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Switching-based stabilization of aperiodic sampled-data Boolean control networks with all subsystems unstable*

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Abstract: We aim to further study the global stability of Boolean control networks (BCNs) under aperiodic sampled-data control (ASDC). According to our previous work, it is known that a BCN under ASDC can be transformed into a switched Boolean network (SBN), and further global stability of the BCN under ASDC can be obtained by studying the global stability of the transformed SBN. Unfortunately, since the major idea of our previous work is to use stable subsystems to offset the state divergence caused by unstable subsystems, the SBN considered has at least one stable subsystem. The central thought in this paper is that switching behavior also has good stabilization; i.e., the SBN can also be stable with appropriate switching laws designed, even if all subsystems are unstable. This is completely different from that in our previous work. Specifically, for this case, the dwell time (DT) should be limited within a pair of upper and lower bounds. By means of the discretized Lyapunov function and DT, a sufficient condition for global stability is obtained. Finally, the above results are demonstrated by a biological example.

Key words: Aperiodic sampled-data control; Boolean control networks; Unstable subsystem; Discretized Lyapunov function; Dwell time
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1 Introduction

The Boolean network (BN) is a class of discrete-time systems, whose nodes can take only “1” or “0” at each time. A series of logical functions are used to update the states of these nodes. Kauffman (1969) first used a BN to quantify the interactions of gene regulatory systems. The interactions among diverse genes complicate living cells. Genes collectively (as a network) act on the production of cells, tissues, and organisms. Very often, instead of studying one single gene, the global (holistic) behavior of a genetic network is more crucial and significant. Actually, many formalisms of learning genetic regulatory networks have been proposed. Among such mathematical models, BNs have received much attention. Using BNs, one can study and investigate many practical questions related to the complex behavior of large genetic networks in real biological systems.

BNs have been extensively studied in recent years. Apart from their broad applications in systems biology (Akutsu et al., 1999; Shmulevich et al., 2002a, 2002b), other applications such as game theory (Cheng, 2014; Ding XY et al., 2017), cryptography (Lu et al., 2018a, 2018c), modeling of systems of machines (Torres et al., 2018a, 2018b), and correlated default risk (Gu et al., 2013) can be found in the literature.

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In particular, BNs with logical expressions can be translated into algebraic forms by means of a semi-tensor product (STP) of matrices (Cheng and Qi, 2010; Cheng et al., 2011a). This method also has many applications (Lu et al., 2017; Li HT et al., 2018). All information about the logic and structure is contained in the corresponding network transition matrices. Based on this powerful tool, research findings on the control field of BNs emerge continually, ranging from controllability (Ding Y et al., 2017; Tong et al., 2018; Zhong et al., 2019), observability (Yu et al., 2019), optimal control (Wu and Shen, 2017; Zhu QX et al., 2018), output tracking (Li YY et al., 2019), synchronization (Zhong et al., 2016; Yang et al., 2019), and reachability (Guo, 2018), to disturbance decoupling (Liu et al., 2017; Li BW et al., 2019a, 2019b). A Boolean control network (BCN) is a BN with binary inputs.

Stability and stabilization are important research fields of BNs (BCNs), and they have been deeply studied (Cheng et al., 2011b; Li R et al., 2014; Guo et al., 2015; Li BW et al., 2018; Li YY et al., 2018; Liu RJ et al., 2018, 2019; Sun et al., 2018; Zhu SY et al., 2018; Li HT et al., 2019). Most of the above results are obtained using STP and the matrix expression of logic. Wang and Li (2012) and Li and Wang (2017) constructed Lyapunov functions for BNs and presented the Lyapunov-based stability analysis for BNs. After that, the Lyapunov function has been effectively adopted to research the stability and stabilization problems of BNs (BCNs). For example, by designing a co-positive Lyapunov function, the weighted l_1-gain analysis was considered, and then the l_1 model reduction problem for BCNs was studied in Meng et al. (2016). Meng et al. (2017b) analyzed stability and l_1 gain of BNs with Markovian jump parameters. For time-dependent switched Boolean networks (SBNs), Meng et al. (2017a) considered their stability and guaranteed the cost. A BCN under aperiodic sampled-data control (ASDC) was converted into an SBN in Lu et al. (2018b). At the same time, by means of the switching-based Lyapunov function, global stability was investigated. Based on the control Lyapunov function, Li and Ding (2019) investigated the feedback stabilization problem of logical control networks. It is noteworthy that SBNs considered in Meng et al. (2017a) and Lu et al. (2018b) contain at least one stable subsystem. To the best of our knowledge, there is no work on the global stability of SBNs containing all unstable subsystems. Moreover, how to design a switching policy which can make a switching system with all subsystems unstable achieve stability has always been a challenging issue.

Based on the above statements, we consider the global stability of SBNs containing all unstable subsystems. In Meng et al. (2017a) and Lu et al. (2018b), using the Lyapunov function and the method of average dwell time (DT), some sufficient conditions for global stability of the SBN with at least one stable subsystem were obtained. The main idea of this method is to use stable subsystems to offset the state divergence caused by unstable subsystems. That is to say, the existence of stable subsystems ensures the global stability of the SBN. Therefore, the above method is not applicable to the situation in which all subsystems are unstable. As is well known, all the subsystems being unstable does not mean that the switched system must be unstable. According to Xiang and Xiao (2014), Feng et al. (2017), and Liu Z et al. (2018), even if all subsystems are unstable, the switched system can be stabilized with appropriate switching laws designed, which makes a switched system carefully switch between unstable subsystems. Here, to achieve the global stability of an SBN containing all unstable subsystems, the DT should be limited within a pair of upper and lower bounds, and the discretized Lyapunov function is used.

We summarize our main contributions as follows: First, the proposed method can not only solve the global stability problem of the BCN under ASDC when all subsystems of the transformed SBN are unstable, but also adapt to work on the global stability of SBNs with all subsystems unstable. Thus, the problems left in Meng et al. (2017a) and Lu et al. (2018b) are solved. Second, compared with direct research on the global stability of SBNs with all unstable subsystems, the problem considered in this study is more complex. Although we transform this problem into studying the global stability of the transformed SBN with all subsystems unstable, the switching instant must be the sampling instant of the original BCN under ASDC. Therefore, the construction of the Lyapunov function and the definition of DT are different from those in Xiang and Xiao (2014), Feng et al. (2017), Meng et al. (2017a), Liu Z et al. (2018), and Lu et al. (2018b).

2 Preliminaries

2.1 Notations and definitions

The basic notations used in this study are listed in Table 1.

Table 1 Basic notations

Notation	Description
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	Set of n-dimensional column vectors
$\boldsymbol{I}_k$	$k \times k$ identity matrix
$\boldsymbol{\mathcal{L}}_{m\times n}$	Set of all $m \times n$ logical matrices

In addition, denote $\mathcal{D} := \{1, 0\}$, besides $\mathcal{D}^n := \underbrace{\mathcal{D} \times \mathcal{D} \ldots \times \mathcal{D}}_{n}$. $\Delta_k := \left\{\boldsymbol{\delta}_k^1, \boldsymbol{\delta}_k^2, \ldots, \boldsymbol{\delta}_k^k\right\}$, where $\boldsymbol{\delta}_k^i$ is the i^{th} column of $\boldsymbol{I}_k$ with degree k. In particular, $\Delta := \Delta_2$ and $\Delta^n := \underbrace{\Delta \times \Delta \times \cdots \times \Delta}_{n}$. Here, $1 \sim \boldsymbol{\delta}_2^1$ and $0 \sim \boldsymbol{\delta}_2^2$. An $m \times n$ logical matrix $\boldsymbol{A}$ is defined as follows:

$$\boldsymbol{A} = [\boldsymbol{\delta}_m^{i_1}, \boldsymbol{\delta}_m^{i_2}, \ldots, \boldsymbol{\delta}_m^{i_n}],$$

where $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, m\}$, and for simplicity, $\boldsymbol{A} = \boldsymbol{\delta}_m[i_1, i_2, \ldots, i_n]$.

Let $\boldsymbol{A} \in \mathbb{R}^m$ and $\boldsymbol{B} \in \mathbb{R}^n$. Then $\boldsymbol{W}_{[m,n]}\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$, where $\boldsymbol{W}_{[m,n]} = [\boldsymbol{I}_n \otimes \boldsymbol{\delta}_m^1, \boldsymbol{I}_n \otimes \boldsymbol{\delta}_m^2, \ldots, \boldsymbol{I}_n \otimes \boldsymbol{\delta}_m^m]$ is an $mn \times mn$ swap matrix. $\lfloor x \rfloor = \max\{n \in \mathbb{Z} | n \leq x, x \in \mathbb{R}\}$.

Let $\boldsymbol{x} = \boldsymbol{x}_1\boldsymbol{x}_2 \cdots \boldsymbol{x}_n$ with $\boldsymbol{x}_i \in \Delta$, $i = 1, 2, \ldots, n$. Then $\boldsymbol{x}^2 = \boldsymbol{\Phi}_n\boldsymbol{x}$, where $\boldsymbol{\Phi}_n = \boldsymbol{\delta}_{2^{2n}}[1, 2^n+2, 2 \times 2^n + 3, \ldots, (2^n - 2) \cdot 2^n + 2^n - 1, 2^{2n}]$.

Definition 1 The STP of two matrices $\boldsymbol{A} \in \boldsymbol{M}_{m\times n}$ and $\boldsymbol{B} \in \boldsymbol{M}_{p\times q}$ is defined as (Cheng et al., 2011a)

$$\boldsymbol{A} \ltimes \boldsymbol{B} = (\boldsymbol{A} \otimes \boldsymbol{I}_{\alpha/n})(\boldsymbol{B} \otimes \boldsymbol{I}_{\alpha/p}),$$

where $\otimes$ is the tensor (or Kronecker) product and $\alpha = \text{lcm}(n, p)$ is the least common multiple of n and p.

In this study, symbol $\ltimes$ can be omitted, since there is no confusion.

Lemma 1 Let $f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) \in \mathcal{D}$ be an n-ary logical function. It can be expressed in a multi-linear form as follows (Cheng et al., 2011a):

$$f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) = \boldsymbol{M}_f\boldsymbol{x}_1\boldsymbol{x}_2 \ldots \boldsymbol{x}_n,$$

where the structure matrix $\boldsymbol{M}_f \in \boldsymbol{\mathcal{L}}_{2\times 2^n}$ is determined uniquely.

2.2 Converting a BCN under ASDC into an SBN

Consider the following BCN under ASDC:

$$\begin{cases} X(t+1) = f\left(X(t), U(t)\right), \\ U(t) = e\left(X(t_k)\right),\ t_k \leq t < t_{k+1}, \end{cases} \tag{1}$$

where $X(t) \in \mathcal{D}^n$ is the state variable, $U(t) \in \mathcal{D}^m$ is the ASDC input variable, and t_k ($k = 0, 1, \ldots$) are sampling instants. The mappings $f : \mathcal{D}^{n+m} \to \mathcal{D}^n$ and $e : \mathcal{D}^n \to \mathcal{D}^m$ are logical functions.

Using Lemma 1, we can represent logical functions f and e by their unique structure matrices $\boldsymbol{M}$ and $\boldsymbol{E}$, respectively. Here $X(t)$ and $U(t)$ are represented by their vector forms $\boldsymbol{x}(t) \in \Delta_{2^n}$ and $\boldsymbol{u}(t) \in \Delta_{2^m}$, respectively. System (1) is given as follows:

$$\boldsymbol{x}(t+1) = \boldsymbol{M}\boldsymbol{u}(t)\boldsymbol{x}(t), \tag{2}$$

$$\boldsymbol{u}(t) = \boldsymbol{E}\boldsymbol{x}(t_k),\ t_k \leq t < t_{k+1}, \tag{3}$$

where $\boldsymbol{M} \in \boldsymbol{\mathcal{L}}_{2^n\times 2^{n+m}}$ and $\boldsymbol{E} \in \boldsymbol{\mathcal{L}}_{2^m\times 2^n}$.

Denote $h_k \triangleq t_{k+1} - t_k$ as the k^{th} sampling interval, where $h_k \in Z_h \triangleq \{i_1, i_2, \ldots, i_l\}$ ($i_1 < i_2 < \cdots < i_l$) and i_j ($j = 1, 2, \ldots, l$) are positive integers. According to Lu et al. (2018b), system (2) under ASDC (Eq. (3)) can be translated into an SBN, which can be described as follows:

$$\begin{aligned} \boldsymbol{x}(t_{k+1}) &= (\boldsymbol{M}\boldsymbol{W}_{[2^n,2^m]})^{h_k}\boldsymbol{x}(t_k)\boldsymbol{\Phi}_m^{h_k-1}\boldsymbol{u}(t_k) \\ &= (\boldsymbol{M}\boldsymbol{W}_{[2^n,2^m]})^{h_k}(\boldsymbol{I}_{2^n} \otimes \boldsymbol{\Phi}_m^{h_k-1}\boldsymbol{E})\boldsymbol{\Phi}_n\boldsymbol{x}(t_k) \\ &\triangleq \boldsymbol{F}_{\sigma(t_k)}\boldsymbol{x}(t_k), \end{aligned} \tag{4}$$

where the switching signal $\sigma(t_k) \in Z_\sigma \triangleq \{1, 2, \ldots, l\}$. Note that Eq. (4) switches only at the sampling instant, but the switch may not occur at every sampling instant. The corresponding details have been well discussed in Lu et al. (2018b). Thus, the switching time sequence is given below: $0 = t_0 = t_{k^{(0)}} < t_{k^{(1)}} < t_{k^{(2)}} < \cdots < t_{k^{(i)}} < t_n$, where $t_{k^{(j)}} \in \{t_0, t_1, \ldots, t_n\}$ ($j = 0, 1, \ldots, i$) and $t_0, t_1, \ldots, t_n$ are sampling instants.

3 Main results

In this section, we analyze the global stability of system (2) under Eq. (3). We consider the case in which all subsystems of the corresponding Eq. (4) are unstable.

Definition 2 System (2) is said to be globally stable at $\boldsymbol{x}_e$, if for any initial state $\boldsymbol{x}(0) \in \Delta_{2^n}$, the corresponding trajectory $\boldsymbol{x}(t)$ converges to $\boldsymbol{x}_e$.

Here, we assume $\boldsymbol{x}_e = \boldsymbol{\delta}_{2^n}^{2^n}$ (a coordinate transformation in Cheng et al. (2011a) can ensure this).

Lemma 2 System (2) is globally stable at $\boldsymbol{x}_e$ if and only if the corresponding Eq. (4) is globally stable at $\boldsymbol{x}_e$ and $\boldsymbol{\delta}_{2^n}^{2^n} = \boldsymbol{ME}\boldsymbol{\delta}_{2^n}^{2^n}\boldsymbol{\delta}_{2^n}^{2^n}$.

Here, we assume that system (2) satisfies $\boldsymbol{\delta}_{2^n}^{2^n} = \boldsymbol{ME}\boldsymbol{\delta}_{2^n}^{2^n}\boldsymbol{\delta}_{2^n}^{2^n}$.

Let $Z \triangleq \{k^{(0)}, k^{(1)}, \ldots, k^{(i-1)}\}$. Define

$$\tau_{k^{(j)}} = k^{(j+1)} - k^{(j)}$$

as the DT, where $j = 0, 1, \ldots, i-1$ and $\tau_{k^{(j)}} \in [\tau_{\min}, \tau_{\max}]$ with $\tau_{\min} = \inf_{k^{(j)} \in Z} \tau_{k^{(j)}}$ and $\tau_{\max} = \sup_{k^{(j)} \in Z} \tau_{k^{(j)}}$.

Remark 1 Note that the DT should be limited within a pair of upper and lower bounds to ensure global stability, since too small or too large DT may lead Eq. (4) containing all unstable subsystems to be unstable.

Definition 3 The set of vectors $\{\boldsymbol{\beta}_{a,q} | a \in Z_\sigma, q = 0, 1, \ldots, L\}$ is defined as a set of Lyapunov coefficients of Eq. (4) if $\forall r = 1, 2, \ldots, 2^n - 1$ and $\forall a, b \in Z_\sigma$, the following equations/inequalities are satisfied:

$$\boldsymbol{\beta}_{a,q}^{\mathrm{T}} \boldsymbol{\delta}_{2^n}^{2^n} = 0,\ q = 0, 1, \ldots, L, \tag{5}$$

$$\boldsymbol{\beta}_{a,q}^{\mathrm{T}} \boldsymbol{\delta}_{2^n}^{r} > 0,\ q = 0, 1, \ldots, L, \tag{6}$$

$$\Big[\boldsymbol{\beta}_{a,q}^{\mathrm{T}}(\boldsymbol{F}_a - \boldsymbol{I}) + \frac{1}{h}(\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}} - \boldsymbol{\beta}_{a,q}^{\mathrm{T}})\boldsymbol{F}_a - \lambda_a \boldsymbol{\beta}_{a,q}^{\mathrm{T}}\Big]\boldsymbol{\delta}_{2^n}^{r} < 0,\ q = 0, 1, \ldots, L-1, \tag{7}$$

$$\Big[\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}}(\boldsymbol{F}_a - \boldsymbol{I}) + \frac{1}{h}(\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}} - \boldsymbol{\beta}_{a,q}^{\mathrm{T}})\boldsymbol{F}_a - \lambda_a \boldsymbol{\beta}_{a,q+1}^{\mathrm{T}}\Big]\boldsymbol{\delta}_{2^n}^{r} < 0,\ q = 0, 1, \ldots, L-1, \tag{8}$$

$$\boldsymbol{F}_a \boldsymbol{\delta}_{2^n}^{2^n} = \boldsymbol{\delta}_{2^n}^{2^n}, \tag{9}$$

$$\Big[\boldsymbol{\beta}_{a,L}^{\mathrm{T}}(\boldsymbol{F}_a - \boldsymbol{I}) - \lambda_a \boldsymbol{\beta}_{a,L}^{\mathrm{T}}\Big]\boldsymbol{\delta}_{2^n}^{r} < 0, \tag{10}$$

$$\boldsymbol{\beta}_{b,0}^{\mathrm{T}} \le \mu_b \boldsymbol{\beta}_{a,L}^{\mathrm{T}},\ 0 < \mu_b < 1,\ a \ne b, \tag{11}$$

where $\lambda_a > 0$ and $h = \lfloor \frac{\tau_{\min}}{L} \rfloor$.

A result on the global stability of system (2) is obtained below. Here, note that all subsystems in Eq. (4) can be unstable.

Theorem 1 Consider system (2) under Eq. (3). If there exist a set of Lyapunov coefficients $\{\boldsymbol{\beta}_{a,q} > 0 | a \in Z_\sigma, q = 0, 1, \ldots, L\}$ as defined in Definition 3 and a constant $\tau_{\max} \ge \tau_{\min}$ such that for any $a, b \in Z_\sigma,\ a \ne b$, the following inequality holds:

$$\ln \mu_b + \tau_{\max} \ln(\lambda_a + 1) < 0, \tag{12}$$

then system (2) is globally stable at $\boldsymbol{x}_e$.

Proof We first prove that Eq. (4) is globally stable at $\boldsymbol{x}_e$. For any $\sigma(t_k) \in Z_\sigma$, we construct the following Lyapunov function of Eq. (4):

$$V_{\sigma(t_k)}(t_k) = \boldsymbol{\beta}_{\sigma(t_k)}^{\mathrm{T}}(k)\boldsymbol{x}(t_k). \tag{13}$$

Denote $t_{k^{(1)}}, t_{k^{(2)}}, \ldots, t_{k^{(i)}}$ as the switching instants. For any $t \in [t_{k^{(j)}}, t_{k^{(j+1)}}),\ j = 0, 1, \ldots i-1$, the a^{th} subsystem is activated, i.e., $\sigma(t_{k^{(j)}}) = a$.

The interval $[k^{(j)}, k^{(j)} + \tau^*)$ where $\tau^* = L\lfloor \frac{\tau_{\min}}{L} \rfloor$ is divided into L segments described as

$$N_{k^{(j)},q} = [k^{(j)} + qh, k^{(j)} + (q+1)h),\ q = 0, 1, \ldots, L-1,$$

of equal length $h = \lfloor \frac{\tau_{\min}}{L} \rfloor$. Thus,

$$[k^{(j)}, k^{(j)} + \tau^*) = \bigcup_{q=0}^{L-1} N_{k^{(j)},q}.$$

The vector function $\boldsymbol{\beta}_a(k)$ where $k \in [k^{(j)}, k^{(j)} + \tau^*)$ is chosen to be linear within each segment $N_{k^{(j)},q},\ q = 0, 1, \ldots, L-1$. Let

$$\boldsymbol{\beta}_{a,q} = \boldsymbol{\beta}_a(k^{(j)} + qh) > 0,\ q = 0, 1, \ldots, L-1.$$

Then

$$\begin{aligned}\boldsymbol{\beta}_a(k) &= \boldsymbol{\beta}_a(k^{(j)} + qh + r)\\ &= \left(1 - \frac{r}{h}\right)\boldsymbol{\beta}_{a,q} + \frac{r}{h}\boldsymbol{\beta}_{a,q+1}\\ &= \boldsymbol{\beta}_{a,q} + \frac{1}{h}(\boldsymbol{\beta}_{a,q+1} - \boldsymbol{\beta}_{a,q})(k - k^{(j)} - qh),\end{aligned}$$

where $k \in N_{k^{(j)},q},\ r \in \{0, 1, \ldots, h-1\}$, and $q = 0, 1, \ldots, L-1$. Then we have

$$\begin{aligned}&\boldsymbol{\beta}_a(k+1)\\ =&\boldsymbol{\beta}_{a,q} + \frac{1}{h}(\boldsymbol{\beta}_{a,q+1} - \boldsymbol{\beta}_{a,q})(k + 1 - k^{(j)} - qh)\\ =&\boldsymbol{\beta}_{a,q} + \frac{1}{h}(\boldsymbol{\beta}_{a,q+1} - \boldsymbol{\beta}_{a,q})(k - k^{(j)} - qh)\\ &+ \frac{1}{h}(\boldsymbol{\beta}_{a,q+1} - \boldsymbol{\beta}_{a,q})\\ =&\left(1 - \frac{r}{h}\right)\boldsymbol{\beta}_{a,q} + \frac{r}{h}\boldsymbol{\beta}_{a,q+1} + \frac{1}{h}(\boldsymbol{\beta}_{a,q+1} - \boldsymbol{\beta}_{a,q}),\end{aligned}$$

where $k \in N_{k^{(j)},q}$.

Afterwards, for $k \in [k^{(j)}+\tau^*, k^{(j+1)})$, we establish the vector function $\boldsymbol{\beta}_a(k)=\boldsymbol{\beta}_{a,L}$, where $\boldsymbol{\beta}_{a,L}$ is a constant vector. Hence, $\boldsymbol{\beta}_a(k)$ with $a \in Z_\sigma$ is described as follows:

$$\boldsymbol{\beta}_a(k)=\begin{cases}\left(1-\dfrac{r}{h}\right)\boldsymbol{\beta}_{a,q}+\dfrac{r}{h}\boldsymbol{\beta}_{a,q+1},\ k\in N_{k^{(j)},q},\\ \boldsymbol{\beta}_{a,L},\ k\in[k^{(j)}+\tau^*,k^{(j+1)}),\end{cases}$$

where $r\in\{0,1,\ldots,h-1\}$.

When $k \in N_{k^{(j)},q}$, we know that

$$\begin{aligned}
&\Delta V_a(t_k)\\
=&V_a(t_{k+1})-V_a(t_k)\\
=&\boldsymbol{\beta}_a^{\mathrm{T}}(k+1)\boldsymbol{x}(t_{k+1})-\boldsymbol{\beta}_a^{\mathrm{T}}(k)\boldsymbol{x}(t_k)\\
=&\boldsymbol{\beta}_a^{\mathrm{T}}(k+1)\boldsymbol{F}_a\boldsymbol{x}(t_k)-\boldsymbol{\beta}_a^{\mathrm{T}}(k)\boldsymbol{x}(t_k)\\
=&\left\{\left[\left(1-\frac{r}{h}\right)\boldsymbol{\beta}_{a,q}^{\mathrm{T}}+\frac{r}{h}\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}}+\frac{1}{h}(\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}}-\boldsymbol{\beta}_{a,q}^{\mathrm{T}})\right]\boldsymbol{F}_a\right.\\
&\left.-\left[\left(1-\frac{r}{h}\right)\boldsymbol{\beta}_{a,q}^{\mathrm{T}}+\frac{r}{h}\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}}\right]\right\}\boldsymbol{x}(t_k)\\
=&\left[\left(1-\frac{r}{h}\right)\boldsymbol{\beta}_{a,q}^{\mathrm{T}}(\boldsymbol{F}_a-\boldsymbol{I})+\frac{r}{h}\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}}(\boldsymbol{F}_a-\boldsymbol{I})\right.\\
&\left.+\frac{1}{h}(\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}}-\boldsymbol{\beta}_{a,q}^{\mathrm{T}})\boldsymbol{F}_a\right]\boldsymbol{x}(t_k)\\
=&\left\{\left(1-\frac{r}{h}\right)\left[\boldsymbol{\beta}_{a,q}^{\mathrm{T}}(\boldsymbol{F}_a-\boldsymbol{I})+\frac{1}{h}(\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}}-\boldsymbol{\beta}_{a,q}^{\mathrm{T}})\boldsymbol{F}_a\right]\right.\\
&\left.+\frac{r}{h}\left[\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}}(\boldsymbol{F}_a-\boldsymbol{I})+\frac{1}{h}(\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}}-\boldsymbol{\beta}_{a,q}^{\mathrm{T}})\boldsymbol{F}_a\right]\right\}\boldsymbol{x}(t_k).
\end{aligned}$$

For $\boldsymbol{x}(t_k)\neq\boldsymbol{\delta}_{2^n}^{2^n}$, according to inequalities (7) and (8), we can obtain

$$\begin{aligned}
\Delta V_a(t_k)&<\left(1-\frac{r}{h}\right)\lambda_a\boldsymbol{\beta}_{a,q}^{\mathrm{T}}\boldsymbol{x}(t_k)+\frac{r}{h}\lambda_a\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}}\boldsymbol{x}(t_k)\\
&=\lambda_a\left[\left(1-\frac{r}{h}\right)\boldsymbol{\beta}_{a,q}^{\mathrm{T}}+\frac{r}{h}\boldsymbol{\beta}_{a,q+1}^{\mathrm{T}}\right]\boldsymbol{x}(t_k)\\
&=\lambda_a\boldsymbol{\beta}_a^{\mathrm{T}}(k)\boldsymbol{x}(t_k)\\
&=\lambda_a V_a(t_k),
\end{aligned}$$

where

$$k\in\bigcup_{q=0}^{L-1}N_{k^{(j)},q}=[k^{(j)},k^{(j)}+\tau^*).$$

When $k\in[k^{(j)}+\tau^*,k^{(j+1)})$, for $\boldsymbol{x}(t_k)\neq\boldsymbol{\delta}_{2^n}^{2^n}$, from inequality (10), we have

$$\begin{aligned}
\Delta V_a(t_k)&=V_a(t_{k+1})-V_a(t_k)\\
&=\boldsymbol{\beta}_{a,L}^{\mathrm{T}}\boldsymbol{x}(t_{k+1})-\boldsymbol{\beta}_{a,L}^{\mathrm{T}}\boldsymbol{x}(t_k)\\
&=\boldsymbol{\beta}_{a,L}^{\mathrm{T}}(\boldsymbol{F}_a-\boldsymbol{I})\boldsymbol{x}(t_k)\\
&<\lambda_a\boldsymbol{\beta}_{a,L}^{\mathrm{T}}\boldsymbol{x}(t_k)\\
&=\lambda_a V_a(t_k).
\end{aligned}$$

For $\boldsymbol{x}(t_k)=\boldsymbol{\delta}_{2^n}^{2^n}$, we have $V_a(t_{k+1})=V_a(t_k)$, where $k\in[k^{(j)},k^{(j+1)})$.

Thus, for any $k\in[k^{(j)},k^{(j+1)})$, we can obtain

$$V_a(t_{k+1})\leq(1+\lambda_a)V_a(t_k),$$

which implies

$$\begin{aligned}
V_a(t_k)&\leq(1+\lambda_a)V_a(t_{k-1})\\
&\leq(1+\lambda_a)^2V_a(t_{k-2})\\
&\leq\cdots\\
&\leq(1+\lambda_a)^{k-k^{(j)}}V_a(t_{k^{(j)}}),\ k\in[k^{(j)},k^{(j+1)}).
\end{aligned}\tag{14}$$

On the other hand, by inequality (11), one can obtain

$$V_b(t_{k^{(j+1)}})\leq\mu_b V_a(t_{k^{(j+1)}}),\tag{15}$$

where $b=\sigma(t_{k^{(j+1)}})\in Z_\sigma$ and $a\neq b$.

Then combined with inequalities (14) and (15) and $k^{(i)}<n<k^{(i+1)}$, we have

$$\begin{aligned}
&V_{\sigma(t_n)}(t_n)\\
=&V_{\sigma(t_{k^{(i)}})}(t_n)\\
\leq&(1+\lambda_{\sigma(t_{k^{(i)}})})^{n-k^{(i)}}V_{\sigma(t_{k^{(i)}})}(t_{k^{(i)}})\\
\leq&(1+\lambda_{\sigma(t_{k^{(i)}})})^{n-k^{(i)}}\mu_{\sigma(t_{k^{(i)}})}V_{\sigma(t_{k^{(i)}-1})}(t_{k^{(i)}})\\
=&\mu_{\sigma(t_{k^{(i)}})}(1+\lambda_{\sigma(t_{k^{(i)}})})^{n-k^{(i)}}V_{\sigma(t_{k^{(i-1)}})}(t_{k^{(i)}})\\
\leq&\cdots\\
\leq&\mu_{\sigma(t_{k^{(i)}})}\cdots\mu_{\sigma(t_{k^{(1)}})}(1+\lambda_{\sigma(t_{k^{(i)}})})^{n-k^{(i)}}\cdots\\
&\cdot(1+\lambda_{\sigma(t_{k^{(0)}})})^{k^{(1)}-k^{(0)}}V_{\sigma(t_{k^{(0)}})}(t_{k^{(0)}})\\
=&(1+\lambda_{\sigma(t_{k^{(i)}})})^{n-k^{(i)}}\left[\prod_{j=0}^{i-1}\mu_{\sigma(t_{k^{(j+1)}})}\right.\\
&\left.\cdot(1+\lambda_{\sigma(t_{k^{(j)}})})^{k^{(j+1)}-k^{(j)}}\right]V_{\sigma(t_{k^{(0)}})}(t_{k^{(0)}})\\
<&\frac{1}{\mu_{\sigma(t_{k^{(i+1)}})}}\left[\prod_{j=0}^{i}\mu_{\sigma(t_{k^{(j+1)}})}(1+\lambda_{\sigma(t_{k^{(j)}})})^{k^{(j+1)}-k^{(j)}}\right]\\
&\cdot V_{\sigma(t_{k^{(0)}})}(t_{k^{(0)}}).
\end{aligned}\tag{16}$$

From inequality (12), we can derive

$$\mu_b(1+\lambda_a)^{\tau_{\max}}<1,\ a\neq b,\ \forall a,b\in Z_\sigma.$$

Thus, when

$$\mu_1=\max\mu_b,\rho_1=\max(1+\lambda_a)^{\tau_{\max}},\ a,b\in Z_\sigma,$$

we have $\rho = \mu_1\rho_1 < 1$. Let $\mu_2 = \min \mu_b,\ b \in Z_\sigma$. Then inequality (16) can be converted to

$$V_{\sigma(t_n)}(t_n) < \frac{1}{\mu_2}\rho^{i+1}V_{\sigma(t_{k(0)})}(t_{k^{(0)}}) = \frac{1}{\mu_2}\rho^{i+1}V_{\sigma(0)}(0). \quad (17)$$

Therefore, in view of Eqs. (5) and (13) and inequality (6), if $n \to \infty$, i.e., $i \to \infty$, one can conclude that $\boldsymbol{x}(t_n) \to \boldsymbol{\delta}_{2^n}^{2^n}$, which further implies that Eq. (4) is globally stable at $\boldsymbol{x}_e$. It means $\boldsymbol{x}(t) \to \boldsymbol{\delta}_{2^n}^{2^n}$ as $t \to \infty$. The proof is thus completed.

Remark 2 For the global stability analysis of an SBN $\boldsymbol{x}(t+1) = \boldsymbol{F}_{\sigma(t)}\boldsymbol{x}(t)$, the above method is still applicable by constructing the following Lyapunov function:

$$V_{\sigma(t)}(t) = \boldsymbol{\beta}_{\sigma(t)}^{\mathrm{T}}(t)\boldsymbol{x}(t).$$

Remark 3 Because the transformed SBN switches only at the sampling instant, but not at each sampling instant, the problem considered in this study is more complex than directly studying the global stability of SBNs with all subsystems unstable. The construction of the Lyapunov function and the definition of DT are also different from those in Xiang and Xiao (2014), Feng et al. (2017), Meng et al. (2017a), Liu Z et al. (2018), and Lu et al. (2018b).

4 A biological example

A biological example is shown to demonstrate the validity of Theorem 1.

Example 1 Consider the BCN model studied in Li et al. (2013), which is a reduced model for the lac operon in the bacterium *Escherichia coli*:

$$\begin{cases} x_1(t+1) = \neg u_1(t) \wedge (x_2(t) \vee x_3(t)), \\ x_2(t+1) = \neg u_1(t) \wedge u_2(t) \wedge x_1(t), \\ x_3(t+1) = \neg u_1(t) \wedge (u_2(t) \vee (u_3(t) \wedge x_1(t))). \end{cases}$$

Here, x_1, x_2, and x_3 are state variables, representing the lac mRNA, high-concentration lactose, and medium-concentration lactose, respectively; u_1, u_2, and u_3 are control inputs, representing the extracellular glucose, high-concentration extracellular lactose, and medium-concentration extracellular lactose, respectively.

Setting

$$\boldsymbol{x}(t) = \ltimes_{i=1}^{3}\boldsymbol{x}_i(t)$$

and

$$\boldsymbol{u}(t) = \ltimes_{j=1}^{3}\boldsymbol{u}_j(t),$$

we obtain

$$\boldsymbol{x}(t+1) = \boldsymbol{M}\boldsymbol{u}(t)\boldsymbol{x}(t), \quad (18)$$

where

$$\begin{aligned} \boldsymbol{M} = \boldsymbol{\delta}_8[&8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8, \\ &8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8, \\ &1,1,1,5,3,3,3,7,1,1,1,5,3,3,3,7, \\ &3,3,3,7,4,4,4,8,4,4,4,8,4,4,4,8]. \end{aligned}$$

The ASDC for Eq. (18) is given in the form of Eq. (3) as

$$\boldsymbol{u}(t) = \boldsymbol{E}\boldsymbol{x}(t_k),\ t_k \le t < t_{k+1}, \quad (19)$$

where $\boldsymbol{E} = \boldsymbol{\delta}_8[1,1,4,7,4,6,8,8]$.

Consider a sampling period $h_k \in \{1,2\}$. Then an SBN can be obtained as follows:

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{F}_{\sigma(t_k)}\boldsymbol{x}(t_k), \quad (20)$$

where $\boldsymbol{F}_1 = \boldsymbol{\delta}_8[8,8,8,7,8,3,4,8]$ and $\boldsymbol{F}_2 = \boldsymbol{\delta}_8[8,8,8,4,8,1,8,8]$.

By calculating the state transition for any initial state $\boldsymbol{x}(0) \in \Delta_8$, the first subsystem has the following two attractors of length 1 and length 2, respectively, i.e., $\boldsymbol{\delta}_8^8$ and $(\boldsymbol{\delta}_8^4, \boldsymbol{\delta}_8^7)$. Similarly, it follows that the second subsystem has two point attractors, $\boldsymbol{\delta}_8^8$ and $\boldsymbol{\delta}_8^4$. Based on the definition of stability in Cheng et al. (2011a), one has that both subsystems are unstable. Let

$$L = 1,\ \lambda_1 = \lambda_2 = 0.16,\ \mu_1 = \mu_2 = 0.86,\ \tau_{\min} = 1.$$

By constraints (5)–(12), we obtain the following feasible solution:

$$\begin{cases} \tau_{\max} = 1, \\ \boldsymbol{\beta}_{1,0}^{\mathrm{T}} = (130,132,100,146,137,100,148,0), \\ \boldsymbol{\beta}_{1,1}^{\mathrm{T}} = (176,170,100,171,172,154,169,0), \\ \boldsymbol{\beta}_{2,0}^{\mathrm{T}} = (100,146,86,147,146,132,145,0), \\ \boldsymbol{\beta}_{2,1}^{\mathrm{T}} = (152,154,117,170,160,176,173,0). \end{cases}$$

According to Theorem 1, we know that Eq. (20) can be globally stabilized by the switching signal $\sigma(t_k)$, and that Eq. (18) under Eq. (19) is globally stable at $\boldsymbol{\delta}_8^8$.

Choose the initial state $\boldsymbol{x}(0) = \boldsymbol{\delta}_8^7$. Figs. 1–3 show the corresponding state trajectory, controller $\boldsymbol{u}(t)$, and switching signal $\sigma(t_k)$. In Fig. 3, we can see that the sampling instants are

$$t_0 = 0, t_1 = 1, t_2 = 3, t_3 = 4, t_4 = 6, t_5 = 7, t_6 = 9,$$

and the switching time sequence is $t_0, t_1, t_2, t_3, t_4, t_5$, satisfying $k^{(j+1)} - k^{(j)} = 1$.

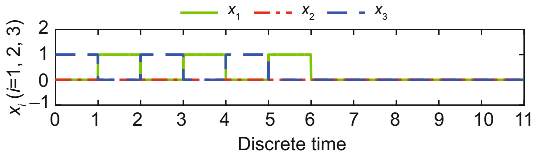

Fig. 1 State trajectory of Eq. (18) under initial state $x(0) = \delta_8^7$

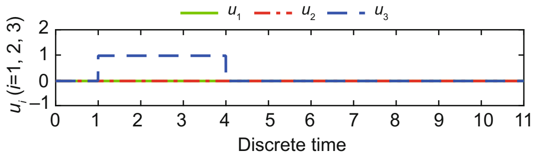

Fig. 2 Trajectory of controller $u(t)$ under initial state $x(0) = \delta_8^7$

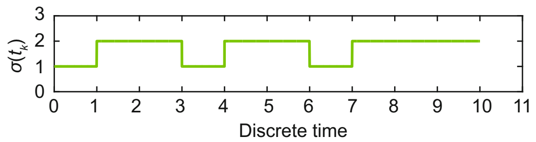

Fig. 3 Trajectory of switching signal $\sigma(t_k)$ under initial state $x(0) = \delta_8^7$

5 Conclusions

In this paper, the global stability of BCNs under ASDCs has been studied. Using STP, we converted a BCN under ASDC into an SBN, whose subsystems are all unstable. Some results for global stability of BCNs under ASDC have been obtained by means of a discretized Lyapunov function and DT. The validity has been demonstrated by a biological example.

Compliance with ethics guidelines

Liang-jie SUN, Jian-quan LU, and Wai-Ki CHING declare that they have no conflict of interest.

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