tau_\omega}{G}\mathrm{d}s = \frac{\mathrm{d}z}{2G}\int_s \tau_\omega^2 t \mathrm{d}s. \quad (44)$$

The total energy must be equal to the work done by the warping torque that rotates the cross section, i.e.,

$$M_\omega \theta_s' = \frac{1}{G}\int_s \tau_\omega^2 t \mathrm{d}s. \quad (45)$$

By substituting Eq. (23) into Eq. (45), one obtains

$$\theta_s' = \frac{M_\omega}{I_\omega^2 G}\int_s \frac{S_\omega^2}{t}\mathrm{d}s = \frac{f_\omega M_\omega}{GI_{\rho s}}, \quad (46)$$

where f_ω is a new parameter, which is called the torsion shear coefficient, and $I_{\rho s}$ is the tangential polar moment of inertia. They are given by

$$f_\omega = \frac{I_{\rho s}}{I_\omega^2}\int_s \frac{S_\omega^2}{t}\mathrm{d}s, \quad (47)$$

$$I_{\rho s} = \int_s \rho_s^2 t \mathrm{d}s. \quad (48)$$

As can be seen from Eq. (47), f_ω is dimensionless and only related to geometrical properties of the cross section.

In current theory, a torsion shear coefficient is presented. The torsion shear coefficient attempts to overcome the inability of this first-order torsion theory, to account for the true warping shear stress distribution in the cross section. Due to the ignorance of torsion shear coefficient, Benscoter theory underestimates the value of rotation for a thin-walled beam.

5 Torsion differential equation and boundary conditions

5.1 Torsion differential equation

In the restrained torsion, the total rotation is divided into the free warping rotation and the restrained shear rotation. Consequently, the total torque is the sum of that generated by the free warping rotation and that due to the restrained shear rotation, i.e.,

$$M_k + M_\omega = M_z. \quad (49)$$

By substituting Eqs. (22) and (38) into Eq. (49), the total torque of the beam is given as follows:

$$GI_k \theta_\omega' - EI_\omega \theta_\omega''' = M_z. \quad (50)$$

The equations of equilibrium associated with the torsion behavior can be obtained by considering the infinitesimal element of the beam of length dz as shown in Fig. 5. Summing all the moment about the axis z yields the torsion equilibrium equation:

$$\frac{\mathrm{d}M_z}{\mathrm{d}z} = -m. \quad (51)$$

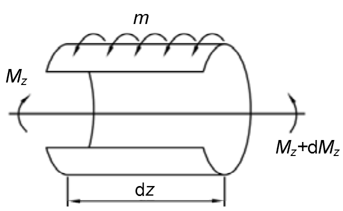

Fig. 5 Equilibrium of a infinitesimal element

In conjunction with Eq. (50), one obtains the differential equation as follows:

$$GI_k\theta''_\omega - EI_\omega\theta_\omega^{\mathrm{IV}} = -m. \tag{52}$$

According to Eq. (46), the restrained shear rotation has the following relationship with the warping torque:

$$\theta'_s = \frac{f_\omega M_\omega}{GI_{\rho s}}. \tag{53}$$

Eq. (25) gives

$$\theta(z) = \theta_\omega(z) + \theta_s(z). \tag{54}$$

Eqs. (52)–(54) form the governing equations of an open thin-walled beam under the restrained torsion. As shown in Eq. (52), it is a differential equation about free warping rotation θ_ω, rather than the total rotation in Vlasov theory or the combination of total rotation and another function in Benscoter theory. Unlike other torsion theories, a differential equation about restrained shear rotation should be given as shown in Eq. (53). In this case, there is no need to find a relation between the free warping rotation and the total rotation to build a differential equation about the total rotation for the solution of total rotation.

In view of Eqs. (22), (53), and (54), the relationship between total rotation and free warping rotation is obtained as

$$\theta' = \theta'_\omega - \mu\theta'''_\omega, \tag{55}$$

where

$$\mu = \frac{f_\omega EI_\omega}{GI_{\rho s}}. \tag{56}$$

5.2 Initial parameter method and its influence function

Compared with the direct method, it is efficient to use the initial parameter method to solve the differential equation due to its simple style and definite mechanical meaning (Chen and Hu, 1998).

To ease the solution of Eq. (52), a parameter κ is introduced by

$$\kappa^2 = \frac{GI_k}{EI_\omega}. \tag{57}$$

The differential Eq. (52) becomes

$$\theta_\omega^{\mathrm{IV}} - \kappa^2\theta''_\omega = \frac{m}{EI_\omega}. \tag{58}$$

When external distributed torque m=0, the homogeneous solution of Eq. (58) is

$$\theta_\omega = C_1 + C_2 z + C_3\sinh\kappa z + C_4\cosh\kappa z. \tag{59}$$

The first-order and higher-order derivatives of Eq. (59) are

$$\theta'_\omega = C_2 + C_3\kappa\cosh\kappa z + C_4\kappa\sinh\kappa z, \tag{60}$$

$$\theta''_\omega = C_3\kappa^2\sinh\kappa z + C_4\kappa^2\cosh\kappa z, \tag{61}$$

$$\theta'''_\omega = C_3\kappa^3\cosh\kappa z + C_4\kappa^3\sinh\kappa z. \tag{62}$$

And the bimoment and warping torque are given by

$$B = -EI_\omega(C_3\kappa^2\sinh\kappa z + C_4\kappa^2\cosh\kappa z), \tag{63}$$

$$M_z = GI_k C_2. \tag{64}$$

In view of Eqs. (22), (53), and (62), the restrained shear rotation of the thin-walled beam is obtained as

$$\theta_s = -\mu(C_3\kappa^2\sinh\kappa z + C_4\kappa^2\cosh\kappa z) + C_5. \tag{65}$$

By substituting Eqs. (59) and (65), the total rotation is given by

$$\theta = C_1 + C_5 + C_2 z + \mu_1 C_3\sinh\kappa z + \mu_1 C_4\cosh\kappa z, \tag{66}$$

where

$$\mu_1 = 1 - \mu\kappa^2. \tag{67}$$

The state parameters of cross section at z=0 are given as follows:

$$\theta(0)=\theta_0,\quad \theta_\omega(0)=\theta_{\omega 0},\\ \theta'_\omega(0)=\theta'_{\omega 0},\ B(0)=B_0,\ M_z(0)=M_{z0}. \tag{68}$$

Substituting Eqs. (59), (60), (63), (64), and (66) into Eq. (68) yields

$$\theta_0=C_1+C_5+\mu_1 C_4,\quad \theta_{\omega 0}=C_1+C_4,\\ \theta'_{\omega 0}=C_2+C_3\kappa,\ B_0=-EI_\omega C_4\kappa^2,\ M_{z0}=GI_k C_2, \tag{69}$$

and the undetermined coefficients of solution of differential equations can be obtained as

$$\begin{cases} C_1=\theta_{\omega 0}+B_0/(GI_k), \\ C_2=M_{z0}/(GI_k), \\ C_3=\dfrac{1}{\kappa}\left(\theta'_{\omega 0}-\dfrac{M_{z0}}{GI_k}\right), \\ C_4=-B_0/(GI_k), \\ C_5=\theta_0-\theta_{\omega 0}-\mu\kappa^2 B_0/(GI_k). \end{cases} \tag{70}$$

Substituting Eq. (70) into Eqs. (60), (63), (64), and (66) yields the following forms:

$$\theta=\theta_0+\frac{\mu_1}{\kappa}\theta'_{\omega 0}\sinh\kappa z+\mu_1(1-\cosh\kappa z)\frac{B_0}{GI_k}\\ +\left(z-\frac{\mu_1}{\kappa}\sinh\kappa z\right)\frac{M_{z0}}{GI_k}, \tag{71}$$

$$\theta_\omega=\theta_{\omega 0}+\frac{1}{\kappa}\theta'_{\omega 0}\sinh\kappa z+(1-\cosh\kappa z)\frac{B_0}{GI_k}\\ +\left(z-\frac{1}{\kappa}\sinh\kappa z\right)\frac{M_{z0}}{GI_k}, \tag{72}$$

$$\theta'_\omega=\theta'_{\omega 0}\cosh\kappa z-\kappa\frac{B_0}{GI_k}\sinh\kappa z+(1-\cosh\kappa z)\frac{M_{z0}}{GI_k}, \tag{73}$$

$$\frac{B}{GI_k}=-\frac{1}{\kappa}\theta'_{\omega 0}\sinh\kappa z+\frac{B_0}{GI_k}\cosh\kappa z+\frac{1}{\kappa}\frac{M_{z0}}{GI_k}\sinh\kappa z. \tag{74}$$

In general cases of loading, the expression of initial method is shown in the Appendix.

5.3 Boundary conditions

The various boundary conditions can be determined as follows:

(1) Simply supported ends (where the cross section can be free to warp without rotation):

$$\theta=0,\quad B_\omega=0\ (\text{i.e., }\theta''_\omega=0). \tag{75}$$

(2) Clamped ends (where the cross section generates shear deformation without warping deformation and rotation):

$$\theta=0,\quad \theta'_\omega=0\ (\text{i.e., }w=-\theta'_\omega\omega=0). \tag{76}$$

(3) Free ends (where the cross section can be free to warp and rotate without bimoment):

$$\theta''_\omega=0\ (\text{i.e., }B_\omega=0),\\ M_z=\bar{M}_z\ (\text{if there is no end torque, }\bar{M}_z=0). \tag{77}$$

6 Simplified theory of restrained torsion

When the St. Venant constant is negligible, the simplified theory of restrained torsion is obtained. In this case, there is a total analogy between the simplified theory and the first-order flexure theory. The parameter $\lambda=1/(\kappa L)$ can be indirectly justified if the St. Venant constant is negligible. The method for flexure including the effect of shear deformation can be applied to the simplified theory.

The differential equations of restrained torsion of a thin-walled beam are simplified as

$$EI_\omega\theta_\omega^{\mathrm{IV}}=m, \tag{78}$$

$$\theta'_s=\frac{f_\omega M_\omega}{GI_{\rho s}}, \tag{79}$$

$$\theta=\theta_\omega+\theta_s. \tag{80}$$

The simplified theory of restrained torsion is applied to a cantilever beam of length L subjected to external distributed torque, for example. The solution of state parameter is given as follows:

$$\theta=\frac{m}{24EI_\omega}z^4-\frac{m}{6EI_\omega}Lz^3+\frac{m}{4EI_\omega}L^2z^2\\ -\frac{f_\omega m}{2GI_{\rho s}}z^2+\frac{f_\omega mL}{GI_{\rho s}}z, \tag{81}$$

$$\theta_\omega = \frac{m}{24EI_\omega}z^4 - \frac{m}{6EI_\omega}Lz^3 + \frac{m}{4EI_\omega}L^2z^2, \quad (82)$$

$$B = -\frac{m}{2}z^2 + mLz - \frac{m}{2}L^2, \quad (83)$$

$$M_\omega = M_z = -mz + mL. \quad (84)$$

The maximum rotation of a thin-walled beam is at the free end as follows:

$$\theta_{\max} = \frac{mL^4}{8EI_\omega}\left(1 + \frac{4f_\omega EI_\omega}{GI_{\rho s}L^2}\right). \quad (85)$$

The solution of the simplified theory of thin-walled beams is similar to the solution of the first-order flexure theory and the second term in brackets of Eq. (85) denotes the effect of restrained shear deformation on the total rotation.

7 Numerical examples

On the basis of the analytical method and the simplified approach developed in the previous sections, several problems of thin-walled beams with external torque applied on shear center are presented to illustrate the applicability and fidelity of the method.

7.1 Example 1

As shown in Fig. 6, a channel-shaped shear wall with a length of 18 m is fixed at the bottom end and subjected to an external torque M_z at the free end. The problem was studied by Back and Will (1998) and Kim and Kim (2005). The external torque is 1000 kN·m, and the elastic modulus E and shear modulus G are 30 and 13 kN/mm^2, respectively.

As the parameter λ is 1.77, the simplified theory is not applicable for this problem. A comparison between the results of current theory and those of other methods is listed in Table 1. Six solutions are provided by using Vlasov theory, Benscoter theory, Back and Will (1998)'s theory, Kim and Kim (2005)'s theory, the finite element method, and current theory.

A finite element package, ABAQUS, is used to create a finite element model of the shear wall model, as shown in Fig. 7. A concentrated torque is applied to the shear center of the wall at the free end. The element type is S4R, which is a four-node, quadrilateral, stress/displacement shell element with reduced integration. The finite element model is believed to give an accurate prediction because S4R is a general-purpose conventional shell element which allows transverse shear deformations.

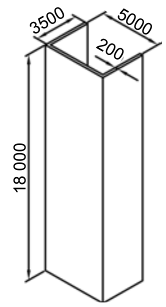

Fig. 6 A channel-shaped shear wall (unit: mm)

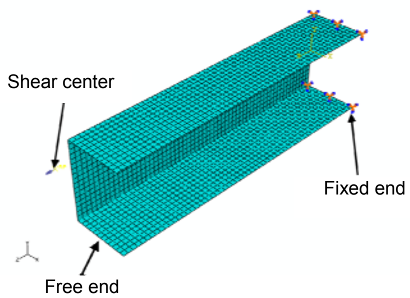

Fig. 7 Finite element model of the shear wall

Table 1 Rotations of cross sections at different locations*

Solution	Rotation of cross section ($\times10^{-3}$ rad)					
	3 m	6 m	9 m	12 m	15 m	18 m
Vlasov theory	0.1631	0.6111	1.284	2.124	3.075	4.081
Benscoter theory	0.1836	0.6505	1.341	2.198	3.165	4.186
Back and Will (1998)'s theory	0.1882	0.6595	1.354	2.215	3.185	4.210
Kim and Kim (2005)'s theory	0.1933	0.6692	1.368	2.233	3.207	4.236
Current theory	0.1941	0.6718	1.374	2.242	3.220	4.253
Finite element model	0.1987	0.6769	1.381	2.252	3.234	4.284

* The locations (3, 6, 9, 12, 15, and 18 m) denote the distance from cross sections to the fixed end

The predicted rotations of various cross sections of the shear wall are listed in Table 1. Due to ignorance of the effect of shear deformations, there exist large errors in the results of Vlasov theory. In comparison with the results of Vlasov theory, the percentage differences of other solutions are plotted in Fig. 8. It can be seen that the current theory gives the most accurate solution since its results are closer to the finite element model results than others. The percentage difference between Vlasov theory and the current theory is up to 9.93%. The difference becomes more significant near the fixed end. The Benscoter solution is not so accurate due to the ignorance of torsion shear coefficient.

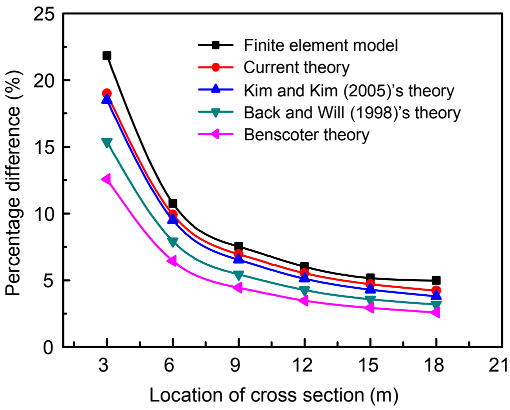

Fig. 8 Percentage differences with Vlasov theory at six different cross sections of a shear wall

7.2 Example 2

The influence of shear deformation on the behavior of a thin-walled I-beam is investigated in this example. Fig. 9a shows that the I-beam is fixed at both ends and a concentrated torque 2000 N·m is applied on the shear center at the midspan. The dimensions of the cross section of the beam are shown in Fig. 9b. The elastic modulus E and Poisson's ratio v of the beam material are 206 GPa and 0.3, respectively. The length of the beam is 1.5 m. A finite element model of the thin-walled beam with b=0.3 m is shown in Fig. 10. The variation of rotation along the longitudinal direction of the beam is shown in Fig. 11. It can be seen from Fig. 11 that the current predictions and the finite element results are approximately the same. It indicates that both Benscoer theory and Vlasov theory underestimate the beam rotation.

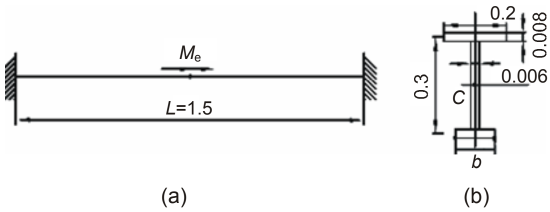

Fig. 9 I-beam subjected to a concentrated torque on the shear center at the midspan (unit: m)
(a) Thin-walled beam; (b) Cross section

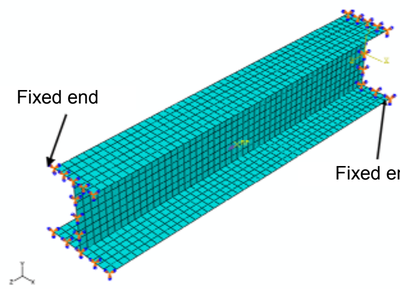

Fig. 10 Finite element model of the thin-walled I-beam with b=0.3 m

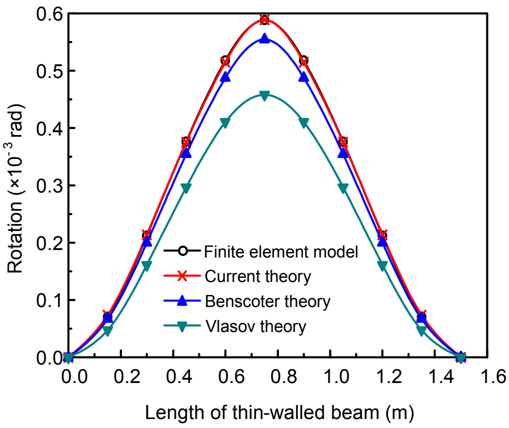

Fig. 11 Rotations along the longitudinal direction of the thin-walled beam

To study the effect of torsion shear deformation on the total rotation of open thin-walled beam, the variation of the percentage difference between Vlasov theory and other theories including shear deformation with respect to width of bottom flange are given in Fig. 12.

As shown in Fig. 12, difference between current theory and Benscoter theory increases as the width of bottom flange increases. It is interesting to note that due to ignorance of torsion shear coefficient and the utilization of total rotation in Benscoter theory, the difference between current theory and Benscoter theory can reach 7.5%, when the width is 0.3 m. In addition, the difference of variation tendency between two curves is obvious when the width of bottom flange increases, and it can be concluded that Benscoter theory will give more inaccurate results for wide flange beams.

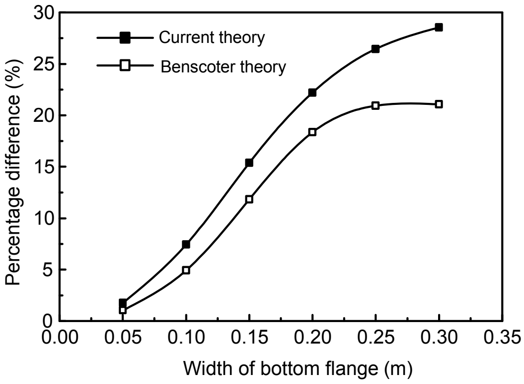

Fig. 12 Percentage differences of rotations at the middle location with Vlasov theory

7.3 Example 3

A channel-shaped cantilever beam subjected to distributed torque is shown in Fig. 13. The external distributed torque is 1000 N·m/m, and the elastic modulus E and Poisson's ratio v are 2.06×10^{11} N/m^2 and 0.3, respectively, and the length of the thin-walled beam is 1.2 m. The torsion shear coefficient is 1.4526. The dimensions of cross section are shown in Fig. 13.

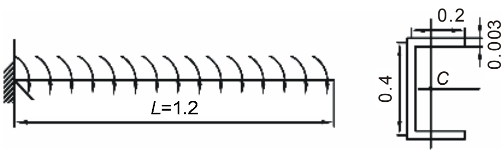

Fig. 13 Channel-shaped cantilever beam (unit: m)

In this example the parameter λ becomes 8.375. St. Venant constant is negligible, and the simplified theory can be used for this problem. The analytical results of the present methods at two cross sections are given in Table 2, compared with those of Vlasov theory.

It can be seen from Table 2 that the results of theories considering shear deformations are different from those of the classical Vlasov theory, and the effect of the shear deformation is very significant, especially at cross sections near the fixed end. The percentage difference of restrained shear rotation to that of Vlasov theory reaches 5.4% at the free end and 11.4% at the middle location. The solution of the simplified theory is sufficiently accurate to use, which is closer to the solution of the first-order torsion theory. The results excluding shear deformations underestimate the rotation of the thin-walled beam. It can also be concluded that the effect of the shear deformation is more significant for short beams than long beams. For this example, the classical Benscoter theory underestimates the total rotation. The percentage difference between current theory and Benscoter theory reaches 3.6% at middle location. The difference becomes more significant near the fixed end.

Table 2 Rotations of cross sections

Theory	Rotation at middle location ($\times10^{-3}$ rad)			Rotation at free end ($\times10^{-3}$ rad)		
	θ	θ_ω	θ_s	θ	θ_ω	θ_s
Vlasov theory	1.584			4.469		
Benscoter theory	1.708			4.634		
Current theory	1.765	1.584	0.181	4.709	4.469	0.240
Simplified theory	1.772	1.592	0.181	4.735	4.494	0.241

In order to determine the applicability of simplified theory, the parameter λ is studied based on this example. Fig. 14 shows the relationship of the parameter λ and the percentage difference of tip rotation between the first-order torsion theory and simplified theory.

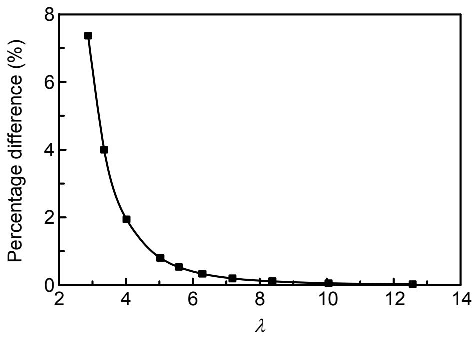

Fig. 14 Effects of parameter λ on the applicability of simplified theory

As shown in Fig. 14, as the parameter λ decreases, the difference between current theory and simplified theory becomes large, and the simplified theory is not applicable. As the parameter λ increases, the difference becomes small. When λ=5, the difference has been less than 1%, and the simplified theory provided a satisfactory approximation.

8 Conclusions

In this paper, the first-order torsion theory was developed for the restrained torsion of open thin-walled beams on the basis of Vlasov theory. The theory considered the effects of both the warping deformation and the restrained shear deformation of a cross section. The simplified theory of restrained torsion is presented.

In the first-order torsion theory, the effects of the restrained shear rotation on total rotation and St. Venant torque were studied. To consider the definite effect of shear deformations, the torsion shear coefficient is developed, which is similar to the flexure shear coefficient in the first-order flexure theory, i.e., Timoshenko beam theory. The initial parameter vector and influence matrix are given to solve the problem efficiently. Unlike the Vlasov and Benscoter theories, the total rotation, the free warping rotation, and the restrained rotation can be obtained in current theory. When the value of parameter λ is large enough, the St. Venant constant can be negligible and the simplified theory is applicable for the torsion problems of thin-walled beams. Comparison shows that the first-order theory provides more accurate results. The parameter λ is studied for the channel-shaped cantilever beam and the corresponding applicability of simplified theory is determined. When λ is not less than 5, the solution of simplified theory of a thin-walled beam is closer to the solution of the first-order torsion theory, and the numerical results validate the first-order torsion and simplified theories.

References

Back, S.Y., Will, K.M., 1998. A shear-flexible element with warping for thin-walled open beams. *International Journal for Numerical Methods in Engineering*, **43**(7): 1173-1191. [doi:10.1002/(SICI)1097-0207(19981215)43:7<1173::AID-NME340>3.0.CO;2-4]

Back, S.Y., Will, K.M., 2008. Shear-flexible thin-walled element for composite I-beams. *Engineering Structures*, **30**(5):1447-1458. [doi:10.1016/j.engstruct.2007.08.002]

Chen, B.Z., Hu, Y.R., 1998. Thin-Walled Structural Mechanics. Shanghai Jiao Tong University Press, Shanghai, China, p.51-67 (in Chinese).

El Fatmi, R.E., 2007a. Non-uniform warping including the effects of torsion and shear forces. Part I: a general beam theory. *International Journal of Solids and Structures*, **44**(18-19):5912-5929. [doi:10.1016/j.ijsolstr.2007.02.006]

El Fatmi, R.E., 2007b. Non-uniform warping including the effects of torsion and shear forces. Part II: analytical and numerical applications. *International Journal of Solids and Structures*, **44**(18-19):5930-5952. [doi:10.1016/j.ijsolstr.2007.02.005]

Erkmen, E.R., Mohareb, M., 2006. Torsion analysis of thin-walled beams including shear deformation effects. *Thin-Walled Structures*, **44**(10):1096-1108. [doi:10.1016/j.tws.2006.10.012]

Gjelsvik, A., 1981. The Theory of Thin-Walled Bars. John Wiley & Sons Inc., New York, p.1-10.

Kim, N.I., 2011. Shear deformable doubly- and monosymmetric composite I-beams. *International Journal of Mechanical Sciences*, **53**(1):31-41. [doi:10.1016/j.ijmecsci.2010.10.004]

Kim, N.I., Kim, M.Y., 2005. Exact dynamic/static stiffness matrices of non-symmetric thin-walled beams considering coupled shear deformation effects. *Thin-Walled Structures*, **43**(5):701-734. [doi:10.1016/j.tws.2005.01.004]

Lee, J., 2005. Flexural analysis of thin-walled composite beams using shear-deformable beam theory. *Composite Structures*, **70**(2):212-222. [doi:10.1016/j.compstruct.2004.08.023]

Librescu, L., Song, O., 2006. Thin-Walled Composite Beams: Theory and Application. Springer, Berlin, p.xxi-xxvi.

Mohareb, M., Nowzartash, F., 2003. Exact finite element for nonuniform torsion of open sections. *Journal of Structural Engineering*, **129**(2):215-223. [doi:10.1061/(ASCE)0733-9445(2003)129:2(215)]

Mokos, V.G., Sapountzakis, E.J., 2011. Secondary torsional moment deformation effect by BEM. *International Journal of Mechanical Sciences*, **53**(10):897-909. [doi:10.1016/j.ijmecsci.2011.08.001]

Park, S.W., Fujii, D., Fujitani, Y., 1997. A finite element analysis of discontinuous thin-walled beams considering nonuniform shear warping deformation. *Computers & Structures*, **65**(1):17-27. [doi:10.1016/S0045-7949(96)00325-2]

Pavazza, R., 2005. Torsion of thin-walled beams of open cross-section with influence of shear. *International Journal of Mechanical Sciences*, **47**(7):1099-1122. [doi:10.1016/j.ijmecsci.2005.02.007]

Reddy, J.N., Wang, C.M., Lee, K.H., 1997. Relationships between bending solutions of classical and shear deformation beam theories. *International Journal of Solids and Structures*, **34**(26):3373-3384. [doi:10.1016/S0020-7683(96)00211-9]

Roberts, T.M., Al-Ubaidi, H., 2001. Influence of shear deformation on restrained torsional warping of pultruded

FRP bars of open cross-section. *Thin-Walled Structures*, **39**(5):395-414. [doi:10.1016/S0263-8231(01)00009-X]

Saadé, K., Espion, B., Warzee, G., 2004. Non-uniform torsional behavior and stability of thin-walled elastic beams with arbitrary cross sections. *Thin-Walled Structures*, **42**(6):857-881. [doi:10.1016/j.tws.2003.12.003]

Shakourzadeh, H., Guo, Y.Q., Batoz, J.L., 1994. A torsion bending element for thin-walled beams with open and closed cross sections. *Computers & Structures*, **55**(6): 1045-1054. [doi:10.1016/0045-7949(94)00509-2]

Vlasov, V.Z., 1961. Thin-Walled Elastic Beams. Israel Program for Scientific Translations, Jerusalem, p.7-10.

Wang, C.M., 1995. Timoshenko beam-bending solutions in terms of Euler-Bernoulli solutions. *Journal of Engineering Mechanics*, **121**(6):763-765. [doi:10.1061/(ASCE)0733-9399(1995)121:6(763)]

Yang, Q.S., Wang, X.F., 2010. A geometrical and physical nonlinear finite element model for spatial thin-walled beams with arbitrary section. *Science China Technological Sciences*, **53**(3):829-838. [doi:10.1007/s11431-010-0065-9]

Appendix

In general cases of loading including external concentrated torque M_e, external distributed torque m, and external bimoment B_e, as shown in Fig. A1, Eqs. (71)–(74) of initial method is expressed in the following matrix form as shown in Eq. (A1).

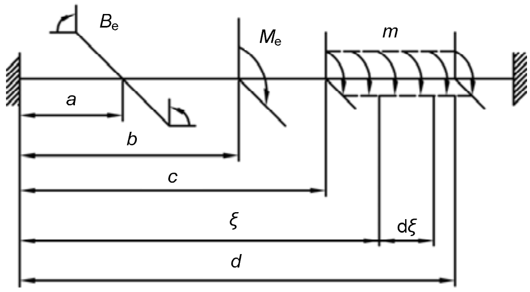

Fig. A1 General cases of loading

$$\begin{Bmatrix} \theta \\ \theta'_\omega \\ \dfrac{B}{GI_k} \\ \dfrac{M_z}{GI_k} \end{Bmatrix} = \begin{bmatrix} 1 & \dfrac{\mu_1}{\kappa}\sinh\kappa z & \mu_1(1-\cosh\kappa z) & z-\dfrac{\mu_1}{\kappa}\sinh\kappa z \\ 0 & \cosh\kappa z & -\kappa\sinh\kappa z & 1-\cosh\kappa z \\ 0 & -\dfrac{1}{\kappa}\sinh\kappa z & \cosh\kappa z & \dfrac{1}{\kappa}\sinh\kappa z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_0 \\ \theta'_{\omega 0} \\ \dfrac{B_0}{GI_k} \\ \dfrac{M_{z0}}{GI_k} \end{Bmatrix} + \begin{Bmatrix} \Big\|_a \dfrac{B_e}{GI_k}\mu_1[1-\cosh\kappa(z-a)] \\ -\Big\|_a \dfrac{B_e}{GI_k}\kappa\sinh\kappa(z-a) \\ \Big\|_a \dfrac{B_e}{GI_k}\cosh\kappa(z-a) \\ 0 \end{Bmatrix}$$

$$+\begin{Bmatrix} \Big\|_b \dfrac{M_e}{GI_k}\left[(z-b)-\dfrac{\mu_1}{\kappa}\sinh\kappa(z-b)\right] \\ \Big\|_b \dfrac{M_e}{GI_k}[1-\cosh\kappa(z-b)] \\ \Big\|_b \dfrac{M_e}{GI_k}\dfrac{1}{\kappa}\sinh\kappa(z-b) \\ \Big\|_b \dfrac{M_e}{GI_k} \end{Bmatrix} + \begin{Bmatrix} \Big\|_c \displaystyle\int_c^z \dfrac{m}{GI_k}\left[(z-\xi)-\dfrac{\mu_1}{\kappa}\sinh\kappa(z-\xi)\right]d\xi \\ \Big\|_c \displaystyle\int_c^z \dfrac{m}{GI_k}[1-\cosh\kappa(z-\xi)]d\xi \\ \Big\|_c \displaystyle\int_c^z \dfrac{m}{GI_k}\dfrac{1}{\kappa}\sinh\kappa(z-\xi)d\xi \\ \Big\|_c \displaystyle\int_c^z \dfrac{m}{GI_k}d\xi \end{Bmatrix}, \quad \text{(A1)}$$

where the symbol $\|_a$ denotes that the corresponding member should be counted when $z>a$, and the symbols $\|_b$ and $\|_c$ have the same case.