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A framework for analysis of extended fuzzy logic

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Abstract: We address a framework for the analysis of extended fuzzy logic (FLe) and elaborate mainly the key characteristics of FLe by proving several qualification theorems and proposing a new mathematical tool named the A-granule. Specifically, we reveal that within FLe a solution in the presence of incomplete information approaches the one gained by complete information. It is also proved that the answers and their validities have a structural isomorphism within the same context. This relationship is then used to prove the representation theorem that addresses the rationality of FLe-based reasoning. As a consequence of the developed theoretical description of FLe, we assert that in order to solve a problem, having complete information is not a critical need; however, with more information, the answers achieved become more specific. Furthermore, reasoning based on FLe has the advantage of being computationally less expensive in the analysis of a given problem and is faster.

Key words: Extended fuzzy logic, Fuzzy logic, *f*-Transformation, *S*-answer, Validity
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1 Introduction

Fuzzy logic is a fascinating and dynamic field of study and research that pertains to a broadening of perspectives on the basis of information and the philosophy of science. Over the years, it has advanced often rapidly and at times slowly from theoretical to practical aspects. There are different mathematical tools to study fuzzy logic to secure a mathematical foundation, such as set theory and possibility theory (Zadeh, 1965), as well as algebraic logic (Hájek, 2006). In this paper, we consider fuzzy logic based on fuzzy set theory.

Fuzzy logic considers approximate reasoning in its framework; however, fuzzy logic is itself a precise logic due to the related definitions of membership functions and generalized constraints (Zadeh, 2009). What remains widely unrecognized is that, results in fuzzy logic carry a necessary condition of being provably valid. Yet, many real world applications do not satisfy this condition. Therefore, the validity term is added to fuzzy logic to meet this condition, which leads to extended fuzzy logic (FLe).

In fact, by surrounding the given problem with severe conjectures, the model of the problem becomes necessarily limited. However, considering those conjectures in a wider scene by combining logical and pragmatic facets of the given problem, we can capture more uncertainty. This can be done within FLe (Sabahi and Akbarzadeh-T, 2013).

FLe is a combination of precisiated fuzzy logic (FLp) and unprecisiated fuzzy logic (FLu). The unprecisiated viewpoint lowers the standards of precision in FLp explanation, and thus FLe emerges from FLp (Zadeh, 2009). In fact, FLe has two different modules: the first deals with truth values and the second deals with information states (Sabahi and Akbarzadeh-T, 2013). Since truth values cannot go along with uncertainty (Dubois and Prade, 2012), each module requires to be expressed by different sets (Sabahi and Akbarzadeh-T, 2013). In other

words, the module that deals with truth functionality of a given problem can be expressed by conjunction sets because the role played by truth values is gradual, and conjunction sets (Dubois and Prade, 2012) precisely represent objective entities. The module that deals with the given problem, however, can be expressed by disjunction sets because the problem may relate to incomplete information, and disjunction sets (Dubois and Prade, 2012) can characterize incomplete information.

To the best of our knowledge, few studies have been performed to develop a theoretical description of FLe. Niskanen (2009; 2010) considered probability and statistical explanation, as well as the degree of confirmation in the context of explanatory hypotheses based on FLe. Niskanen (2012) studied reasoning by different forms including inductive and deductive. It was shown that reasoning based on FLe leads to more (approximate) conclusions in comparison with the standpoint of science philosophy. Wilke (2009) focused on FLe to fuzzify a version of David Hilbert's axiomatic logical calculus and prompted fuzzy geometry, which was used later by Aliev *et al.* (2010) to propose an approach based on FLe to deal with unprecisiated information in decision making. In addition, Imran and Beg (2011) used a set of appropriate exponential functions to develop a kind of fuzzy geometry based on FLe with respect to its classical geometry prototype. Imran and Beg (2012) extended this work from the standpoint of *f*-similarity and *f*-validity for the identification process.

To design a gadget based on FLe, Tolosa and Guadarrama (2010) considered non-expert user's perception and its confidence as a cloud of points, which leads to a suitable approximation being selected for fuzzy sets in the design. Sabahi and Akbarzadeh-T (2013) provided a qualified description of FLe by giving definitions and proving theorems about the different aspects of the basic tools of FLe, such as validity and *f*-transformation. Approximate reasoning (Raha and Ray, 1999) within FLe was also considered by Sabahi and Akbarzadeh-T (2013). Niskanen (2013) considered a regression analysis based on FLe philosophy, applied to a qualitative research. In Sabahi and Akbarzadeh-T (2014a), a medical application of FLe-based reasoning was applied to the actual database of coronary heart disease. Sabahi and Akbarzadeh-T (2014b) introduced a new distribution based on FLe, which is a combination of probability, possibility, and validity.

Although most of the above studies focus on the various aspects of FLe, there is a need to describe the existing challenges of FLe in mathematical detail. In fact, to gain a clear description of what FLe's structural explanation aspires, we need to interpret the intricate mathematical aspects of FLe's modules in uncertainty management and reasoning. Thus, in this paper, we consider various constitutive behaviors within FLe and mathematically justify those conditions that arise from these behaviors. In fact, our objective is to present a framework for the analysis of FLe's reasoning and a mathematical interpretation of FLe by proposing several theorems that consider essential terms such as validity.

More specifically, in this paper, the existence of structural isomorphism between the answers and their validities is proved. This isomorphism structure allows defining scalar measures even in the presence of incomplete information. We also propose a representation theorem expressing that, through *f*-transformation, an unsolvable problem over C can be associated with a set of solutions based on the relevant rules and constraints. We take advantage of this notion to prove the rationality of FLe-based reasoning. We also show that in the presence of incomplete information, the solutions obtained are near the true ones when applying FLe.

The notations used in this paper are listed in Table 1.

2 Main concepts of FLe and its basic terminology

Reasoning based on extended fuzzy logic, as described by Zadeh (2009), is likened to "a venture into unchartered territory", in which exploration is based mainly on quasi-mathematical interpretation. FLe is especially valuable when just far less precise data is available. In FLe, membership functions and generalized constraints are determined mainly based on perception; i.e., they are not precisiated (Zadeh, 2009). To describe the framework, we define briefly the main concepts. For details, readers can refer to Zadeh (2009) and Sabahi and Akbarzadeh-T (2013).

Table 1 Notations used in this paper

Notation	Description	Notation	Description
$\|\cdot\|$	Absolute value	$f\text{-}s_i(P)\|_v$	f-Answer with validity v respect to P
$\|\|\cdot\|\|$	Norm	$f\text{-}s_i(Q)\|_v$	f-Answer with validity v respect to Q
Q	A-granule	$F_Q(S_i)$	Functional of S_i with respect to Q
d, δ, ρ	Positive quantities	$S_i(P)$	Solution (answer) of the problem in the C-space
P	Problem	$S_i(Q)$	Solution (answer) in the C-space with respect to Q
f-C	f-Transform of C	$s_i(P)$	S-answer of the problem
f-=	Approximately equal	$s_i(Q)$	S-answer with respect to Q
v	Validity	$=\|_{v'}$	Equal to a degree v'

2.1 *f*-Transformation

f-Transformation is defined as a map of one-to-many, in which the main characteristics of 'one' need to be preserved in the *f*-transformation process. Zadeh (2009) used geometric shapes such as lines, triangles, and circles to explain this phenomenon. As shown in Fig. 1, three geometric shapes sprayed by hand with a spray pen show the *f*-transformation process. A line is *f*-transformed to an *f*-line, a triangle to an *f*-triangle, and a circle to an *f*-circle. As can be seen from these examples, we can define *f*-transformation for any mapping of any structure, but a valid result is achieved if we keep the main features of the problem during the *f*-transformation process (Sabahi and Akbarzadeh-T, 2013). Note that we use this concept in the same way as Zadeh (2009) defined it, which is different from that introduced by Perfilieva (2006).

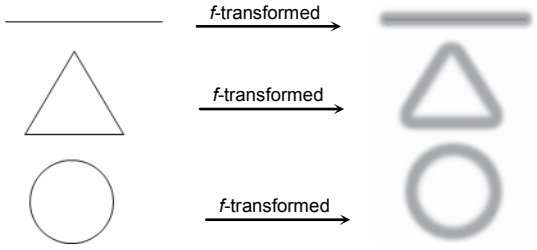

Fig. 1 Examples of *f*-transformation (Zadeh, 2009)

2.2 Validation principle

In FLe, solutions need to have a sufficient validity degree. Therefore, Zadeh (2009) considered the following validation principle: Letting p be a p-valid conclusion drawn from a chain of premises $p_1, p_2, \ldots, p_n$, then f-p is an f-valid conclusion drawn from f-p_1, f-$p_2, \ldots, f$-p_n, and f-p has a high validity index.

2.3 Cointensive assumption

The validation principle is based on this condition that if the contrary is not stated, *f*-transform is always considered cointensive; i.e., f-C is close-fitting to its true prototype, C (Zadeh, 2009). This condition is referred to as the cointensive assumption.

2.4 P/I principle

In spite of FLe's flexibility, computing the *f*-transform function is problematic. To explain this (Zadeh, 2009), let us consider h as a function, a functional, or an operator, where an *f*-transform f-C is an argument of h, i.e., $h(f\text{-}C)$. Then computing h is problematic. Fortunately, using the precisiation/imprecisiation principle (or P/I principle), we can provide an *f*-valid approximation. More specifically, we can compute f-$h(C)$ instead of $h(f\text{-}C)$ since based on the P/I principle, we have $h(f\text{-}C)$ f-= f-$h(C)$, where f-= means approximately equal (read it *f*-equal). For example, in Fig. 2, where h is a differentiation operator and f-C is an *f*-function, the *f*-derivative of f-C is *f*-function $f\text{-}\dfrac{\mathrm{d}C}{\mathrm{d}x}$ as indicated.

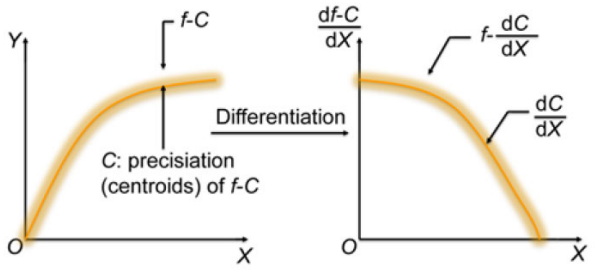

Fig. 2 Using the P/I principle for the derivative of an *f*-transform (Zadeh, 2009)

Note that using the P/I principle several times in one relation decreases the relation's validity value,

but the relation is still valid to a degree. This issue is described as the following remark:

Remark 1 (Sabahi and Akbarzadeh-T, 2013) $\underbrace{f\text{-}f\text{-}\ldots f\text{-}}_{n}A\Big|_{v_n}(f\text{-}=)\ldots(f\text{-}=)f\text{-}f\text{-}A\big|_{v_2}(f\text{-}=)f\text{-}A\big|_{v_1}$, then $v_n<\ldots<v_2<v_1$, where A and v represent a relation and validity index, respectively, and n is the number of times using the P/I principle.

To illustrate the above remark, note that if we denote approximate equality with f-=, we can write $\underbrace{f\text{-}f\text{-}\ldots f\text{-}}_{n}A\Big|_{v_n}(f\text{-}=)\ldots(f\text{-}=)f\text{-}f\text{-}A\big|_{v_2}(f\text{-}=)f\text{-}A\big|_{v_1}$. In the f-C space, we can consider approximate equality as equality to a degree v', i.e., $(f\text{-}=)\overset{\text{def}}{\equiv}(=\big|_{v'})$; therefore, we can write

$$\underbrace{f\text{-}f\text{-}\ldots f\text{-}}_{n}A(=\big|_{v_n'})\ldots(=\big|_{v_2'})f\text{-}f\text{-}A(=\big|_{v_1'})f\text{-}A. \quad (1)$$

Eq. (1) can be rewritten as $\underbrace{f\text{-}f\text{-}\ldots f\text{-}}_{n}A\Big|_{v_n}(f\text{-}=)\ldots(f\text{-}=)f\text{-}f\text{-}A\big|_{v_2}(f\text{-}=)f\text{-}A\big|_{v_1}$. As the P/I principle has been used n times in the relation A, it is clear that $v_n<\ldots<v_2<v_1$.

2.5 *S*-answer

In FLe, we have two types of answers: *S*-answer and *f*-answer (Sabahi and Akbarzadeh-T, 2013). An *S*-answer could be considered the centroid of a spray pen that is chosen based on the validity index, and an *f*-answer could be expressed by a conceptual visual data entry named *Z*-mouse (Zadeh, 2010). As a matter of fact, an *f*-answer comprises many *S*-answers, and an *S*-answer is the centroid (precisiated) of the relevant *f*-answer, one that is chosen as the outcome of the given problem (Fig. 3). For example, suppose we intend to define a circle. The *f*-answer then contains different geometric shapes including different polygons and ellipses, while the relevant *S*-answer is the one we choose as substitute for a circle. In fact, an *S*-answer has two degrees, which are not known a priori and operationally differ from each other. *S*-answer's validity degree expresses the validity value of the framework within which the given problem can be treated and the other degree expresses the method's ability to handle incomplete information. In the mentioned example, the former degree relates to how much the chosen object is similar to the circle and the latter degree relates to the conditions of choosing objects to build the *f*-answer that links to the possibility and/or probability degree.

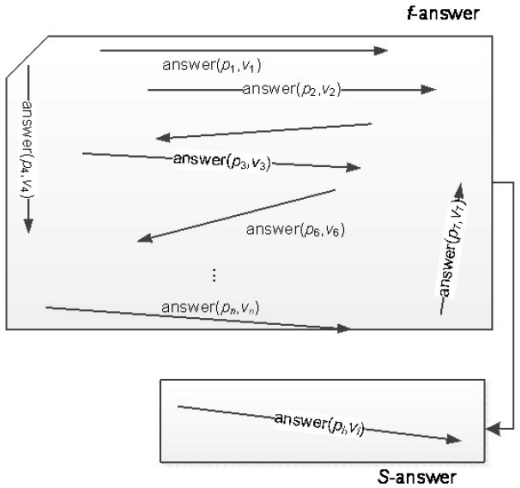

Fig. 3 Definition of *S*-answers and *f*-answers (p_i and v_i represent possibility and validity degrees, respectively)

2.6 A-granule

Let us first define a granule. According to Zadeh (1997), a granule is a group of objects drawn together by indistinguishability, similarity, proximity, or functionality. In *f*-*C*, we define a new concept named A-granule:

Definition 1 (A-granule) A-granule Q is the smallest granule that contains the *f*-transform of the object P such that $f\text{-}P\ f\text{-}=Q$.

We can consider an A-granule as a well-defined *f*-transform. In fact, due to existing uncertainty in the given problem P, we recognize some uncertainty about its *f*-transform's boundaries. Therefore, we can assume that there is a granule that can become large enough to include the complete information about the *f*-transform, yet small enough to exclude irrelevant information about the *f*-transform; hence, we can call this granule an A-granule. For example, suppose there is a triangular membership function over [0.2, 0.4] with a height of 0.5. Then its A-granule containing its *f*-transform can be defined as an interval over [0, 1] with a height of 1. The relationship among P, f-P, and Q is described in Fig. 4.

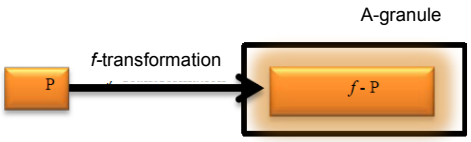

Fig. 4 A-granule *Q* which contains the *f*-transform of *P*

Note that we can consider an A-granule as a tracer attached to the *f*-transform, so that we can now use scalar measures such as distance measures even in the presence of incomplete information. Note that when we do not have complete information, we cannot define scalar measures (Dubois and Prade, 2012; Sabahi and Akbarzadeh-T, 2013).

The following conditions confirm that an A-granule and its relevant *f*-transform differ just slightly:

1. The problem is based on the realities; therefore, its *f*-transform reflects those realities, and the relevant A-granule represents the smallest granule that contains the entire *f*-transform.

2. The validity measure controls the amount of deviation from full validity.

3. The maximum validity is expected due to the validation principle and the cointensive assumption.

3 Relationship between problems in *C* and answers in *f-C*

At first, suppose we have a rule that can be considered as 'if solution is *f-D* then validity is *f-G*', where *f-D* represents the possibility of the solution and *f-G* represents the possibility of validity. Since the conjunction-based model of rules can be interpreted by a Cartesian product (Dubois and Prade, 1996), we have $f\text{-}D\times f\text{-}G$ which is approximately equal to $f\text{-}(D\times G)$ based on the P/I principle. In addition, let the rule 'if x is $f\text{-}A$ then y is $f\text{-}B$' represent the solution. Then

$$\begin{cases} H_1\text{: } x \text{ is } f\text{-}A \overset{\text{using P/I}}{\cong} f\text{-}(x \text{ is } A); \\ H_2\text{: } y \text{ is } f\text{-}B \overset{\text{using P/I}}{\cong} f\text{-}(y \text{ is } B). \end{cases} \tag{2}$$

We assume that the possibility distribution is known a priori; therefore, the likelihood of the rule can be described by possibility and necessity measures. Any approximation of H: $H_1\times H_2$ based on the measures is valid if it is encapsulated in the $f\text{-}(D\times G)$.

With the following theorems, we explain the relationship between problems in *C* and their answers that are now computed in *f-C*. Theorems 1 and 2 indicate that if *S*-answers in the A-granule are similar, the equivalent answers to the problem in *f-C* would be similar as well.

Theorem 1 Suppose Q is an A-granule, $f\text{-}s_i(P)$ the *f*-answer of problem P, s_i the *S*-answer, v_{s_i} a validity measure, d a bounded positive quantity, $f\text{-}s_i(P)\big|_{v_{s_i(P)}} = s_i(P)$, i=1, 2 and $\|s_1(Q)-s_2(Q)\|<d$. Then $\|s_1(P)-s_2(P)\|<f\text{-}d$, where $f\text{-}d$ is the *f*-transform of d and is bounded and positive.

Proof We know

$$\| s_1(P)-s_2(P)\| = \| f\text{-}s_1(P)\big|_{v^1_{s_1(P)}} - f\text{-}s_2(P)\big|_{v^1_{s_2(P)}} \|. \tag{3}$$

By the P/I principle, we have $f\text{-}s(P)$ $f\text{-}= s(f\text{-}P)$. Hence, with regard to Remark 1, we have

$$\begin{aligned} &\| s_1(P)-s_2(P)\| \\ &= \| s_1(f\text{-}P)\big|_{v^2_{s_1(P)}} - s_2(f\text{-}P)\big|_{v^2_{s_2(P)}} \| \\ &= \| s_1(f\text{-}Q)\big|_{v^2_{s_1(P)}} - s_2(f\text{-}Q)\big|_{v^2_{s_2(P)}} \| \\ &= \| f\text{-}s_1(Q)\big|_{v^3_{s_1(P)}} - f\text{-}s_2(Q)\big|_{v^3_{s_2(P)}} \|, \end{aligned} \tag{4}$$

where $v^1_{s_i} > v^2_{s_i} > v^3_{s_i}$. Note that in the above equation, we substitute $f\text{-}P$ $f\text{-}= Q$ with $f\text{-}P = f\text{-}Q$. Then,

$$\begin{aligned} \left\|s_1(P)-s_2(P)\right\| &= \left\| f\text{-}(s_1(Q)-s_2(Q))\big|_{v^4} \right\| \\ &= f\text{-}\left\|s_1(Q)-s_2(Q)\right\|_{v^5}, \end{aligned} \tag{5}$$

where v^4 is less than $v^3_{s_1(P)}$ and $v^3_{s_2(P)}$ but more than v^5. Since $\|s_1(Q)-s_2(Q)\|<d$, we conclude

$$\left\|s_1(P)-s_2(P)\right\| < f\text{-}d. \tag{6}$$

Note that $f\text{-}d$ is not definite, but due to the cointensive assumption, since d is a bounded positive quantity, $f\text{-}d$ would be bounded and positive as well.

As a consequence of Theorem 1, when complete information is not available, the obtained solution $s_i(P)$ is in the vicinity of the one gained with complete information, denoted by $s_i(Q)$.

Theorem 2 Suppose Q is an A-granule, v a validity measure, $f\text{-}s_i(P)$ the f-answer of problem P, s_i the S-answer, $f\text{-}s_i(P)\big|_{v_{s_i(P)}} = s_i(P)$, S_i the solution in the C-space, $F_Q(S_i)$ the functional of S_i with respect to Q such that $F_Q(S_i) = f\text{-}s_i(Q)\big|_{v_{s_i(Q)}} = s_i(Q)$, and $v(\|F_Q(S_1) - F_Q(S_2)\| < d) > \delta$, where i=1, 2, and δ, d are positive quantities and are bounded. Then there exists a positive quantity ρ different from δ such that $v(\|s_1(P) - s_2(P)\| < d) > \rho$.

Proof We have

$$\begin{aligned} & v\left(\|s_1(P) - s_2(P)\| < d\right) \\ & = v\left(\left\| f\text{-}s_1(P)\big|_{v^1_{s_1(P)}} - f\text{-}s_2(P)\big|_{v^1_{s_2(P)}} \right\| < d\right) \\ & = v\left(\left\| s_1(f\text{-}P)\big|_{v^2_{s_1(P)}} - s_2(f\text{-}P)\big|_{v^2_{s_2(P)}} \right\| < d\right) \\ & = v\left(\left\| f\text{-}s_1(Q)\big|_{v^3_{s_1(P)}} - f\text{-}s_2(Q)\big|_{v^3_{s_2(P)}} \right\| < d\right) \\ & = v\left(\left\| F_Q(S_1) - F_Q(S_2) \right\|_{v'} < d\right) \\ & > \rho. \end{aligned} \tag{7}$$

Due to using the P/I principle several times, based on Remark 1, we have $v^1_{s_i(P)} > v^2_{s_i(P)} > v^3_{s_i(P)}$; therefore, ρ is different from δ. However, the main aspects of $\|\cdot\|$, d, and δ are to be positive, and we are committed to the cointensive assumption. Thus, ρ>0.

As a consequence of the above theorem, a structural isomorphism exists between validities and answers within the same context. Therefore, by considering the validity index, scalar measures can be used to evaluate a problem even in the presence of incomplete information.

4 Mathematical study of FLe-based reasoning

Now we discuss the rationality of reasoning based on FLe from a mathematical perspective. Suppose that the given problem P is solvable. If we can define a function (solution) like $S(P)$ for the problem P on C, then f-transformation produces a function $s(P)$ defined on $f\text{-}s(P)$ in $f\text{-}C$ that is over C; we have $s(P)f\text{-}=S(P)$. When uncertainty disappears, we have $s(P)=S(P)$. Now, if we prove that $s(P)$ exists, then we can easily obtain $S(P)$.

If we assume the availability of complete information, we may regard $f\text{-}s$ as a set of functional F's as they share the main properties of problem P, which remain preserved due to the cointensive assumption. We can symbolize this as follows, considering Q as an A-granule defined for the problem P in $f\text{-}C$:

$$F_Q(S) = f\text{-}s(Q)\big|_{v_{S(Q)}} = s(Q), \tag{8}$$

where $v_{S(Q)}$ represents the degree to which the above equality holds. In fact, s is characterized as precisiation of $f\text{-}s$ by a membership function, which is distributed over $f\text{-}s$ and expresses the validity of the solutions. We can consider this validity membership function as an f-validity measure.

Now we consider the similarity to S. We can find a pair of S_1 with respect to s_1 (where $f\text{-}s_1$ is an f-transform of S_1) and S_2 with respect to s_2 (where $f\text{-}s_2$ is an f-transform of S_2). Both S_1 and S_2 resemble S as they are the possible solutions of the given problem by considering a different perspective.

Since problem P is a model of the realities and the A-granule contains the complete information of its f-transform, we can conclude that $\|F_Q(S_1) - F_Q(S_2)\|$ is bounded, where $F_Q(S_1) = f\text{-}s_1(Q)\big|_{v_{S_1(Q)}}$ and $F_Q(S_2) = f\text{-}s_2(Q)\big|_{v_{S_2(Q)}}$. Moreover, there exist positive quantities d and δ, where d is bounded such that

$$v(\| F_Q(S_1) - F_Q(S_2) \| < d) > \delta. \tag{9}$$

Without loss of generality, we can assume that d represents the distance between functions S_1, S_2, and S in terms of similarity considering the condition of preserving the main characteristics of S in S_1, S_2. This assumption is reasonable due to the definition of the A-granule. According to Theorem 2, there exists a positive quantity ρ such that

$$v(\| s_1(P) - s_2(P) \| < d) > \rho. \tag{10}$$

Note that we can find a function S_i with respect to s_i. Due to the definition of the A-granule, we have

$$\delta < v(\| F_Q(S_1) - F_Q(S_i) + F_Q(S_i) - F_Q(S_2) \| < d). \tag{11}$$

Considering a metric space M with metric v, we know that the scalar measures satisfy the triangle inequality $v(x, z)\leq v(x, y)+v(y, z)$ for all x, y, z in M. Therefore, considering $|\cdot|$ as absolute value, we have

$$\begin{aligned}&\left\|F_Q(S_1)-F_Q(S_i)+F_Q(S_i)-F_Q(S_2)\right\|\\&\leq\left\|\left\|F_Q(S_1)-F_Q(S_i)\right\|\right\|+\left\|\left\|F_Q(S_i)-F_Q(S_2)\right\|\right\|.\end{aligned}\tag{12}$$

On the other hand, since $d>0$ and $\|F_Q(S_1)-F_Q(S_i)+F_Q(S_i)-F_Q(S_2)\|<d$, there exists a bounded positive quantity d' such that

$$\begin{aligned}&\left\|F_Q(S_1)-F_Q(S_i)+F_Q(S_i)-F_Q(S_2)\right\|\\&<\left\|\left|F_Q(S_1)-F_Q(S_i)\right|\right\|+\left\|\left|F_Q(S_i)-F_Q(S_2)\right|\right\|<d'.\end{aligned}\tag{13}$$

Hence, there exist bounded positive quantities d_1 and d_2 such that $d'\leq d_1+d_2$:

$$\begin{aligned}&\left\|\left|F_Q(S_1)-F_Q(S_i)\right|\right\|+\left\|\left|F_Q(S_i)-F_Q(S_2)\right|\right\|<d'\\&\equiv\left\|\left|F_Q(S_1)-F_Q(S_i)\right|\right\|<d_1\ \&\ \left\|\left|F_Q(S_i)-F_Q(S_2)\right|\right\|<d_2.\end{aligned}\tag{14}$$

According to Eq. (11), we can find a positive bounded quantity δ' such that

$$\delta'<v\left(\left\|\left|F_Q(S_1)-F_Q(S_i)\right|\right\|+\left\|\left|F_Q(S_i)-F_Q(S_2)\right|\right\|<d'\right).\tag{15}$$

In addition, regarding Eq. (14), we can assume that there exist bounded quantities δ_1 and δ_2 such that

$$\begin{cases}\delta_1<v\left(\left\|\left|F_Q(S_1)-F_Q(S_i)\right|\right\|<d_1\right),\\\delta_2<v\left(\left\|\left|F_Q(S_i)-F_Q(S_2)\right|\right\|<d_2\right).\end{cases}\tag{16}$$

According to Eq. (8), we have

$$\begin{cases}\delta_1<v\left(\left\|\left|s_1(Q)-s_i(Q)\right|\right\|<d_1\right),\\\delta_2<v\left(\left\|\left|s_i(Q)-s_2(Q)\right|\right\|<d_2\right).\end{cases}\tag{17}$$

Since $s_i(P)$ relates to incomplete information, we use a validity measure to define a distance measure for it. Regarding Theorem 2, there exist bounded quantities ρ_1 and ρ_2 such that

$$\begin{cases}\rho_1<v\left(\left\|\left|s_1(P)-s_i(P)\right|\right\|<d_1\right),\\\rho_2<v\left(\left\|\left|s_i(P)-s_2(P)\right|\right\|<d_2\right).\end{cases}\tag{18}$$

Now, note that at least one of ρ_i (i=1, 2) in Eq. (18) should be positive with a reasonable value to make Eq. (15) hold. Regarding this notion and due to the structural isomorphism as discussed in Section 3, one of the following equalities holds or both hold:

$$\left\|\left|s_1(P)-s_i(P)\right|\right\|<d_1,\tag{19}$$

$$\left\|\left|s_i(P)-s_2(P)\right|\right\|<d_2.\tag{20}$$

Using Eq. (16), we can form a chain of $s_i(P)$, for which Eq. (19) or (20) holds or both hold. These related/attached $s_i(P)$'s construct f-answer f-$s(P)$. Now we can compute the centroid of f-$s(P)$ based on the highest validity degree, that is $s(P)$; hence, its prototype $S(P)$ exists. We might naturally call $s(P)$ the solution $S(P)$, where $s(P)$ f-= $S(P)$. Note that through the validity index, $s(P)$ becomes precisiated.

Now, to justify that the S-answer (i.e., $s(P)$) and the actual solution of problem P (i.e., $S(P)$) can be regarded as equal, we can write $v(s(P)=S(P))=\min(v(s(P)\subseteq S(p)),\ v(s(P)\supseteq S(P)))$. Since $S(P)$ is the real solution and based on the above discussion, we have $v(s(P)\subseteq S(p))=1$ and $v(s(P)=S(P))=v(s(P)\supseteq S(P))$. Now, if we assume that there is another answer $s'(P)$ such that $s'(P)\subseteq s(P)$, then intuitively and regarding the cointensive assumption and noticing that $s(P)$ is achieved based on the highest validity degree, $v(s(P)\supseteq S(P))\geq v(s'(P)\supseteq S(P))$. Hence, $v(s(P))>v(s'(P))$. Therefore, $s(P)$ is the best approximation of $S(P)$ in terms of similarity. The above discussion is summarized in the following representation theorem:

Theorem 3 For a problem P on C through f-transformation, there exists a function $s(P)$ defined in f-C that is the centroid (precisiated) of the f-transform of $S(P)$, where $S(P)$ is the solution of problem P on C and $s(P)$ is the best approximation of $S(P)$ in terms of similarity.

Note that when a problem is provably valid, the map would be one-to-one, and the proof is trivial. In the above discussion, an unsolvable case is just considered in the sense of not being provably valid.

5 Conclusions

In this paper, for the first time, a mathematical description of FLe-based reasoning has been considered. The issues related to FLe are discussed and a new tool named the A-granule is introduced. An A-granule is the smallest granule retaining the entire *f*-transform's characteristics. We also prove that within FLe, in the presence of incomplete information, we can reach answers that are in the neighborhood of the true ones and that this vicinity is addressed by a validity degree. Then we suggest that there is a structural isomorphism between answers and their validities in the sense of *f*-valid philosophy. This point is used to mathematically address the rationality of FLe-based reasoning.

As a consequence of this study, in FLe, by trading accuracy for speed and simplicity, we deal with the kinds of solutions that are ill-posed; i.e., only to a degree are the solutions and their validities satisfactory. These kinds of solutions may reflect a sense of doubt, but it should be noted that, as Zadeh (2009) stated, an approximate (*f*-valid) solution based on the realistic model is more valuable when we are forced to consider the unrealistic model in order to achieve a *p*-valid solution.

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