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Direct adaptive regulation of unknown nonlinear systems with analysis of the model order problem

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Abstract: A new method for the direct adaptive regulation of unknown nonlinear dynamical systems is proposed in this paper, paying special attention to the analysis of the model order problem. The method uses a neuro-fuzzy (NF) modeling of the unknown system, which combines fuzzy systems (FSs) with high order neural networks (HONNs). We propose the approximation of the unknown system by a special form of an NF-dynamical system (NFDS), which, however, may assume a smaller number of states than the original unknown model. The omission of states, referred to as a model order problem, is modeled by introducing a disturbance term in the approximating equations. The development is combined with a sensitivity analysis of the closed loop and provides a comprehensive and rigorous analysis of the stability properties. An adaptive modification method, termed 'parameter hopping', is incorporated into the weight estimation algorithm so that the existence and boundedness of the control signal are always assured. The applicability and potency of the method are tested by simulations on well known benchmarks such as 'DC motor' and 'Lorenz system', where it is shown that it performs quite well under a reduced model order assumption. Moreover, the proposed NF approach is shown to outperform simple recurrent high order neural networks (RHONNs).

Key words: Neuro-fuzzy systems, Direct adaptive regulation, Model order problems, Parameter hopping
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1 Introduction

It is common knowledge that neural networks and fuzzy inference systems are universal approximators (Hornik *et al.*, 1989; Wang, 1994; Passino and Yurkovich, 1998), in their ability to approximate nonlinear functions to any degree of accuracy if sufficient numbers of hidden neurons and fuzzy rules are available. The combination of these two different technologies has given rise to neuro-fuzzy (NF) approaches (Lin, 1995; Nounou and Passino, 2004; Li *et al.*, 2009; Theodoridis *et al.*, 2009a; 2009b), which are capable of capturing the advantages of both fuzzy logic and neural networks.

Many researchers have been active in the adaptive control area (Tong and Chai, 1999; Ordonez and Passino, 2001; Diao and Passino, 2002; Ge and Jing, 2002; Haddada *et al.*, 2003; Kim and Bien, 2004; Nounou and Passino, 2004; Yang, 2004; Ioannou and Fidan, 2006; Chemachema and Belarbi, 2007; Zhang *et al.*, 2007). In NF adaptive control, two main approaches are followed. First, we have the indirect adaptive control schemes separated into two steps: (1) the dynamics of the system are identi-

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fied and (2) a control input is generated according to the certainty equivalence principle. Secondly, we have the direct adaptive control schemes (Ordonez and Passino, 2001; Diao and Passino, 2002; Ioannou and Fidan, 2006) where the controller is directly estimated and the control input is generated to guarantee stability without knowledge of the system dynamics. Furthermore, hybrid schemes developed using NF approaches solve difficult problems, which require relatively short development time and are robust (Tong and Chai, 1999; Ordonez and Passino, 2001; Diao and Passino, 2002; Ge and Jing, 2002; Haddada *et al.*, 2003; Kim and Bien, 2004; Yang, 2004; Ioannou and Fidan, 2006; Chemachema and Belarbi, 2007; Zhang *et al.*, 2007).

Recently, fuzzy-recurrent high order neural networks (F-RHONNs) have been proposed for the identification of nonlinear dynamical systems of the form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}$, approximated by a recurrent model using two independent fuzzy sub-systems (Theodoridis *et al.*, 2009b). This approximation depends on the fact that fuzzy rules could be identified with the help of HONNs in conjunction with the centers determined from the output fuzzy partitioning. The same approach has been followed in Boutalis *et al.* (2009) and Theodoridis *et al.* (2009a) where an NF approach for the indirect (Boutalis *et al.*, 2009) and direct (Theodoridis *et al.*, 2009a) control of square unknown systems has been introduced, assuming only parameter uncertainty.

In this paper, an HONN-based NF controller is proposed and used for the direct control of nonlinear dynamical systems under the presence of model order problems. The proposed NF approximator may assume a smaller number of states than the original unknown model. The omission of states, referred to as a model order problem, is modeled by introducing in the approximating equations a disturbance term which depends on the unknown (omitted) states. The development is combined with a sensitivity analysis of the closed loop and provides a thorough analysis of the stability properties. In the proposed approach, the underlying fuzzy model is of Mamdani type (Mamdani, 1976).

The nonlinear system is considered to be initially approximated using two independent fuzzy subsystems. Every fuzzy subsystem is in the sequel approximated from a group of HONNs, each one being associated with a family of fuzzy rules. Next, weight updating laws are derived, and it is proved that when a proper structural identification has been performed and the unmodeled dynamics are within a certain region, depending on the measured states or some constant values, then the error goes to zero very fast. Also, asymptotic regulation of the states is accomplished by constructing suitable state feedback, keeping at the same time all signals in the closed loop bounded. A technique known as hopping (Boutalis *et al.*, 2009; Theodoridis *et al.*, 2009a) is incorporated into the weight updating laws in order to guarantee the existence and boundedness of the control signal.

2 Preliminaries and neuro-fuzzy representation

2.1 Preliminaries

Consider a nonlinear function $f(\boldsymbol{x}) \in \mathbb{R}^n$, $\boldsymbol{x} \in X \subset \mathbb{R}^n$ approximately described by a Mamdani-type fuzzy system (FS). Let $\Omega^{l_1,l_2,\ldots,l_n}_{j_1,j_2,\ldots,j_n}$ be defined as the subset of $\boldsymbol{x} \in X$ belonging to the $(j_1, j_2, ..., j_n)$th input fuzzy patch and pointing—through the vector field $f(\cdot)$—to the subset which belongs to the $(l_1, l_2, ..., l_n)$th output fuzzy patch. In other words, $\Omega^{l_1,l_2,\ldots,l_n}_{j_1,j_2,\ldots,j_n}$ contains input values $\boldsymbol{x}$ that are associated through a fuzzy rule with output values $f(\boldsymbol{x})$.

Furthermore, the FS receiving as input the n-tuple of $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ gives as output an approximate of $f(\boldsymbol{x})$ using fuzzy rules and a well known fuzzy inference procedure.

According to the above notation, the rule firing indicator function (RFIF) or simply the indicator function (IF) connected to $\Omega^{l_1,l_2,\ldots,l_n}_{j_1,j_2,\ldots,j_n}$ is defined as

$$I^{l_1,l_2,\ldots,l_n}_{j_1,j_2,\ldots,j_n}(\boldsymbol{x}(t)) = \begin{cases} \alpha(\boldsymbol{x}(t)), & \text{if } \boldsymbol{x}(t) \in \Omega^{l_1,l_2,\ldots,l_n}_{j_1,j_2,\ldots,j_n}, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $\alpha(\boldsymbol{x}(t))$ denotes the firing strength of the rule. According to standard fuzzy system description, this strength depends on the membership value of each x_i in the corresponding input membership functions μ_{Fj_i}, and more specifically (Wang, 1994), $\alpha(\boldsymbol{x}(t)) = \min(\mu_{Fj_1}(x_1(k)), \mu_{Fj_2}(x_2(k)), ..., \mu_{Fj_n}(x_n(k)))$.

Then assuming a standard defuzzification procedure (e.g., weighted average), the functional representation of the fuzzy system can be written as

$$f(\boldsymbol{x}(t)) = \sum (\bar{\boldsymbol{x}}_f)^{l_1,l_2,\ldots,l_n}_{j_1,j_2,\ldots,j_n} \cdot (I')^{l_1,l_2,\ldots,l_n}_{j_1,j_2,\ldots,j_n}(\boldsymbol{x}(t)), \tag{2}$$

where the summation is carried out over all the available fuzzy rules. $(\bar{\boldsymbol{x}}_f)^{l_1,l_2,...,l_n}_{j_1,j_2,...,j_n}$ is any constant vector consisting of the centers of fuzzy partitions of f determined by $l_1, l_2, ..., l_n$, and $(I')^{l_1,l_2,...,l_n}_{j_1,j_2,...,j_n}(\boldsymbol{x}(t))$ is the IF defined in Eq. (1) divided by the sum of all IF participating in the summation of Eq. (2).

Based on Christodoulou *et al.* (2007), Eq. (2) can be rewritten as

$$f(\boldsymbol{x}(t)) = \sum (\bar{\boldsymbol{x}}_f)^{l_1,l_2,...,l_n}_{j_1,j_2,...,j_n} \cdot (N_f)^{l_1,l_2,...,l_n}_{j_1,j_2,...,j_n}(\boldsymbol{x}(t)), \tag{3}$$

where $(N_f)^{l_1,l_2,...,l_n}_{j_1,j_2,...,j_n}(\boldsymbol{x}(t))$ are the appropriately selected HONNs. Details of a HONN definition can be found in Boutalis *et al.* (2009) and Theodoridis *et al.* (2009b). Taking also into account the form of a HONN (Boutalis *et al.*, 2009; Theodoridis *et al.*, 2009b), Eq. (3) can be written as

$$f(\boldsymbol{x}(t)) = \sum_{p=1}^{m} \bar{\boldsymbol{x}}_{f_p} \cdot \left(\sum_{\text{hot}=1}^{k} w_{p,\text{hot}} \cdot s_{\text{hot}}(\boldsymbol{x}(t)) \right), \tag{4}$$

where m is the number of fuzzy output partitions.

From the above definitions and Eq. (4), it is obvious that the accuracy of the approximation of $f(\boldsymbol{x})$ depends on the approximation abilities of HONN's and on an initial estimate of the centers of the output membership functions. These centers can be obtained by experts or by off-line techniques based on gathered data. Any other information related to the input membership functions is not necessary because it is replaced by the HONNs.

2.2 Neuro-fuzzy modeling

We consider affine in the control, nonlinear dynamical system with the following form:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x}) \cdot \boldsymbol{u} + \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\text{ud}}), \tag{5}$$

$$\dot{\boldsymbol{x}}_{\text{ud}} = B(\boldsymbol{x}, \boldsymbol{x}_{\text{ud}}), \tag{6}$$

where the true plant is of order $N \geq n$, the control $\boldsymbol{u} \in \mathbb{R}^n$, f is an unknown smooth vector field called the drift term, and $\boldsymbol{G}$ is a diagonal matrix with elements of the unknown smooth controlled vector field, $\boldsymbol{g}_i$, $i = 1, 2, ..., n$, $\boldsymbol{G} = \text{diag}\{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$. Also, $\boldsymbol{x}_{\text{ud}} \in \mathbb{R}^p$ is the state of the unmodeled dynamics and $\boldsymbol{\phi}(\cdot)$, $B(\cdot)$ are unknown vector fields of $\boldsymbol{x}$ and $\boldsymbol{x}_{\text{ud}}$. Obviously, $p = N - n$.

Following the approach presented in Boutalis *et al.* (2009) and Theodoridis *et al.* (2009b) and using Eq. (4) for approximating the unknown vector fields, the unknown system in Eq. (5) is approximated by an NF representation assuming the following equivalent affine recurrent HONN, which is termed F-RHONN and depends on the centers of the fuzzy output partitions $\bar{\boldsymbol{x}}_f$ and $\bar{\boldsymbol{x}}_g$:

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}_i &= -a_i \hat{\boldsymbol{x}}_i + \sum_{l=1}^{N_p f_i} \bar{\boldsymbol{x}}_{f_i} \cdot N^l_{f_i}(\boldsymbol{x}) \\ &\quad + \left(\sum_{l=1}^{N_p g_i} \bar{\boldsymbol{x}}_{\boldsymbol{g}_i} \cdot N^l_{\boldsymbol{g}_i}(\boldsymbol{x}) \right) \boldsymbol{u}_i, \end{aligned} \tag{7}$$

where $a_i > 0$ and $N_p\, f_i$, $N_p\, g_i$ are the numbers of the corresponding fuzzy output partitions; or in a more compact form,

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{X}_f\, \boldsymbol{W}_f\, \boldsymbol{s}_f(\boldsymbol{x}) + \boldsymbol{X}_g\, \boldsymbol{W}_g \boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u}, \tag{8}$$

where $\boldsymbol{A}$ is an $n \times n$ stable matrix, $\boldsymbol{X}_f$ and $\boldsymbol{X}_g$ are matrices containing the centers of the membership functions of every fuzzy output variable of $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$, respectively, $\boldsymbol{s}_f(\boldsymbol{x})$ is a column vector, $\boldsymbol{S}_g(\boldsymbol{x})$ is a matrix containing high order combinations of sigmoid functions of the state $\boldsymbol{x}$. For simplicity, $\boldsymbol{A}$ can be taken to be diagonal as $\boldsymbol{A} = \text{diag}\{-a_1, -a_2, ..., -a_n\}$. Finally, $\boldsymbol{W}_f$, $\boldsymbol{W}_g$ are matrices containing respective neural weights according to Eq. (7). The dimensions and the contents of all the above matrices or vectors are chosen so that $\boldsymbol{X}_f\, \boldsymbol{W}_f\, \boldsymbol{s}_f(\boldsymbol{x})$ is an $n \times 1$ vector and $\boldsymbol{X}_g\, \boldsymbol{W}_g \boldsymbol{S}_g(\boldsymbol{x})$ is an $n \times n$ matrix. Details about the vectors and matrix dimensions can be found in Theodoridis *et al.* (2009a).

Motivated from the above analysis, it is easy to extend our algorithm to systems with non-square form, with the number of inputs being different from the number of states.

3 Direct adaptive neuro-fuzzy control for systems with model order problems

3.1 Problem formulation

State regulation aims at driving system states to zero from arbitrary initial values. To this end, appropriate feedback control has to be designed and applied to the plant input. However, since the plant is considered unknown and there are dynamics that have not been taken into account, we may take advantage of the proposed neuro-fuzzy approximator and its approximation capabilities. With no loss of generality, we assume that the unknown plant (5), (6) can be completely described by the proposed NF approximation plus an unmodeled dynamics term $\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\rm ud})$. This is equivalent to assuming that, there exist weight values $\boldsymbol{W}_f^*$ and $\boldsymbol{W}_g^*$ such that the system Eq. (5), Eq. (6) can be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{X}_f \boldsymbol{W}_f^* \boldsymbol{s}_f(\boldsymbol{x}) + \boldsymbol{X}_g \boldsymbol{W}_g^* \boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\rm ud}), \tag{9}$$

$$\dot{\boldsymbol{x}}_{\rm ud} = \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{x}_{\rm ud}). \tag{10}$$

Under this assumption, the state regulation can be analyzed and designed for the system (9), (10) instead of (5), (6). But $\boldsymbol{W}_f^*$ and $\boldsymbol{W}_g^*$ are unknown. Therefore, our solution embodies the design of a control law $\boldsymbol{u}(\boldsymbol{W}_f, \boldsymbol{W}_g, \boldsymbol{x})$ and updating laws for $\boldsymbol{W}_f$ and $\boldsymbol{W}_g$, which guarantee the driving of the states to zero and the boundedness of $\boldsymbol{x}$ and of all signals in the closed loop.

Before proceeding further, in order to guarantee the existence and uniqueness of solution of Eq. (5) for any finite initial condition and $\boldsymbol{u} \in U$, the following mild assumptions are imposed:

Assumption 1 The states $\boldsymbol{x}_{\rm ud} \in \mathbb{R}^p$ of the unmodeled dynamics with a finite initial condition are uniformly bounded for any finite time $t > 0$.

Assumption 2 Given a class $U \subset \mathbb{R}^n$ of admissible inputs, for any $\boldsymbol{u} \in U$ and any finite initial condition, the state trajectories are uniformly bounded for any finite $t > 0$, meaning that we do not allow systems processing trajectories which escape at infinite, in finite time t, t being arbitrarily small. That is, $|\boldsymbol{x}(t)| < \infty$.

Assumption 3 The vector fields f and $\boldsymbol{g}_i$ ($i = 1, 2, ..., n$) are continuous with respect to their arguments and satisfy a local Lipchitz condition so that the solution $\boldsymbol{x}(t)$ of Eq. (5) is unique for any finite initial condition and $\boldsymbol{u} \in U$.

3.2 Adaptive regulation with model order problems

In this subsection a solution to the adaptive regulation problem is presented and the model order problem is investigated. Assuming the presence of a modeling error, the unknown system can be written as Eq. (9). In the sequel, we define $\boldsymbol{\nu}$ as

$$\boldsymbol{\nu} \triangleq \boldsymbol{X}_f \boldsymbol{W}_f \boldsymbol{s}_f(\boldsymbol{x}) + \boldsymbol{X}_g \boldsymbol{W}_g \boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u} - \dot{\boldsymbol{x}} + \boldsymbol{A}\boldsymbol{x}. \tag{11}$$

Substituting Eq. (9) into Eq. (11) we have

$$\boldsymbol{\nu} \triangleq \boldsymbol{X}_f \tilde{\boldsymbol{W}}_f \boldsymbol{s}_f(\boldsymbol{x}) + \boldsymbol{X}_g \tilde{\boldsymbol{W}}_g \boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u} - \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\rm ud}), \tag{12}$$

where $\tilde{\boldsymbol{W}}_f = \boldsymbol{W}_f - \boldsymbol{W}_f^*$ and $\tilde{\boldsymbol{W}}_g = \boldsymbol{W}_g - \boldsymbol{W}_g^*$. $\boldsymbol{W}_f$ and $\boldsymbol{W}_g$ are estimates of $\boldsymbol{W}_f^*$ and $\boldsymbol{W}_g^*$, respectively, and can be determined by weight updating laws, which will be derived. $\boldsymbol{\nu}$ cannot be measured since $\dot{\boldsymbol{x}}$ is unknown. Alternatively, we may use the following filtered version of $\boldsymbol{\nu}$:

$$\boldsymbol{\nu} = \dot{\boldsymbol{\xi}} + \boldsymbol{K}\boldsymbol{\xi},$$

where $\boldsymbol{K} = {\rm diag}\{k_1, k_2, ..., k_n\}$ is a positive definite matrix and $\boldsymbol{\xi} \in \mathbb{R}^n$. According to Eq. (11) we have

$$\dot{\boldsymbol{\xi}} + \boldsymbol{K}\boldsymbol{\xi} = -\dot{\boldsymbol{x}} + \boldsymbol{A}\boldsymbol{x} + \boldsymbol{X}_f \boldsymbol{W}_f \boldsymbol{s}_f(\boldsymbol{x}) + \boldsymbol{X}_g \boldsymbol{W}_g \boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u}, \tag{13}$$

and after substituting Eq. (9) we have

$$\dot{\boldsymbol{\xi}} = -\boldsymbol{K}\boldsymbol{\xi} + \boldsymbol{X}_f \tilde{\boldsymbol{W}}_f \boldsymbol{s}_f(\boldsymbol{x}) + \boldsymbol{X}_g \tilde{\boldsymbol{W}}_g \boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u} - \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\rm ud}). \tag{14}$$

To implement Eq. (14), we take

$$\boldsymbol{\xi} \triangleq \boldsymbol{\zeta} - \boldsymbol{x}. \tag{15}$$

Employing Eq. (15), Eq. (13) can be written as

$$\dot{\boldsymbol{\zeta}} + \boldsymbol{K}\boldsymbol{\zeta} = \boldsymbol{K}\boldsymbol{x} + \boldsymbol{A}\boldsymbol{x} + \boldsymbol{X}_f \boldsymbol{W}_f \boldsymbol{s}_f(\boldsymbol{x}) + \boldsymbol{X}_g \boldsymbol{W}_g \boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u}, \tag{16}$$

with state $\boldsymbol{\zeta} \in \mathbb{R}^n$. This approach is known as error filtering.

The regulation of the system can be achieved by selecting the control input:

$$\boldsymbol{u} = -\left[\boldsymbol{X}_g \boldsymbol{W}_g \boldsymbol{S}_g(\boldsymbol{x})\right]^+ \left[\boldsymbol{X}_f \boldsymbol{W}_f \boldsymbol{s}_f(\boldsymbol{x}) + \boldsymbol{v}\right], \tag{17}$$

where $[\cdot]^+$ means pseudoinverse in the Moore-Penrose sense, and

$$\boldsymbol{v} = (\boldsymbol{K} + \boldsymbol{A})\boldsymbol{x}. \tag{18}$$

To proceed, the following lemma has to be stated:

Lemma 1 The control law (17), (18) guarantees the following properties:

(1) $\boldsymbol{\zeta}(t) \leq \boldsymbol{0}, \quad \forall t \geq 0$;

(2) $\lim_{t\to\infty} \boldsymbol{\zeta}(t) = \boldsymbol{0}$ exponentially fast provided that $\boldsymbol{\zeta}(t) < \boldsymbol{0}$.

Proof Using the control laws (17), (18) and substituting in Eq. (16), it becomes

$$\dot{\boldsymbol{\zeta}} = -\boldsymbol{K}\boldsymbol{\zeta}, \quad \forall t \geq 0,$$

which is a homogeneous differential equation with solution

$$\boldsymbol{\zeta}(t) = \boldsymbol{\zeta}(0)\mathrm{e}^{-\boldsymbol{K}t}.$$

Therefore, if $\boldsymbol{\zeta}(0)$, the initial value of $\boldsymbol{\zeta}(t)$, is chosen negative, we obtain

$$\boldsymbol{\zeta}(t) \leq \boldsymbol{0}, \quad \forall t \geq 0.$$

Moreover, $\boldsymbol{\zeta}(t)$ converges to zero exponentially fast.

To proceed, consider the following candidate Lyapunov function:

$$\begin{aligned} V \;=\; & \frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi} + \frac{1}{2}\boldsymbol{\zeta}^{\mathrm{T}}\boldsymbol{\zeta} + L\left(\boldsymbol{x}_{\mathrm{ud}}\right) \\ & + \frac{1}{2}\mathrm{tr}\left\{ \tilde{\boldsymbol{W}}_f^{\mathrm{T}} \boldsymbol{D}_f^{-1} \tilde{\boldsymbol{W}}_f \right\} + \frac{1}{2}\mathrm{tr}\left\{ \tilde{\boldsymbol{W}}_g^{\mathrm{T}} \boldsymbol{D}_g^{-1} \tilde{\boldsymbol{W}}_g \right\}, \end{aligned} \tag{19}$$

where $\boldsymbol{D}_f$ and $\boldsymbol{D}_g$ are positive definite diagonal gain matrices. Taking the derivative of Eq. (19) in respect to time, we obtain

$$\begin{aligned} \dot{V} \;=\; & -\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{\xi} - \boldsymbol{\zeta}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{\zeta} + \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{X}_f\,\tilde{\boldsymbol{W}}_f\,\boldsymbol{s}_f(\boldsymbol{x}) \\ & + \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{X}_g\,\tilde{\boldsymbol{W}}_g\boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u} - \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\phi}\left(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}}\right) + \dot{L}\left(\boldsymbol{x}_{\mathrm{ud}}\right) \\ & + \mathrm{tr}\left\{ \dot{\boldsymbol{W}}_f^{\mathrm{T}} \boldsymbol{D}_f{}^{-1}\,\tilde{\boldsymbol{W}}_f \right\} + \mathrm{tr}\left\{ \dot{\boldsymbol{W}}_g{}^{\mathrm{T}} \boldsymbol{D}_g{}^{-1}\,\tilde{\boldsymbol{W}}_g \right\}. \end{aligned}$$

Hence, if we choose

$$\mathrm{tr}\left\{ \dot{\boldsymbol{W}}_f^{\mathrm{T}} \boldsymbol{D}_f{}^{-1}\,\tilde{\boldsymbol{W}}_f \right\} = -\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{X}_f\,\tilde{\boldsymbol{W}}_f\,\boldsymbol{s}_f(\boldsymbol{x}), \tag{20}$$

$$\mathrm{tr}\left\{ \dot{\boldsymbol{W}}_g{}^{\mathrm{T}} \boldsymbol{D}_g{}^{-1}\,\tilde{\boldsymbol{W}}_g \right\} = -\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{X}_g\,\tilde{\boldsymbol{W}}_g\boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u}, \tag{21}$$

$\dot{V}$ becomes

$$\begin{aligned} \dot{V} \;\leq\; & -\lambda_{\min}\left(\boldsymbol{K}\right)\left\|\boldsymbol{\xi}\right\|^2 - \lambda_{\min}\left(\boldsymbol{K}\right)\left\|\boldsymbol{\zeta}\right\|^2 \\ & -\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\phi}\left(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}}\right) + \dot{L}\left(\boldsymbol{x}_{\mathrm{ud}}\right). \end{aligned} \tag{22}$$

It is easy to verify that Eqs. (20) and (21), after making appropriate straightforward operations, can be element-wise written as:

(a) for the elements of $\boldsymbol{W}_f$,

$$\dot{w}_{f_i}^{pl} = -\bar{\boldsymbol{x}}_{f_i}^{p}\,\boldsymbol{\xi}_i\boldsymbol{s}_l(\boldsymbol{x})d_{f_l}, \tag{23}$$

or equivalently,

$$\dot{W}_{f_i}^{l} = -\left(\bar{\boldsymbol{x}}_{f_i}\right)^{\mathrm{T}}\boldsymbol{\xi}_i\boldsymbol{s}_l(\boldsymbol{x})d_{f_l}, \tag{24}$$

for all $i = 1, 2, ..., n$, $p = 1, 2, ..., m$, and $l = 1, 2, ..., k$.

(b) for the elements of $\boldsymbol{W}_g$

$$\dot{w}_{g_i}^{p} = -\bar{\boldsymbol{x}}_{g_i}^{p}\boldsymbol{\xi}_i u_i\boldsymbol{s}_i(\boldsymbol{x})d_{g_i}, \tag{25}$$

or equivalently,

$$\dot{W}_{g_i} = -\left(\bar{\boldsymbol{x}}_{g_i}\right)^{\mathrm{T}}\boldsymbol{\xi}_i u_i\boldsymbol{s}_i(\boldsymbol{x})d_{g_i}, \tag{26}$$

for all $i = 1, 2, ..., n$ and $p = 1, 2, ..., m$.

In a compact form, Eqs. (23) and (25) can be written as

$$\dot{\boldsymbol{W}}_f = -\boldsymbol{X}_f{}^{\mathrm{T}}\boldsymbol{\xi}\boldsymbol{s}_f^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{D}_f, \tag{27}$$

$$\dot{\boldsymbol{W}}_g = -\boldsymbol{X}_g{}^{\mathrm{T}}\boldsymbol{\xi}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{S}_g^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{D}_g. \tag{28}$$

Furthermore, we cannot conclude anything about the weight convergence if the existence and boundedness of signal $\boldsymbol{u}$ is not assured. To this end, the weight updating laws (23), (25) have to be modified by introducing a method of parameter 'Hopping' (Boutalis *et al.*, 2009; Theodoridis *et al.*, 2009a), as explained below.

The weight updating laws presented previously in Section 3.2 are valid when the control law signal in Eqs. (17) and (18) exists. Therefore, the existence of $[\boldsymbol{X}_g\,\boldsymbol{W}_g\boldsymbol{S}_g(\boldsymbol{x})]^+$ has to be assured. Since the matrix of $\boldsymbol{S}_g(\boldsymbol{x})$ is diagonal with the diagonal elements $\boldsymbol{s}_i(\boldsymbol{x}) \neq 0$ and $\boldsymbol{X}_g$, $\boldsymbol{W}_g$ are block diagonal, the existence of the pseudoinverse is assured when $\bar{\boldsymbol{x}}_{g_i} \cdot \boldsymbol{W}_{g_i} \neq 0$, $\forall\, i = 1, 2, ..., n$. Therefore, $\boldsymbol{W}_{g_i}$ has to be confined such that $|\bar{\boldsymbol{x}}_{g_i} \cdot \boldsymbol{W}_{g_i}| \geq \theta_i > 0$, with θ_i being a small positive design parameter (usually in the range of $[0.001, 0.01]$). In case the boundary defined by the above confinement is nonlinear, the updating $\boldsymbol{W}_{g_i}$ can be modified by using a projection algorithm (Ioannou and Fidan, 2006). This procedure has been extensively explained in Boutalis *et al.* (2009) and Theodoridis *et al.* (2009a) and is depicted in Fig. 1 in a simplified 2D representation. This hopping in the weight updating law of $\boldsymbol{W}_{g_i}$ can be expressed as

$$\dot{\boldsymbol{W}}_{g_i} = \begin{cases} -(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}\boldsymbol{\xi}_i u_i s_i(\boldsymbol{x})d_{g_i}, \text{ if } |\bar{\boldsymbol{x}}_{g_i} \cdot \boldsymbol{W}_{g_i}| > \theta_i \\ \text{or } |\bar{\boldsymbol{x}}_{g_i} \cdot \boldsymbol{W}_{g_i}| = \pm\theta_i \text{ and } \bar{\boldsymbol{x}}_{g_i} \cdot \dot{\boldsymbol{W}}_{g_i} <> 0, \\ -(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}\boldsymbol{\xi}_i u_i s_i(\boldsymbol{x})d_{g_i} - \dfrac{2\bar{\boldsymbol{x}}_{g_i}\boldsymbol{W}_{g_i}(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}}{\mathrm{tr}\{(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}\bar{\boldsymbol{x}}_{g_i}\}}, \\ \text{otherwise,} \end{cases}$$

where <> assumes a one-to-one correspondence with $\pm$, meaning that $<$ corresponds to $+$ and $>$ corresponds to $-$. This updating law assures the existence of the control signal.

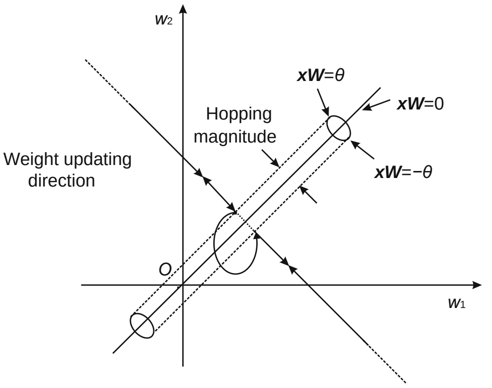

Fig. 1 Pictorial representation of parameter hopping

Including hopping in the weights updating law will prevent the control signal from going to infinity. Apart from that, it also has to be assured that $\boldsymbol{X}_g\boldsymbol{W}_g\boldsymbol{S}_g(\boldsymbol{x})$ does not achieve even temporarily very large values. Otherwise, the computation of $\boldsymbol{u}$ may become numerically unstable and drive the control signal to zero, failing to control the system. To avoid this situation it has to be assured that $|\bar{\boldsymbol{x}}_{g_i}\cdot\boldsymbol{W}_{g_i}|<\rho_i$, where ρ_i is a design parameter determining an external limit for $\bar{\boldsymbol{x}}_{g_i}\cdot\boldsymbol{W}_{g_i}$. Following the same rationale as the case of weight hopping introduced above, we could now consider two forbidden hyperplanes, which are defined by the equation $|\bar{\boldsymbol{x}}_{g_i}\cdot\boldsymbol{W}_{g_i}|=\rho_i$. When the weight vector reaches one of the forbidden hyperplanes $\bar{\boldsymbol{x}}_{g_i}\cdot\boldsymbol{W}_{g_i}=\rho_i$ with the direction of updating pointing toward it, a new modified hopping is introduced which pushes the weights back, inside the restricting area. A simplified 2D sketch of the procedure is given in Fig. 2.

The size of hopping is given by $-\kappa_g\left(\bar{\boldsymbol{x}}_{g_i}\boldsymbol{W}_{g_i}(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}\right)/\mathrm{tr}\{(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}\bar{\boldsymbol{x}}_{g_i}\}$ and is determined by following the vectorial proof of Theodoridis *et al.* (2009a), with κ_g being a small positive number decided appropriately from the designer. By performing hopping when $\bar{\boldsymbol{x}}_{g_i}\cdot\boldsymbol{W}_{g_i}$ reaches either the inner or outer forbidden planes, $\bar{\boldsymbol{x}}_{g_i}\cdot\boldsymbol{W}_{g_i}$ is confined to lie in space $P=\{\bar{\boldsymbol{x}}_{g_i}\cdot\boldsymbol{W}_{g_i}:|\bar{\boldsymbol{x}}_{g_i}\cdot\boldsymbol{W}_{g_i}|\le\rho_i\ ,\ |\bar{\boldsymbol{x}}_{g_i}\cdot\boldsymbol{W}_{g_i}|>\theta_i\}$ lying between these hyperplanes. The weight updating law for $\boldsymbol{W}_{g_i}$, which embodies both hopping conditions, can now be expressed as

$$\dot{\boldsymbol{W}}_{g_i}=-(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}\boldsymbol{\xi}_i u_i s_i(\boldsymbol{x})d_{g_i}-\frac{2\sigma_i\left(\bar{\boldsymbol{x}}_{g_i}\boldsymbol{W}_{g_i}(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}\right)}{\mathrm{tr}\{(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}\bar{\boldsymbol{x}}_{g_i}\}}-\frac{2(1-\sigma_i)\kappa_g\left(\bar{\boldsymbol{x}}_{g_i}\boldsymbol{W}_{g_i}(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}\right)}{\mathrm{tr}\{(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}\bar{\boldsymbol{x}}_{g_i}\}},\qquad(29)$$

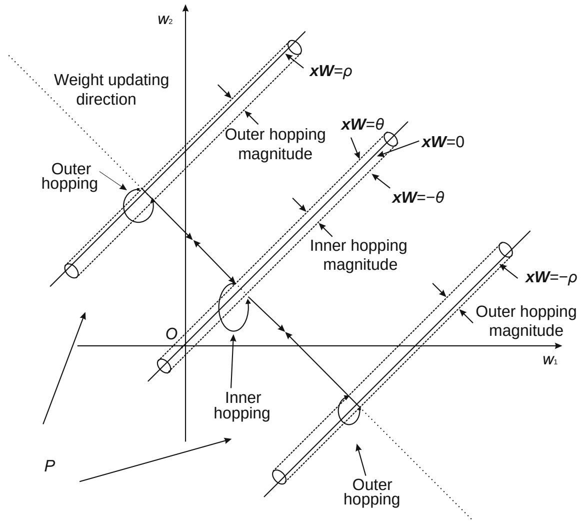

Fig. 2 Two-dimensional sketch of both inner and outer parameter hopping

where

$$\sigma_i=\begin{cases}0, & \text{if } \boldsymbol{x}_{g_i}\cdot\boldsymbol{W}_{g_i}=\pm\rho_i \\ & \text{and } \bar{\boldsymbol{x}}_{g_i}\cdot\dot{\boldsymbol{W}}_{g_i}<>0,\\ 1, & \text{otherwise.}\end{cases}\qquad(30)$$

We may now distinguish two possible cases: the uniform asymptotic stability in the large case and the violation of the uniform asymptotic stability in the large condition.

3.2.1 Uniform asymptotic stability in the large case

For completeness, we introduce from Michel and Miller (1977) the following definitions that are crucial to our discussion:

Definition 1 (Rovithakis and Christodoulou, 2000) The equilibrium point $\boldsymbol{x}_{\mathrm{ud}}=\mathbf{0}$ is said to be uniformly asymptotically stable in the large case if:

(1) for every $M>0$ and any $t_0\in\mathbb{R}^+$, there exists $\bar{\alpha}(M)>0$ such that $|\boldsymbol{x}_{\mathrm{ud}}(t;\boldsymbol{x}_{\mathrm{ud}}(0),t_0)|<M$ for all $t\ge t_0$ whenever $|\boldsymbol{x}_{\mathrm{ud}}(0)|<\bar{\alpha}(M)$;

(2) for every $\bar{\alpha}>0$ and any $t_0\in\mathbb{R}^+$, there exists $M(\bar{\alpha})>0$ such that $|\boldsymbol{x}_{\mathrm{ud}}(t;\boldsymbol{x}_{\mathrm{ud}}(0),t_0)|<M(\bar{\alpha})$ for all $t\ge t_0$ whenever $|\boldsymbol{x}_{\mathrm{ud}}(0)|<\bar{\alpha}$;

(3) for any $\bar{\alpha}$, any $M>0$, and any $t_0\in\mathbb{R}^+$, there exists $T(M,\bar{\alpha})>0$, independent of t_0 such that, if $|\boldsymbol{x}_{\mathrm{ud}}(0)|<\bar{\alpha}$ then $|\boldsymbol{x}_{\mathrm{ud}}(t;\boldsymbol{x}_{\mathrm{ud}}(0),t_0)|<M$ for all $t\ge t_0+T(M,\bar{\alpha})$.

For the unmodeled dynamics the following assumption is made:

Assumption 4 The origin $\boldsymbol{x}_{\mathrm{ud}}=\mathbf{0}$ of the unmodeled dynamics is uniformly stable in the large case.

More specifically, there is a C^1 function $L(\boldsymbol{x}_{\mathrm{ud}})$ from $\mathbb{R}^p \to \mathbb{R}^+$ and continues, strictly increasing, scalar functions $\gamma_i(|\boldsymbol{x}_{\mathrm{ud}}|)$ from $\mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2, 3$, which satisfy

$$\gamma_i(0) = 0, \quad i = 1, 2, 3,$$

$$\lim_{s\to\infty} \gamma_i(s) = \infty, \quad i = 1, 2,$$

such that for $\boldsymbol{x}_{\mathrm{ud}} \in \mathbb{R}^p$

$$\gamma_1(|\boldsymbol{x}_{\mathrm{ud}}|) \le L(\boldsymbol{x}_{\mathrm{ud}}) \le \gamma_2(|\boldsymbol{x}_{\mathrm{ud}}|)$$

and

$$\frac{\vartheta L}{\vartheta \boldsymbol{x}_{\mathrm{ud}}} B(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}}) \le -\gamma_3 (|\boldsymbol{x}_{\mathrm{ud}}|). \tag{31}$$

Employing Assumption 3, Eq. (22) becomes

$$\begin{aligned} \dot{V} \le & -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ & -\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}}) + \frac{\partial L}{\partial \boldsymbol{x}_{\mathrm{ud}}} B(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}}) \\ \le & -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ & -\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}}) - \gamma_3(|\boldsymbol{x}_{\mathrm{ud}}|) \\ \le & -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ & -\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}}) \\ \le & -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ & + \|\boldsymbol{\xi}\| \|\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}})\|. \end{aligned} \tag{32}$$

To continue we consider the following assumption:

Assumption 5 Assume that the unknown vector field $\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}})$ satisfies the condition

$$\|\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}})\| \le k_{\boldsymbol{\phi}} \|\boldsymbol{x}\| \|\boldsymbol{\phi}_{\mathrm{ud}}(\boldsymbol{x}_{\mathrm{ud}})\|, \tag{33}$$

with $\boldsymbol{\phi}_{\mathrm{ud}}(\boldsymbol{x}_{\mathrm{ud}})$ an unknown vector field that depends only on $\boldsymbol{x}_{\mathrm{ud}}$. We also assume that $\boldsymbol{\phi}_{\mathrm{ud}}(\boldsymbol{x}_{\mathrm{ud}})$ is bounded uniformly by a constant θ. Therefore,

$$\|\boldsymbol{\phi}_{\mathrm{ud}}(\boldsymbol{x}_{\mathrm{ud}})\| \le \theta. \tag{34}$$

It can be easily verified that Eq. (33) holds, if $\boldsymbol{X}_f \boldsymbol{W}_f$ is uniformly bounded by a known positive constant ε_i so that $\boldsymbol{X}_f \boldsymbol{W}_f(t)$ is confined to the set $P_2 = \{\bar{\boldsymbol{x}}_{f_i} \cdot \boldsymbol{W}^l_{f_i} : \left|\bar{\boldsymbol{x}}_{f_i} \cdot \boldsymbol{W}^l_{f_i}\right| \le \varepsilon_i\}$ through the use of a hopping mechanism. In particular, following the same rationale with the derivation of Eq. (29), the standard update law (21) is modified to

$$\dot{\boldsymbol{W}}^l_{f_i} = -(\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}} \boldsymbol{\xi}_i s_l(\boldsymbol{x}) d_{f_l} - \frac{\sigma_i \kappa_f \left(\bar{\boldsymbol{x}}_{f_i} \boldsymbol{W}^l_{f_i} (\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}}\right)}{\operatorname{tr}\left\{(\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}} \bar{\boldsymbol{x}}_{f_i}\right\}}, \tag{35}$$

where

$$\sigma_i = \begin{cases} 0, & \text{if } \bar{\boldsymbol{x}}_{f_i} \cdot \boldsymbol{W}^l_{f_i} \in P_2 \text{ or } \bar{\boldsymbol{x}}_{f_i} \cdot \boldsymbol{W}^l_{f_i} = \pm\varepsilon_i \\ & \text{and } \bar{\boldsymbol{x}}_{f_i} \cdot \dot{\boldsymbol{W}}^l_{f_i} >< 0, \\ 1, & \text{otherwise,} \end{cases} \tag{36}$$

and κ_f is a small positive number decided by the designer. Therefore, the following lemma can be derived:

Lemma 2 Provided that the initial weights are chosen such that $\bar{\boldsymbol{x}}_{f_i} \cdot \boldsymbol{W}^l_{f_i}(0) \in P_2$ and $\bar{\boldsymbol{x}}_{f_i} \cdot W^{*l}_{f_i} \in P_2$, then we have $\bar{\boldsymbol{x}}_{f_i} \cdot \boldsymbol{W}^l_{f_i} \in P_2$ for all $t \ge 0$.

Proof The above lemma can be easily proved by observing that whenever $|\bar{\boldsymbol{x}}_{f_i} \cdot (\boldsymbol{W}^l_{f_i})^-| \ge \varepsilon_i$,

$$\frac{\mathrm{d}}{\mathrm{d}t} |\bar{\boldsymbol{x}}_{f_i} \cdot (\boldsymbol{W}^l_{f_i})^+|^2 \le 0, \tag{37}$$

which implies that after hopping occurs, the weights $(\boldsymbol{W}^l_{f_i})^+$ are driven towards the interior of P_2. For simplicity, since we will be working from now on with the time $(\cdot)^+$, we omit the '+' sign from the exponent. It is true that

$$\frac{\mathrm{d}}{\mathrm{d}t} |\bar{\boldsymbol{x}}_{f_i} \cdot \boldsymbol{W}^l_{f_i}|^2 = (\boldsymbol{W}^l_{f_i})^{\mathrm{T}} \dot{\boldsymbol{W}}^l_{f_i} \bar{\boldsymbol{x}}_{f_i} (\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}}. \tag{38}$$

Since $\bar{\boldsymbol{x}}_{f_i}(\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}} > 0$, only $(\boldsymbol{W}^l_{f_i})^{\mathrm{T}} \dot{\boldsymbol{W}}^l_{f_i}$ determines the sign of the above derivative.

Using the adaptive law (35), (36), we obtain

$$\begin{aligned} (\boldsymbol{W}^l_{f_i})^{\mathrm{T}} \dot{\boldsymbol{W}}^l_{f_i} = & -(\boldsymbol{W}^l_{f_i})^{\mathrm{T}} (\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}} \boldsymbol{\xi}_i s_l(\boldsymbol{x}) d_{f_l} \\ & -\kappa_f \varepsilon_i \frac{(\boldsymbol{W}^l_{f_i})^{\mathrm{T}} \boldsymbol{W}^l_{f_i}}{\|\boldsymbol{W}^l_{f_i}\|}, \end{aligned} \tag{39}$$

where $\varepsilon_i = \bar{\boldsymbol{x}}_{f_i} \boldsymbol{W}^l_{f_i} (\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}} / \operatorname{tr}\{(\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}} \bar{\boldsymbol{x}}_{f_i}\}$. As concerning the second part of the above equation it is obvious that $\varepsilon_i, \kappa_f > 0$ and $(\boldsymbol{W}^l_{f_i})^{\mathrm{T}} \boldsymbol{W}^l_{f_i} / \|\boldsymbol{W}^l_{f_i}\| > 0$. Thus,

$$-\kappa_f \varepsilon_i \frac{(\boldsymbol{W}^l_{f_i})^{\mathrm{T}} \boldsymbol{W}^l_{f_i}}{\|\boldsymbol{W}^l_{f_i}\|} < 0.$$

Now, regarding the first part of Eq. (39), we can distinguish two possible cases:

Case 1: $\bar{\boldsymbol{x}}_{f_i} \cdot \boldsymbol{W}^l_{f_i} = \varepsilon_i$ and $\bar{\boldsymbol{x}}_{f_i} \cdot \dot{\boldsymbol{W}}^l_{f_i} < 0$.

From the above notation we have

$$\begin{aligned} \bar{\boldsymbol{x}}_{f_i} \dot{\boldsymbol{W}}^l_{f_i} = -\bar{\boldsymbol{x}}_{f_i} (\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}} \boldsymbol{\xi}_i s_l(\boldsymbol{x}) d_{f_l} < 0 \\ \Rightarrow \boldsymbol{\xi}_i s_l(\boldsymbol{x}) d_{f_l} > 0. \end{aligned} \tag{40}$$

Also, $\bar{\boldsymbol{x}}_{f_i} \boldsymbol{W}^l_{f_i} \ge \varepsilon_i$ and thus the first part of Eq. (39) becomes

$$-(\boldsymbol{W}^l_{f_i})^{\mathrm{T}}(\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}}\boldsymbol{\xi}_i s_l(\boldsymbol{x})d_{f_l} \le -\varepsilon_i\boldsymbol{\xi}_i s_l(\boldsymbol{x})d_{f_l}.$$

According to Eq. (40) and $\varepsilon_i > 0$,

$$-(\boldsymbol{W}^l_{f_i})^{\mathrm{T}}(\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}}\boldsymbol{\xi}_i s_l(\boldsymbol{x})d_{f_l} < 0.$$

Case 2: $\bar{\boldsymbol{x}}_{f_i}\cdot\boldsymbol{W}^l_{f_i} \le -\varepsilon_i$ and $\bar{\boldsymbol{x}}_{f_i}\cdot\dot{\boldsymbol{W}}^l_{f_i} > 0$.

From the above notation we have

$$\bar{\boldsymbol{x}}_{f_i}\dot{\boldsymbol{W}}^l_{f_i} = -\bar{\boldsymbol{x}}_{f_i}(\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}}\boldsymbol{\xi}_i s_l(\boldsymbol{x})d_{f_l} > 0 \Rightarrow \boldsymbol{\xi}_i s_l(\boldsymbol{x})d_{f_l} < 0. \tag{41}$$

Also, $\bar{\boldsymbol{x}}_{f_i}\boldsymbol{W}^l_{f_i} \le -\varepsilon_i$ and thus the first part of Eq. (39) becomes

$$-(\boldsymbol{W}^l_{f_i})^{\mathrm{T}}(\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}}\boldsymbol{\xi}_i s_l(\boldsymbol{x})d_{f_l} \le -(-\varepsilon_i)\boldsymbol{\xi}_i s_l(\boldsymbol{x})d_{f_l}.$$

According to Eq. (41),

$$-(\boldsymbol{W}^l_{f_i})^{\mathrm{T}}(\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}}\boldsymbol{\xi}_i s_l(\boldsymbol{x})d_{f_l} < 0.$$

Therefore, we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\bar{\boldsymbol{x}}_{f_i}\cdot\boldsymbol{W}^l_{f_i}(t)|^2 \le 0.$$

Employing Eqs. (33) and (34), Eq. (32) becomes

$$\dot{V} \le -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 + k_{\boldsymbol{\phi}}\theta\|\boldsymbol{\xi}\|\|\boldsymbol{x}\|. \tag{42}$$

Moreover, if we apply $\|\boldsymbol{x}\| \le \|\boldsymbol{\zeta}\| + \|\boldsymbol{\xi}\|$ then Eq. (42) becomes

$$\dot{V} \le -\begin{bmatrix}\|\boldsymbol{\xi}\|\\ \|\boldsymbol{\zeta}\|\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\lambda_{\min}(K)-k_{\boldsymbol{\phi}}\theta & -k_{\boldsymbol{\phi}}\theta\\ 0 & \lambda_{\min}(K)\end{bmatrix}\begin{bmatrix}\|\boldsymbol{\xi}\|\\ \|\boldsymbol{\zeta}\|\end{bmatrix}. \tag{43}$$

Furthermore, the following lemma can be proved:

Lemma 3 Using the adaptive laws (29), (30) and (35), (36), the additional terms introduced in the expression for $\dot{V}$ can only make $\dot{V}$ more negative.

Proof Let $\boldsymbol{W}^\star_{g_i}$ be the actual unknown weight values such that $\left|\bar{\boldsymbol{x}}_{g_i}\cdot W^\star_{g_i}s_{g_i}\right| >> \theta_i$ and $\tilde{\boldsymbol{W}}_{g_i} = \boldsymbol{W}_{g_i} - \boldsymbol{W}^\star_{g_i}$. Then, in case of inner hopping and according to Boutalis *et al.* (2009), the weight hopping can be equivalently written with respect to $\tilde{\boldsymbol{W}}_{g_i}$ as $-2\theta_i\tilde{\boldsymbol{W}}_{g_i}/\|\tilde{\boldsymbol{W}}_{g_i}\|$. Similarly, in case of outer modified hopping the hopping size is given by $-\kappa_g\rho_i\tilde{\boldsymbol{W}}_{g_i}/\|\tilde{\boldsymbol{W}}_{g_i}\|$.

With this observation, the modified-by-hopping-updating law is rewritten as $\dot{\boldsymbol{W}}_{g_i} = -(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}\boldsymbol{\xi}_i u_i s_i(\boldsymbol{x})d_{g_i} - 2\theta_i\tilde{\boldsymbol{W}}_{g_i}/\|\tilde{\boldsymbol{W}}_{g_i}\|$ or $\dot{\boldsymbol{W}}_g^{\,l} = -(\bar{\boldsymbol{x}}_{g_i})^{\mathrm{T}}\boldsymbol{\xi}_i u_i s_i(\boldsymbol{x})d_{g_i} - k_g\rho_i\tilde{\boldsymbol{W}}_{g_i}/\|\tilde{\boldsymbol{W}}_{g_i}\|$. With this updating law it can be easily verified that Eq. (43) becomes

$$\dot{V} \le -\begin{bmatrix}\|\boldsymbol{\xi}\|\\ \|\boldsymbol{\zeta}\|\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\lambda_{\min}(K)-k_{\boldsymbol{\phi}}\theta & -k_{\boldsymbol{\phi}}\theta\\ 0 & \lambda_{\min}(K)\end{bmatrix}\begin{bmatrix}\|\boldsymbol{\xi}\|\\ \|\boldsymbol{\zeta}\|\end{bmatrix} - I_g\Theta_{g_i}, \tag{44}$$

where I_g is a function such that $I_g = 1$, if the hopping condition is applied, or $I_g = 0$ otherwise. Also, Θ_{g_i} is a positive constant computed as $\Theta_{g_i} = \sum 2\theta_i(\tilde{\boldsymbol{W}}_{g_i})^{\mathrm{T}}\tilde{\boldsymbol{W}}_{g_i}/\|\tilde{\boldsymbol{W}}_{g_i}\| \ge 0$ when we have inner hopping, or $\Theta_{g_i} = \sum\kappa_g\rho_i(\tilde{\boldsymbol{W}}_{g_i})^{\mathrm{T}}\tilde{\boldsymbol{W}}_{g_i}/\|\tilde{\boldsymbol{W}}_{g_i}\| \ge 0$ when we have outer modified hopping for all time, where the summation includes all weight vectors which perform hopping. Apparently, the negativity of $\dot{V}$ is further strengthened due to the last negative term.

Considering the second part of $\boldsymbol{W}^l_{f_i}$, term $-\kappa_f\bar{\boldsymbol{x}}_{f_i}\boldsymbol{W}^l_{f_i}(\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}}/\mathrm{tr}\{(\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}}\bar{\boldsymbol{x}}_{f_i}\}$ is the size of weight modified hopping, which as explained in Boutalis *et al.* (2009) has to be κ_f times the distance of the current weight vector from the forbidden hyperplane, with $0 < \kappa_f < 1$. Regarding the negativity of $\dot{V}$, we proceed as follows.

Once again, $W^{l\star}_{f_i}$ are the actual unknown weight values such that $|\bar{\boldsymbol{X}}_f\cdot\boldsymbol{W}^{l\star}_{f_i}| << \varepsilon_i$ and that $\tilde{\boldsymbol{W}}^l_{f_i} = \boldsymbol{W}^l_{f_i} - \boldsymbol{W}^{l\star}_{f_i}$. Therefore, the modified weight hopping can be equivalently written according to $\tilde{\boldsymbol{W}}^l_{f_i}$ as $-\kappa_f\varepsilon_i\tilde{\boldsymbol{W}}^l_{f_i}/\|\tilde{\boldsymbol{W}}^l_{f_i}\|$. Motivated by the above modification the updating law is rewritten as $\dot{\boldsymbol{W}}^l_{f_i} = -(\bar{\boldsymbol{x}}_{f_i})^{\mathrm{T}}\boldsymbol{\xi}_i s_l(\boldsymbol{x})d_{f_l} - \kappa_f\varepsilon_i\tilde{\boldsymbol{W}}^l_{f_i}/\|\tilde{\boldsymbol{W}}^l_{f_i}\|$. With this updating law it can be easily verified that Eq. (44) becomes

$$\dot{V} \le -\begin{bmatrix}\|\boldsymbol{\xi}\|\\ \|\boldsymbol{\zeta}\|\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\lambda_{\min}(K)-k_{\boldsymbol{\phi}}\theta & -k_{\boldsymbol{\phi}}\theta\\ 0 & \lambda_{\min}(K)\end{bmatrix}\begin{bmatrix}\|\boldsymbol{\xi}\|\\ \|\boldsymbol{\zeta}\|\end{bmatrix} - I_g\Theta_{g_i} - I_f\Theta_{f_i}, \tag{45}$$

where I_f defines an indicator function where $I_f = 1$ if the hopping condition is applied and $I_f = 0$ otherwise. Also, Θ_{f_i} defines a real positive constant expressed as $\Theta_{f_i} = \sum\kappa_f\varepsilon_i(\tilde{\boldsymbol{W}}^l_{f_i})^{\mathrm{T}}\tilde{\boldsymbol{W}}^l_{f_i}/\|\tilde{\boldsymbol{W}}^l_{f_i}\| \ge 0$ for all time. Therefore, we conclude that the negativity of $\dot{V}$ is actually enhanced further due to the last negative terms.

Hence, if we choose $\lambda_{\min}(K) \ge k_{\boldsymbol{\phi}}\theta$ then Eq. (45) becomes negative. Thus, we have

$$\dot{V} \le 0. \tag{46}$$

We are now ready to prove the following theorem:

Theorem 1 The control scheme in the closed loop system

$$\begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x}+\boldsymbol{X}_f\,\boldsymbol{W}_f^{*}\boldsymbol{s}_f(\boldsymbol{x})+\boldsymbol{X}_g\,\boldsymbol{W}_g{}^{*}\boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u}\\ &\quad+\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{x}_{\text{ud}}),\\ \dot{\boldsymbol{x}}_{\text{ud}} &= B(\boldsymbol{x},\boldsymbol{x}_{\text{ud}}),\\ \dot{\boldsymbol{\zeta}} &= -K\boldsymbol{\zeta},\\ u &= -\left[\boldsymbol{X}_g\,\boldsymbol{W}_g\boldsymbol{S}_g(\boldsymbol{x})\right]^{+}\left[\boldsymbol{X}_f\,\boldsymbol{W}_f\,\boldsymbol{s}_f(\boldsymbol{x})+\boldsymbol{v}\right],\\ \boldsymbol{v} &= (K+A)\boldsymbol{x},\\ \boldsymbol{\xi} &= \boldsymbol{\zeta}-\boldsymbol{x},\end{aligned}$$

together with the updating laws (29), (30) and (35), (36) guarantees the following properties:

(1) $\boldsymbol{\xi},\boldsymbol{x},\boldsymbol{x}_{\text{ud}},\,\boldsymbol{W}_f,\,\boldsymbol{W}_g\in L_\infty,|\boldsymbol{\xi}|\in L_2$,

(2) $\lim_{t\to\infty}\boldsymbol{\xi}(t)=\boldsymbol{0},\lim_{t\to\infty}\boldsymbol{x}(t)=\boldsymbol{0}$,

(3) $\lim_{t\to\infty}\dot{\boldsymbol{W}}_f(t)=\boldsymbol{0},\lim_{t\to\infty}\dot{\boldsymbol{W}}_g(t)=\boldsymbol{0}$,

provided that $\lambda_{\min}(K_c)>k_{\boldsymbol{\phi}}\theta$ and Assumption 3 is satisfied.

Proof From Eq. (46) we conclude that $V\in L_\infty$, which implies $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, $\tilde{\boldsymbol{W}}_f,\tilde{\boldsymbol{W}}_g\in L_\infty$. Furthermore, $\boldsymbol{W}_f=\tilde{\boldsymbol{W}}_f+\boldsymbol{W}_f^{*}\in L_\infty$ and $\boldsymbol{W}_g=\tilde{\boldsymbol{W}}_g+\boldsymbol{W}_g{}^{*}\in L_\infty$. Since $\boldsymbol{\xi}$, $\boldsymbol{\zeta}\in L_\infty$ and $\boldsymbol{\xi}=\boldsymbol{\zeta}-\boldsymbol{x}$, this also implies that $\boldsymbol{x}\in L_\infty$. Moreover, V is a monotonically decreasing function of time and is bounded from below, which means that $\lim_{t\to\infty}V(t)=V_\infty$ exists. Therefore, integrating $\dot{V}$ from 0 to ∞ we have $(\lambda_{\min}(K)-k_{\boldsymbol{\phi}}\theta)\int_0^\infty\|\boldsymbol{\xi}\|^2\,\mathrm{d}t+\lambda_{\min}(K)\int_0^\infty\|\boldsymbol{\zeta}\|^2\,\mathrm{d}t-k_{\boldsymbol{\phi}}\theta\int_0^\infty\|\boldsymbol{\xi}\|\,\|\boldsymbol{\zeta}\|\,\mathrm{d}t=|V(0)-V_\infty|<\infty$.

This denotes that $|\boldsymbol{\xi}|\in L_2$. We also have that $\dot{\boldsymbol{\xi}}=-K\boldsymbol{\xi}+\boldsymbol{X}_f\,\tilde{\boldsymbol{W}}_f\boldsymbol{s}_f(\boldsymbol{x})+\boldsymbol{X}_g\,\tilde{\boldsymbol{W}}_g\boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u}-\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{x}_{\text{ud}})$.

Hence, and since $\boldsymbol{u}\in L_\infty$, the sigmoidals and center matrices are bounded by definition, $\tilde{\boldsymbol{W}}_f$, $\tilde{\boldsymbol{W}}_g\in L_\infty$ and Assumption 3 hold. Thus, since $\boldsymbol{\xi}\in L_2\cap L_\infty$ and $\dot{\boldsymbol{\xi}}\in L_\infty$, together with Barbalat's lemma, we conclude that $\lim_{t\to\infty}\boldsymbol{\xi}(t)=\boldsymbol{0}$.

Hence, we have

$$\lim_{t\to\infty}\boldsymbol{x}(t)=\lim_{t\to\infty}(\boldsymbol{\zeta}(t)-\boldsymbol{\xi}(t))=\boldsymbol{0}.$$

Thus,

$$\lim_{t\to\infty}\boldsymbol{x}(t)=\boldsymbol{0}.$$

Finally, according to $\boldsymbol{u},\boldsymbol{s}_f(\boldsymbol{x}),\boldsymbol{S}_g(\boldsymbol{x}),\boldsymbol{x}$ boundedness and the convergence of $\boldsymbol{\zeta}(t)$, $\boldsymbol{\xi}(t)$ to zero, we have that $\dot{\boldsymbol{W}}_f$, $\dot{\boldsymbol{W}}_g$ also converge to zero.

To proceed further we have to make the following assumption concerning the unmodeled dynamics (UD) term $\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{x}_{\text{ud}})$:

Assumption 6 Assume that the unknown UD vector field $\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{x}_{\text{ud}})$ satisfies the condition

$$\|\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{x}_{\text{ud}})\|\le\theta_{\text{ud}},\tag{47}$$

or simply, $\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{x}_{\text{ud}})$ is said to be uniformly bounded by a positive constant.

Thus, Eq. (32) becomes

$$\dot{V}\le-\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2-\lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2+\theta_{\text{ud}}\|\boldsymbol{\xi}\|,$$

which can be rewritten as

$$\dot{V}\le-(\lambda_{\min}(K)\|\boldsymbol{\xi}\|-\theta_{\text{ud}})\|\boldsymbol{\xi}\|-\lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2,\tag{48}$$

and finally gives

$$\dot{V}\le0,\tag{49}$$

provided that

$$\|\boldsymbol{\xi}\|>\frac{\theta_{\text{ud}}}{\lambda_{\min}(K)},\tag{50}$$

with $\lambda_{\min}(K)>0$. In the sequel, inequality (50) together with $\|\boldsymbol{x}\|\le\|\boldsymbol{\zeta}\|+\|\boldsymbol{\xi}\|$ leads to the conclusion that the trajectories of $\boldsymbol{\xi}(t)$ and $\boldsymbol{x}(t)$ are uniformly bounded in a small region around zero (according to selection of a sufficiently large K) and may take values from the sets $\varXi$ and X defined as

$$\varXi=\{\boldsymbol{\xi}(t):\|\boldsymbol{\xi}\|\le\theta_{\text{ud}}/\lambda_{\min}(K),\ \lambda_{\min}(K)>0\},$$

and

$$X=\{\boldsymbol{x}(t):\|\boldsymbol{x}(t)\|\le2\theta_{\text{ud}}/\lambda_{\min}(K),\ \lambda_{\min}(K)>0\}.$$

Therefore, we have proven the following theorem:

Theorem 2 Consider the approximated system (9), (10) and the unmodeled dynamics which satisfies Assumption 5. Also, assume that Eq. (47) holds for the unknown vector field $\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{x}_{\text{ud}})$. Then the control law (17), (18) in conjuction with the update laws (29), (30) and (35), (36) guarantees the uniform ultimate boundedness according to the following sets:

(1) $\varXi=\{\boldsymbol{\xi}(t):\|\boldsymbol{\xi}\|\le\theta_{\text{ud}}/\lambda_{\min}(K),\lambda_{\min}(K)>0\}$;

(2) $X=\{\boldsymbol{x}(t):\|\boldsymbol{x}(t)\|\le2\theta_{\text{ud}}/\lambda_{\min}(K),\lambda_{\min}(K)>0\}$.

Furthermore, $\dot{\boldsymbol{\xi}}=-K\boldsymbol{\xi}+\boldsymbol{X}_f\,\tilde{\boldsymbol{W}}_f\boldsymbol{s}_f(\boldsymbol{x})+\boldsymbol{X}_g\,\tilde{\boldsymbol{W}}_g\boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u}-\boldsymbol{\phi}(x,\boldsymbol{x}_{\text{ud}})$. As could be observed, since $\boldsymbol{X}_f,\boldsymbol{X}_g,\boldsymbol{s}_f,\boldsymbol{S}_g$ are bounded by definition, the boundedness of $\tilde{\boldsymbol{W}}_f$ and $\tilde{\boldsymbol{W}}_g$ is assured by the use of the hopping algorithm and $\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{x}_{\text{ud}})$ is bounded uniformly, we conclude that $\dot{\boldsymbol{\xi}}\in L_\infty$. Thus, we are now ready to state the following remark:

Remark 1 The above analysis shows that the appropriate selection of the design constant matrix $\boldsymbol{K}$ is very important. The wise selection of $\boldsymbol{K}$ elements makes our adaptive controller (regulator) capable of converging the system states $\boldsymbol{x}$ to zero, or at least able to ensure uniform ultimate boundedness of $\boldsymbol{x}$ as for all other signals in the closed loop. Also, by appropriate selection of small values for the constants $k_{\boldsymbol{\phi}}$, θ, $\bar{\theta}_{\text{ud}}$, we can avoid any implementation issue that could arise if the elements of matrix $\boldsymbol{K}$ are required to have large values. Moreover, the more the F-RHONN model matches the input-output behavior of the true unknown nonlinear dynamical system, the better the control results. Finally, it is obvious that when the unknown vector field $\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\text{ud}})$ is uniformly bounded by a constant, any positive value for each k_i element of $\boldsymbol{K}$, guarantees uniform boundedness of states $\boldsymbol{x}$.

3.2.2 Violation of the uniform asymptotic stability in the large condition

In this subsection, we investigate the effect of the unmodeled dynamics on the stability of the closed loop system when the uniform asymptotic stability in the large condition is violated, namely Eq. (31). Therefore, instead of Eq. (31) we assume

$$\frac{\vartheta L}{\vartheta \boldsymbol{x}_{\text{ud}}} B(\boldsymbol{x}, \boldsymbol{x}_{\text{ud}}) \leq -\gamma_3 \left(\|\boldsymbol{x}_{\text{ud}}\|\right) + \rho \|\boldsymbol{x}\|^2, \tag{51}$$

where ρ is a positive constant. Employing Eq. (51) within Eq. (22) we have

$$\begin{aligned} \dot{V} \;\leq\;& -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ & -\boldsymbol{\xi}^{\mathrm{T}} \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\text{ud}}) + \frac{\partial L}{\partial \boldsymbol{x}_{\text{ud}}} B(\boldsymbol{x}, \boldsymbol{x}_{\text{ud}}) \\ \leq\;& -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ & -\boldsymbol{\xi}^{\mathrm{T}} \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\text{ud}}) - \gamma_3(\|\boldsymbol{x}_{\text{ud}}\|) + \rho\|\boldsymbol{x}\|^2 \\ \leq\;& -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ & + \|\boldsymbol{\xi}\| \|\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\text{ud}})\| + \rho\|\boldsymbol{x}\|^2. \end{aligned} \tag{52}$$

As previously stated, we shall consider the following cases:

Case 1: Assume that the unknown vector field $\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\text{ud}})$ satisfies Eqs. (33) and (34). Then Eq. (52) becomes

$$\begin{aligned} \dot{V} \;\leq\;& -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ & + k_{\boldsymbol{\phi}}\theta\|\boldsymbol{\xi}\|\|\boldsymbol{x}\| + \rho\|\boldsymbol{x}\|^2. \end{aligned} \tag{53}$$

Hence, we can distinguish the following cases:

(1) If $\boldsymbol{x} \geq \mathbf{0}$ we have that $\boldsymbol{\zeta}(t) \geq \boldsymbol{\xi}(t)$ but $\boldsymbol{\zeta}(t) \leq \mathbf{0}$, $\forall t \geq 0$, which implies that $\|\boldsymbol{\zeta}(t)\| \leq \|\boldsymbol{\xi}(t)\|$. Thus, we have

$$\|\boldsymbol{x}\| \leq \|\boldsymbol{\zeta}\| + \|\boldsymbol{\xi}\| \leq 2\|\boldsymbol{\xi}\|. \tag{54}$$

Therefore, Eq. (53) becomes

$$\begin{aligned} \dot{V} \;\leq\;& -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ & + 2k_{\boldsymbol{\phi}}\theta\|\boldsymbol{\xi}\|^2 + 4\rho\|\boldsymbol{\xi}\|^2 \\ \leq\;& -\left(\lambda_{\min}(K) - 2k_{\boldsymbol{\phi}}\theta - 4\rho\right)\|\boldsymbol{\xi}\|^2 \\ & -\lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ \leq\;& 0, \end{aligned} \tag{55}$$

provided that

$$\lambda_{\min}(K) > 2\left(k_{\boldsymbol{\phi}}\theta + 2\rho\right). \tag{56}$$

(2) If $\boldsymbol{x} < \mathbf{0}$ we have that $\boldsymbol{\zeta}(t) < \boldsymbol{\xi}(t)$ but $\boldsymbol{\zeta}(t) \leq \mathbf{0}$, $\forall t \geq 0$, which implies that $\|\boldsymbol{\zeta}(t)\| > \|\boldsymbol{\xi}(t)\|$. Thus, we have

$$\|\boldsymbol{x}\| \leq \|\boldsymbol{\zeta}\| + \|\boldsymbol{\xi}\| \leq 2\|\boldsymbol{\zeta}\|. \tag{57}$$

Therefore, Eq. (53) becomes

$$\begin{aligned} \dot{V} \;\leq\;& -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ & + 2k_{\boldsymbol{\phi}}\theta\|\boldsymbol{\xi}\|\|\boldsymbol{\zeta}\| + 4\rho\|\boldsymbol{\zeta}\|^2 \\ \leq\;& -\begin{bmatrix} \|\boldsymbol{\xi}\| \\ \|\boldsymbol{\zeta}\| \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \lambda_{\min}(K) & -2k_{\boldsymbol{\phi}}\theta \\ 0 & \lambda_{\min}(K) - 4\rho \end{bmatrix} \begin{bmatrix} \|\boldsymbol{\xi}\| \\ \|\boldsymbol{\zeta}\| \end{bmatrix} \\ \leq\;& 0, \end{aligned} \tag{58}$$

provided that

$$\lambda_{\min}(K) > 4\rho. \tag{59}$$

Conclusively, $\forall \boldsymbol{x} \in \mathbb{R}^n$, the Lyapunov candidate function becomes negative when $\lambda_{\min}(K) > 2\left(k_{\boldsymbol{\phi}}\theta + 2\rho\right)$.

Case 2: Now assume that the unknown vector field $\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\text{ud}})$ satisfies Eq. (47). Eq. (52) becomes

$$\dot{V} \leq -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 + \theta_{\text{ud}}\|\boldsymbol{\xi}\| + \rho\|\boldsymbol{x}\|^2. \tag{60}$$

Hence, we can distinguish the following cases:

(1) If $\boldsymbol{x} \geq \mathbf{0}$ we have that $\boldsymbol{\zeta}(t) \geq \boldsymbol{\xi}(t)$ but $\boldsymbol{\zeta}(t) \leq \mathbf{0}$, $\forall t \geq 0$, which implies that $\|\boldsymbol{\zeta}(t)\| \leq \|\boldsymbol{\xi}(t)\|$. Thus, we have

$$\|\boldsymbol{x}\| \leq \|\boldsymbol{\zeta}\| + \|\boldsymbol{\xi}\| \leq 2\|\boldsymbol{\xi}\|. \tag{61}$$

Therefore, Eq. (60) becomes

$$\begin{aligned} \dot{V} \;\leq\;& -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ & + \theta_{\text{ud}}\|\boldsymbol{\xi}\| + 4\rho\|\boldsymbol{\xi}\|^2 \\ \leq\;& -\left[\left(\lambda_{\min}(K) - 4\rho\right)\|\boldsymbol{\xi}\| - \theta_{\text{ud}}\right]\|\boldsymbol{\xi}\| \\ & -\lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ \leq\;& 0, \end{aligned} \tag{62}$$

provided that

$$\|\boldsymbol{\xi}\| > \frac{\theta_{\mathrm{ud}}}{\lambda_{\min}(K) - 4\rho}, \tag{63}$$

with $\lambda_{\min}(K) > 4\rho$.

(2) If $\boldsymbol{x} < \mathbf{0}$ we have that $\boldsymbol{\zeta}(t) < \boldsymbol{\xi}(t)$ but $\boldsymbol{\zeta}(t) \leq \mathbf{0}$, $\forall t \geq 0$, which implies that $\|\boldsymbol{\zeta}(t)\| > \|\boldsymbol{\xi}(t)\|$. Thus, we have

$$\|\boldsymbol{x}\| \leq \|\boldsymbol{\zeta}\| + \|\boldsymbol{\xi}\| \leq 2\|\boldsymbol{\zeta}\|. \tag{64}$$

Therefore, Eq. (60) becomes

$$\begin{aligned} \dot{V} &\leq -\lambda_{\min}(K)\|\boldsymbol{\xi}\|^2 - \lambda_{\min}(K)\|\boldsymbol{\zeta}\|^2 \\ &\quad +\theta_{\mathrm{ud}}\|\boldsymbol{\xi}\| + 4\rho\|\boldsymbol{\zeta}\|^2 \\ &\leq -(\lambda_{\min}(K)\|\boldsymbol{\xi}\| - \theta_{\mathrm{ud}})\|\boldsymbol{\xi}\| \\ &\quad -(\lambda_{\min}(K) - 4\rho)\|\boldsymbol{\zeta}\|^2 \\ &\leq 0, \end{aligned} \tag{65}$$

provided that

$$\|\boldsymbol{\xi}\| > \frac{\theta_{\mathrm{ud}}}{\lambda_{\min}(K)}, \tag{66}$$

and $\lambda_{\min}(K) > 4\rho$.

Conclusively, $\forall \boldsymbol{x} \in \mathbb{R}^n$, the Lyapunov candidate function becomes negative when $\|\boldsymbol{\xi}\| > \theta_{\mathrm{ud}}/(\lambda_{\min}(K) - 4\rho)$ and $\lambda_{\min}(K) > 4\rho$.

In the sequel, inequality (63) together with Eqs. (61) and (64) leads to the conclusion that the trajectories of $\boldsymbol{\xi}(t)$ and $\boldsymbol{x}(t)$ are uniformly bounded in a small region around zero (according to sufficiently large eigenvalues of $\boldsymbol{K}$). Thus, sets ξ and X defined as

$$\varXi = \left\{\boldsymbol{\xi}(t) : \|\boldsymbol{\xi}(t)\| \leq \frac{\theta_{\mathrm{ud}}}{\lambda_{\min}(K) - 4\rho},\ \lambda_{\min}(K) > 4\rho\right\}$$

and

$$X = \left\{\boldsymbol{x}(t) : \|\boldsymbol{x}(t)\| \leq \frac{2\theta_{\mathrm{ud}}}{\lambda_{\min}(K) - 4\rho},\ \lambda_{\min}(K) > 4\rho\right\}$$

include the allowable values of $\boldsymbol{\xi}(t)$ and $\boldsymbol{x}(t)$. Therefore, we have proven the following theorem:

Theorem 3 Consider the approximated system (9), (10) and the unmodeled dynamics which satisfies Eq. (52). Also, assume that Eq. (47) holds for the unknown vector field $\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}})$. Then, the control law (17), (18) in conjunction with the update laws (29), (30) and (35), (36) guarantees the uniform ultimate boundedness according to the following sets:

(1) $\varXi = \{\boldsymbol{\xi}(t) : \|\boldsymbol{\xi}(t)\| \leq \theta_{\mathrm{ud}}/(\lambda_{\min}(K) - 4\rho), \lambda_{\min}(K) > 4\rho\}$;

(2) $X = \{\boldsymbol{x}(t) : \|\boldsymbol{x}(t)\| \leq 2\theta_{\mathrm{ud}}/(\lambda_{\min}(K) - 4\rho), \lambda_{\min}(K) > 4\rho\}$.

Furthermore, $\dot{\boldsymbol{\xi}} = -K\boldsymbol{\xi} + \boldsymbol{X}_f\tilde{\boldsymbol{W}}_f\boldsymbol{s}_f(\boldsymbol{x}) + \boldsymbol{X}_g\tilde{\boldsymbol{W}}_g\boldsymbol{S}_g(\boldsymbol{x})\boldsymbol{u} - \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}})$. As could be observed, since $\boldsymbol{X}_f, \boldsymbol{X}_g, \boldsymbol{s}_f, \boldsymbol{S}_g$ are bounded by definition, the boundedness of $\tilde{\boldsymbol{W}}_f$ and $\tilde{\boldsymbol{W}}_g$ is assured by the use of the hopping algorithm, and $\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_{\mathrm{ud}})$ is bounded uniformly, we conclude that $\dot{\boldsymbol{\xi}} \in L_\infty$. Thus, we are now ready to state one more remark as follows:

Remark 2 Theorem 3 demonstrates that, the violation of the uniform asymptotic stability in the large condition makes our adaptive controller (regulator) still able to assure the uniform ultimate boundedness of the states $\boldsymbol{x}$ and of all signals in the closed loop. Thus, the more careful selection of the design constant elements k_i of matrix $\boldsymbol{K}$ leads to better performance of our controller, provided that the implementation constraints are satisfied. Finally, the accuracy of our model (fuzzy-recurrent high order neural network) is a performance index of our adaptive controller.

4 Simulation results

To illustrate the effectiveness of the proposed algorithm we present the well known benchmarks 'DC motor' and 'Lorenz system' with comparisons in the second case, against the well established approach of the use of simple RHONNs (Rovithakis and Christodoulou, 2000). The simulations demonstrate the regulation superiority of the proposed approach.

4.1 DC motor

In this example, our purpose is to control the speed of a 1 kW DC motor by applying the proposed method. The normalized DC motor model that was used is described by the following dynamical equations (Rovithakis and Christodoulou, 2000):

$$\begin{cases} T_a\dfrac{\mathrm{d}I_a}{\mathrm{d}t} = -I_a - \boldsymbol{\phi}\varOmega + V_a, \\ T_m\dfrac{\mathrm{d}\varOmega}{\mathrm{d}t} = \boldsymbol{\phi}I_a - K_0\varOmega - m_L, \\ T_f\dfrac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} = -I_f + V_f, \\ \boldsymbol{\phi} = \dfrac{aI_f}{1 + bI_f}. \end{cases} \tag{67}$$

We define the system states as armature current $x_1 = I_a$, angular speed $x_2 = \varOmega$, stator flux $x_3 = \boldsymbol{\phi}$ and the control inputs as armature voltage $u_1 = V_a$, field voltages $u_2 = V_f$. Motivated by this choice, the above dynamical equations are equivalently given by the following vector-matrix representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{T_a x_1} - \dfrac{1}{T_a} x_2 x_3 \\ \dfrac{1}{T_m} x_1 x_3 - \dfrac{K_0}{T_m} x_2 - \dfrac{m_L}{T_m} \\ -\dfrac{1}{T_f} \dfrac{x_3}{a - \beta x_3} \end{bmatrix} + \begin{bmatrix} \dfrac{1}{T_a} & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{T_f} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad (68)$$

which is nonlinear and affine in the control form.

In the sequel, the system dynamics are described within a degree of accuracy, by a second order NF system of the form Eq. (8), where $x_1 = I_a$ and $x_2 = \phi$. Thus, we have two states ($n = 2$), five fuzzy output partitions of each f_i ($m = 5$) with the ranges of f_1 $[-174.5667, -24.3454]$, f_2 $[-15.3627, 23.0566]$, and second order sigmoidal terms ($k = 5$), which assume high order connections up to the second order. Also, the number of fuzzy output partitions of each g_{ii} is selected to be three ($m = 3$) with the ranges of g_{11} $[148, 150]$ and g_{22} $[0.0128, 0.0131]$, using only first order sigmoid terms.

In the simulations, we use the complete set of Eq. (68) to produce data. The control law described in Eqs. (17) and (18) is applied on this system, which in turn produces states x_1, x_3. The sampling time was chosen as 1 ms and the parameter values of the DC motor can be seen in Table 1.

Table 1 Parameter values of the DC motor

Parameter	Value	Parameter	Value
$1/T_a$ (s^{-1})	148.88	m_L	0
$1/T_m$ (s^{-1})	42.91	a	2.6
K_0/T_m (N·m/rad)	0.0129	β	1.6
T_f (s)	31.88		

T_a: electric time constant; T_m: mechanic time constant; T_f: field time constant; K_0: normalized viscous friction coefficient; m_L: speed independent component of the applied load torque; a and β: constant parameters

Furthermore, we have chosen the initial values of all variables as $[I_a \quad \Omega \quad \phi] = [1 \quad 1 \quad 0.98]$, the initial weights $W_{f_i} = 0$, $W_{g_{ij}} = 1$, and the updating learning rates $d_{f_l} = 0.1$ and $d_{g_i} = 0.5$. Also, the parameters of the sigmoidal terms were chosen to be $\alpha_1 = 0.4$, $\alpha_2 = 1$, $\beta_1 = \beta_2 = 1$, $\gamma_1 = \gamma_2 = 0$, while the diagonal elements of matrix $\boldsymbol{K}$ were $k_1 = 1$, $k_2 = 2$.

Fig. 3a gives the evolution of the states x_1, x_2 and x_3 for the proposed F-RHONN model, with time, where it can be observed that the unknown angular velocity converges to zero very fast. Also, Fig. 3b gives the progression of the control inputs u_1 and u_2.

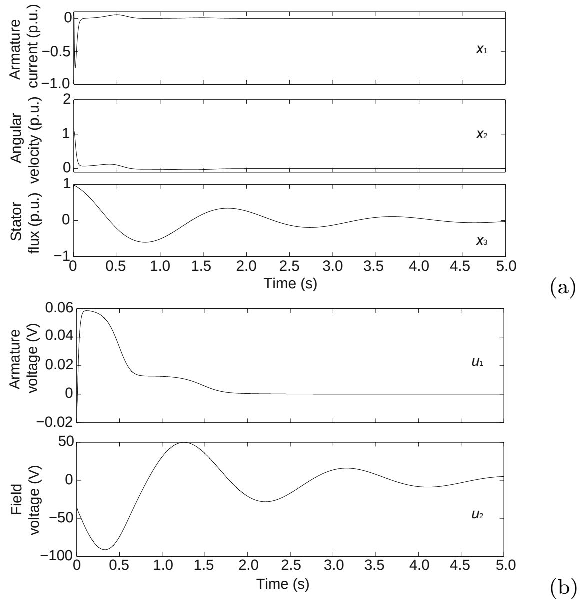

Fig. 3 Convergence of states x_1, x_2, and x_3 to zero (a) and evolution of the control inputs u_1 and u_2 (b) for the fuzzy-recurrent high order neural networks (F-RHONN) model

4.2 Lorenz system

The Lorenz system models the 2D convection flow of a fluid layer heated from below and cooled from above. The model represents the Earth's atmosphere heated by the ground's absorption of sunlight and losing heat into space. The model includes the following dynamical equations:

$$\begin{cases} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2, \end{cases} \quad (69)$$

where x_1, x_2, and x_3 represent values of fluid velocity, horizontal and vertical temperature variations, respectively. The positive constants σ, ρ, and β represent the Prandtl number, Rayleigh number, and geometric factor, respectively. By choosing $\sigma = 10$, $\rho = 28$, $\beta = 8/3$, the system presents three unstable equilibrium points $c_1 = (0, 0, 0)$, $c_2 = (-8.59, -8.59, 27)$, $c_3 = (8.59, 8.59, 27)$, and the system presents a chaotic behavior as its trajectories wander eternally near a strange invariant set called the strange attractor.

The system may be controlled by including control inputs in some or all the dynamical equations. In the full state control its dynamical equations are written as (Yeap and Ahmed, 1994)

$$\begin{cases} \dot{x}_1 &= \sigma(x_2 - x_1) + u_1, \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 + u_2, \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2 + u_3. \end{cases} \tag{70}$$

To test the ability of the proposed approach in controlling the system even if the NF model is of lower order than the real system dynamics and to compare its performance against the use of a simple RHONN scheme, we performed a number of simulations, which are analyzed below.

Simulation 1 Assume that Eq. (69) has the following initial condition:

$$\boldsymbol{x}_0 = [0.39, 0.59, 0.25]^{\mathrm{T}}.$$

Figs. 4a and 4b show the time progression of Lorenz chaotic system states without using control signals.

Our control objective is to produce an appropriate state feedback control law in order to regulate the system states to one of its equilibrium points, here the point $(0, 0, 0)$, when the state x_3 and its dynamics are completely omitted. Therefore, the parameters that have been used in the control law (17), (18) and the learning laws (29), (30) and (35), (36) are as follows:

$$\boldsymbol{A} = \mathrm{diag}\{-15, -15\},\ \boldsymbol{K} = \mathrm{diag}\{3, 4\}.$$

Next, we assume that the system dynamics are described by an NF system of the form Eq. (8) having two states ($n = 2$), the number of fuzzy partitions is selected to be $m = 5$, and the depth of high order sigmoid terms $k = 5$. In this case, $s_i(\boldsymbol{x})$ presumes up to the second order high order connections. Also, the parameters of the sigmoidal terms are selected as $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 1$, $\gamma_1 = \gamma_2 = 0$ and the learning rates $d_{f_l} = 0.1$, $d_{g_i} = 1$.

Figs. 5a and 5b show that the convergence of states x_1, x_2, and x_3 to zero is exponentially fast when we assume the existence of the states x_1, x_2 only.

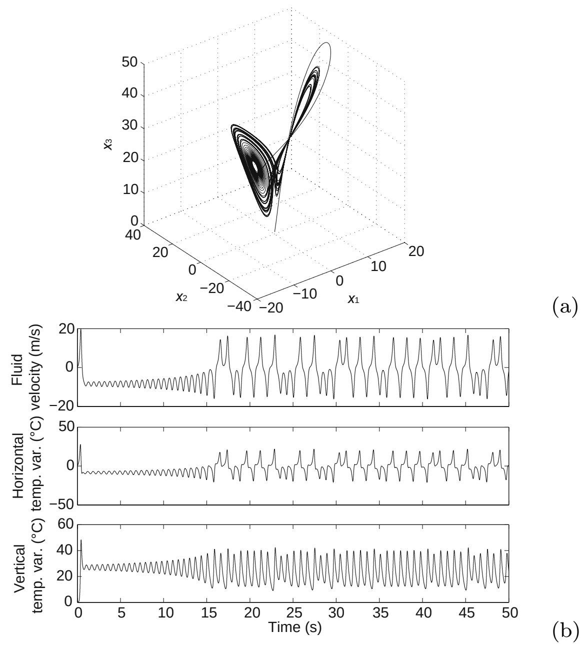

Fig. 4 A phase-space trajectory generated by the Lorenz chaotic system with $\boldsymbol{x}_0$=[0.39, 0.59, 0.25]$^{\mathrm{T}}$ in a 3D plot (a) and evolution of the same Lorenz chaotic system states in a 2D plot (b). temp.: temperature; var.: variation

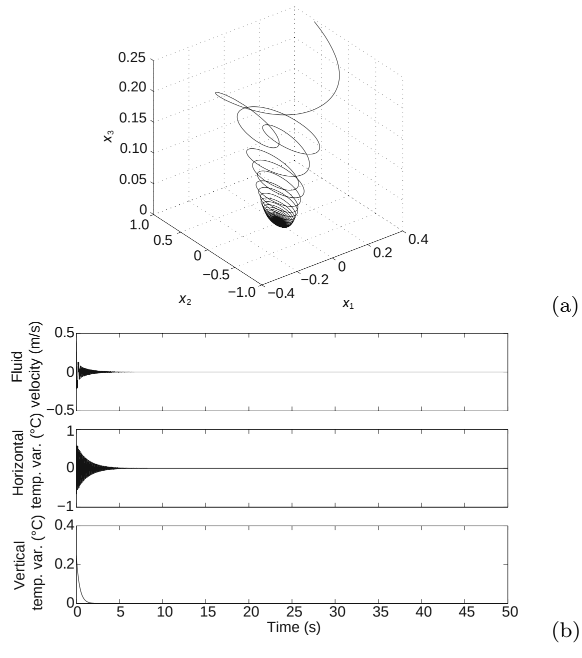

Fig. 5 Regulation of the Lorenz chaotic system when we consider only the states x_1, x_2, and the initial condition $\boldsymbol{x}_0$=[0.39, 0.59, 0.25]$^{\mathrm{T}}$ in a 3D plot (a) and regulation of the same Lorenz chaotic system states in a 2D plot (b)

Simulation 2 Assume that Eq. (69) has the following alternative initial condition:

$$\boldsymbol{x}_0 = [8.39, 8.59, 25]^{\mathrm{T}}.$$

Figs. 6a and 6b show the evolution of Lorenz chaotic system states without using control signals.

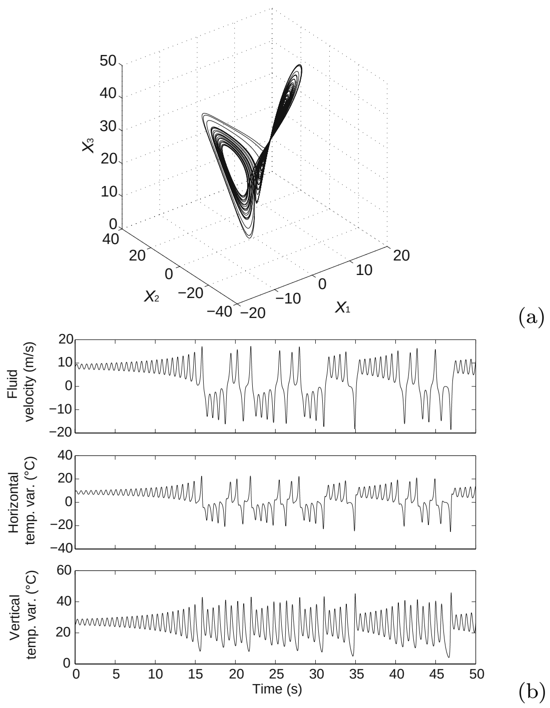

Fig. 6 A phase-space trajectory generated by the Lorenz chaotic system with $\boldsymbol{x}_0$=[8.39, 8.59, 25]$^\mathrm{T}$ in a 3D plot (a) and evolution of the same Lorenz chaotic system states in a 2D plot (b)

The main parameters for the control law (17), (18) and the learning laws (29), (30) and (35), (36) are selected as

$$\boldsymbol{A} = \mathrm{diag}\{-15, -15\},\ \boldsymbol{K} = \mathrm{diag}\{2, 3\}.$$

The parameters of the sigmoidal terms that have been used are $\alpha_1 = \alpha_2 = 2$, $\beta_1 = \beta_2 = 1$, $\gamma_1 = \gamma_2 = 0$, and the learning rates are selected appropriately as $d_{f_l} = d_{g_i} = 0.1$.

Figs. 7a and 7b show that, as before, the convergence of states x_1, x_2, and x_3 to zero is exponentially fast when we assume once again the existence of the states x_1, x_2 only.

Simulation 3 Assume that Eq. (69) has the following random initial condition:

$$\boldsymbol{x}_0 = [-0.5, 0.8, 2]^\mathrm{T}.$$

The main parameters for the control law (17), (18) and the learning laws (29), (30) and (35), (36) are selected as

$$\boldsymbol{A} = \mathrm{diag}\{-15, -15\},\ \boldsymbol{K} = \mathrm{diag}\{4, 5\}.$$

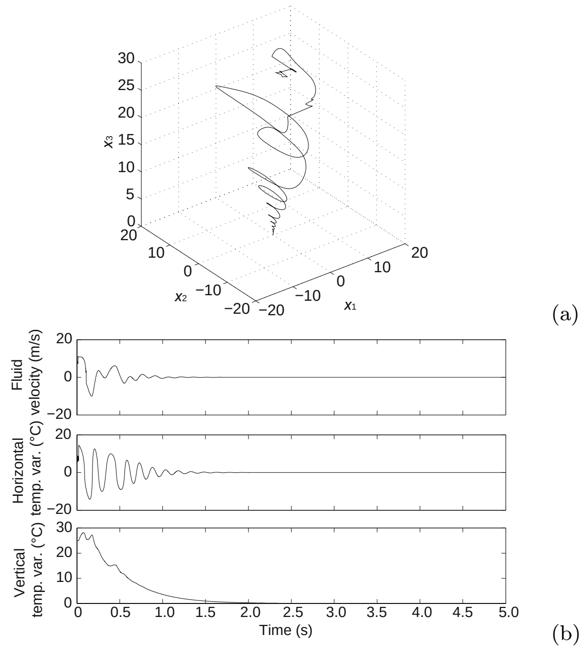

Fig. 7 Regulation of the Lorenz chaotic system when we consider only the states x_1, x_2, and the initial condition $\boldsymbol{x}_0$=[8.39, 8.59, 25]$^\mathrm{T}$ in a 3D plot (a) and regulation of the same Lorenz chaotic system states in a 2D plot (b)

The parameters of the sigmoidal terms used in both methods are $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 1$, $\gamma_1 = \gamma_2 = 0$, and the learning rates are selected appropriately as $d_{f_l} = 0.1$ and $d_{g_i} = 1$. Fig. 8a shows the exponentially fast convergence of states x_1, x_2, and x_3 to zero when we assume the existence of the states x_1, x_2, for our approach in comparison with the simple RHONNs (Rovithakis and Christodoulou, 2000), which denotes the superiority of the proposed NF approach against the simple RHONN scheme. The parameters of RHONNs (which were extensively explained in Rovithakis and Christodoulou (2000)) were selected to have the same values as F-RHONNs concerning the adaptive gains, the sigmoidal, and the matrices $\boldsymbol{A}$, $\boldsymbol{K}$.

Furthermore, we have similar results when we assume the existence of states x_2, x_3 and ignore state x_1 (Fig. 8b). In case x_2 is omitted (only x_1, x_3 are considered), Fig. 9a shows that the proposed NF approach performs equally well, but not the simple RHONN approach, which becomes unstable and cannot be included in the figure. This is due to the faster convergence of the proposed approach, which prevents inherent marginal instabilities of the dynamics of x_2 from occurring. In conclusion, the proposed approach is much more superior compared to RHONNs, working quite well even at small values of k_i, which is desirable in order to avoid large oscillations during the convergence.

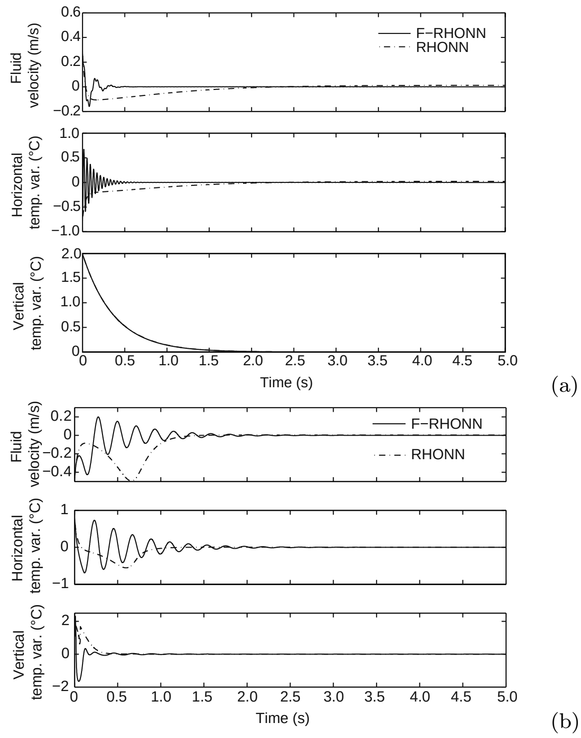

Fig. 8 Evolution of Lorenz states to zero for RHONN and F-RHONN (fuzzy-recurrent high order neural networks) with initial condition x_0=[−0.5, 0.8, 2]$^{\mathrm{T}}$ when only x_1, x_2 dynamics are considered (a) and when only x_2, x_3 dynamics are considered (b)

5 Conclusions

The investigation of regulating unknown nonlinear systems, with their approximation presenting model order problems, is discussed extensively in this paper. The control scheme is a direct approach and it is based on an NF dynamical system definition, which interweaves the concepts of fuzzy systems (FSs) and recurrent high order neural networks (F-RHONNs). We examine two main subjects, concerning first the uniform asymptotic stability in the large case, where the equilibrium point of the unmodeled dynamics is said to be uniformly asymptotically stable in the large case, and then the violation of the uniform asymptotic stability in the large condition. Since the plant is considered unknown, we propose its approximation by a special form of an NF-dynamical system, which, however, may assume a smaller number of states than the original unknown model. This practically transforms the original unknown system into an NF model of known structure, which contains a disturbance term to account for the effect of the omitting states, but contains a number of unknown synaptic weights, which then have to be estimated. Weight updating laws for the synaptic weights of the involved HONNs are provided, assuring that the system states reach zero or a small region around zero exponentially fast, keeping at the same time all signals in the closed loop bounded. Simulations show the performance of the method and it is shown that, by following the proposed procedure, the asymptotic state regulation is obtained quite well, even if some state dynamics are omitted.

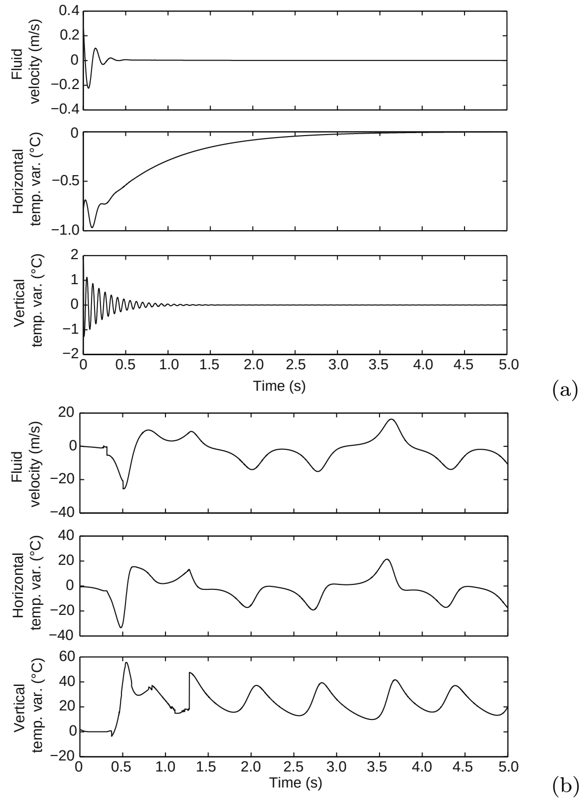

Fig. 9 Evolution of Lorenz states when we use the F-RHONN (fuzzy-recurrent high order neural networks) model and consider only the x_1, x_3 dynamics (a) and when we use the RHONN model and consider the same states (b). In both cases we assume the same initial condition x_0=[−0.5, 0.8, 2]$^{\mathrm{T}}$

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