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Range estimation based on symmetry polynomial aided Chinese remainder theorem for multiple targets in a pulse Doppler radar*

Chenghu CAO[1], Yongbo ZHAO[†‡1,2]

[1]*National Lab of Radar Signal Processing, Xidian University, Xi'an 710071, China*

[2]*Information Sensing and Understanding, Xidian University, Xi'an 710071, China*

[†]E-mail: ybzhao@xidian.edu.cn

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Abstract: To avoid Doppler ambiguity, pulse Doppler radar may operate on a high pulse repetition frequency (PRF). The use of a high PRF can, however, lead to range ambiguity in many cases. At present, the major efficient solution to solve range ambiguity is based on a waveform design scheme. It adds complexity to a radar system. However, the traditional multiple-PRF-based scheme is difficult to be applied in multiple targets because of unknown correspondence between the target range and measured range, especially using the Chinese remainder theorem (CRT) algorithm. We make a study of the CRT algorithm for multiple targets when the residue set contains noise error. In this paper, we present a symmetry polynomial aided CRT algorithm to effectively achieve range estimation of multiple targets when the measured ranges are overlapped with noise error. A closed-form and robust CRT algorithm for single target and the Aitken acceleration algorithm for finding roots of a polynomial equation are used to decrease the computational complexity of the proposed algorithm.

Key words: Range ambiguity; Erroneous range; Multiple targets; Symmetry polynomial aided Chinese remainder theorem

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1 Introduction

One fundamental task of pulse Doppler (PD) radars is range estimation using the received signal coming from the detected target. In general, PD radar with high pulse repetition frequency (PRF) can improve the performance of target detection because of a wider clutter-free area in the frequency domain. For details, readers can refer to Cao et al. (2019). Many PD radars use high PRFs, which are unambiguous in velocity but are likely to have range ambiguities or medium PRFs which are ambiguous in both range and velocity. To date, three approaches to solve range ambiguity for PD radars have been presented. These are phase-coding-based (Levanon, 2009; Xu et al., 2020), frequency-coding-based (Jin et al., 2019), and multi-PRF-based schemes (Kinghorn and Williams, 1997; Zhou R et al., 2002; Wang WJ and Xia, 2010; Liu, 2012; Ma et al., 2012). In the first method, an unambiguous range can be obtained by transmitting a burst of pluses with a coded phase. It may give rise to the sensitivity to the Doppler information, leading to difficulty in detecting the moving target of interest, although the range ambiguity can be resolved by discriminating different pulses. In the second method, the unambiguous range can be derived by processing

‡ Corresponding author

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ORCID: Chenghu CAO, https://orcid.org/0000-0002-2244-7247; Yongbo ZHAO, https://orcid.org/0000-0002-6453-0786

the phase information related to different frequencies in each pulse. However, frequency agility may lead to attenuation in correlation of each signal. The previous two methods adopt a waveform design to disambiguate range but also add complexity to the radar system (Wang WQ, 2013; Wang CH et al., 2017; Zhao et al., 2019). A common approach to find the unfolded range is to obtain multiple measured ranges via transmitting multiple pulses with different PRFs such as remainder look-up table (Lei et al., 1999), one-dimensional clustering, and the Chinese remainder theorem (CRT) (Wang WJ and Xia, 2010; Silva and Fraidenraich, 2018). In the remainder look-up table algorithm, the unambiguous range can be uniquely determined based on the differences of the measured ranges in the multiple PRFs. The remainder look-up table algorithm is a fine compromise between good error-toleration ability and feasible computation. However, it relies on the establishment of the remainder look-up table and is less professional from a mathematical standpoint. That is to say, it is difficult for the remainder look-up table algorithm to provide a complete theoretical solution to solving range ambiguity. The one-dimensional clustering algorithm makes use of all possible unfolded ranges ordered in ascending direction and evaluates the average square error. It has been shown that the one-dimensional clustering algorithm suffers from larger computational burden than the CRT algorithm, although it is perfectly feasible in real time. In fact, a simple and straightforward solution of range ambiguity is the CRT based algorithm.

It is well known that CRT has received considerable attention because it develops the relationship between an integer and the corresponding remainder modulus or several moduli. Hence, CRT has been applied to many fields such as signal processing, coding theory, and radar systems. At present, in Wang WJ and Xia (2010), the range estimation for a single target has been developed using robust and closed-form CRT, where there were errors in the measured ranges due to the restriction on measurement precision. To the best of our knowledge, except for a few studies (Kinghorn and Williams, 1997; Levanon, 2009), there are few available results on the issue of robustly reconstructing ranges of multiple targets with erroneous remainders using the CRT algorithm. As mentioned in Li et al. (2016), Xiao L et al. (2017), and Xiao HS and Xiao (2019), the range estimation for multiple targets is challenging because the determination process of target ranges becomes complicated. On one hand, the difficulty is that the correspondence between multiple targets and range remainders is unknown; i.e., each range residue set is not ordered (Xiao L and Xia, 2014, 2018a; Xiao L et al., 2017; Li et al., 2019). This problem has been studied in Li et al. (2016), where all the remainders were assumed to be error-free. Specifically, a closed-form and simple determination algorithm for two targets was proposed in Li et al. (2016) with a smaller dynamic range. Subsequently, a generalized CRT was presented in Wang W et al. (2015) and Wang WJ et al. (2015) to determine the largest dynamic range for any two targets. A sharpened dynamic range of generalized CRT for multiple targets was presented to further improve the performance (Liao and Xia, 2007; Xiao L and Xia, 2015; Wang CH et al., 2017). The aforementioned algorithms were performed on the assumption that the remainders are at least partly error-free. On the other hand, it inevitably happens that the measured range always has error due to the presence of noise in the PD radar system (Xia, 1999, 2000; Wang WJ et al., 2015). The popular and efficient approach to overcome the sensitivity to error in the reconstruction process is to resort to redundancy (Xiao HS and Xia, 2017; Xiao L et al., 2017). One is to have redundancy in the number of remainders; i.e., most of remainders need to be error-free. For this case, the unambiguous range can be reconstructed within the range of error correction ability. Another has redundancy in each modulus, where all the moduli always have a greatest common divisor (GCD) greater than one and all the quotients of all the moduli divided by the GCD are coprime integers (Li et al., 2018). In this case, the unambiguous range can be uniquely and correctly determined, where all the remainders may have errors but be within the error bound (Xiao L and Xia, 2014, 2015, 2018b).

In this paper, we go further into the robust CRT algorithm to estimate the ranges for multiple targets in a PD radar. We are interested only in the redundancy method in each remainder because any range remainder in practical applications may be corrupted by measurement noise (Xiao L et al., 2014, 2015). The main contributions are listed as follows:

On one hand, we present a symmetry polynomial aided robust CRT algorithm to realize range estimation of multiple targets, where the symmetry polynomial is used to recover the folding numbers. The simulation results verify that the proposed algorithm is effective to estimate ranges where there exists range ambiguity in the PD radar. On the other hand, the closed-form and robust CRT algorithm for a single target and the Aitken acceleration algorithm for finding roots of a polynomial equation are used to decrease the computational complexity of the proposed algorithm. Similar to traditional CRT algorithms, the proposed algorithm performance including the tolerable-error range and maximum decodable range depends on the selected values of the multiple PRFs.

2 Problem statement

As mentioned previously, one of the main problems for PD radars with high PRF is the range ambiguity when the return echoes from several targets related to different transmitted pulses are captured in the same interval. It will lead to overlaid pulses because of the difficulty in recognizing correspondence between transmitted pulse and received pulse. As shown in Fig. 1, echo 1 is the radar return echo from a target with range r_1 due to transmitted pulse 1. Echo 3 could be interpreted as the return echo from the same target due to transmitted pulse 3. A pulse compression technique allows us to obtain range estimation by searching the peak. However, it may be the return echo from a target with range r_2 due to transmitted pulse 1. Consequently, range ambiguity is associated with transmitted pulse 3; i.e., the return echo from a target is always associated with the transmitted pulse at the latest time. In fact, due to the periodicity of the transmitting pulse, the measured range $\hat{r}$ and the unambiguous range R follow

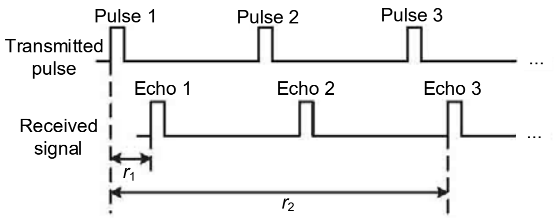

Fig. 1 Illustration of range ambiguity

The reflected signal (echoes 2 and 3) corresponding to the first pulse (pulse 1) is received when the radar has already transmitted other pulses (pulses 2 and 3)

the mathematical expression as

$$\hat{r} = R \mod R^{\max}, \tag{1}$$

where $R^{\max} = c/(2\text{PRF})$ denotes the maximum unambiguous range related to PRF, here c represents the electromagnetic wave velocity. The key idea of the CRT algorithm is to use multiple different PRFs to estimate the unambiguous range of the target from multiple measured ranges.

3 Signal model

We begin with the range estimation problem of multiple targets. As shown in Fig. 2, the PD radar transmits L PRF groups to estimate the ranges of N targets. Given a PRF, there are N measured ranges to be read from the peaks on an output waveform after pulse compression in each coherent processing interval (CPI). The residue set, which represents the union of measured ranges of all targets given one PRF, is given by

$$\begin{aligned} &S_l(R_1, R_2, \cdots, R_n, \cdots, R_N) \\ &= \bigcup_{n=1}^{N} \{\hat{r}_{n,l}\} \quad \text{with} \quad \hat{r}_{n,l} = R_n \mod R_l^{\max}, \end{aligned} \tag{2}$$

where R_n $(n = 1, 2, \cdots, N)$ denotes the unambiguous range of the n^{th} target, $R_l^{\max}$ represents the maximum unambiguous range related to the l^{th} PRF, and $\hat{r}_{n,l}$ is the erroneous remainder of the real range R_n modulo the maximum unambiguous range $R_l^{\max}$. For brevity, suppose that there are no repeated remainders in each residue set (2), i.e., $\hat{r}_{i,l} \neq \hat{r}_{j,l}$ $(1 \leq i, j \leq N,\ i \neq j)$. It must be understood that the remainders $\hat{r}_{n,l}$ in each residue set are not ordered; i.e., the information about the correspondence between the real ranges and the remainders is not specified in advance. In the above formulation, the problem of interest is how to robustly and correctly reconstruct the true ranges from the erroneous residue sets and modulus set.

4 Range estimation for multiple targets based on CRT

Suppose that the maximum detection range of the radar is $R_{\max}$. The maximum occlusion range is defined by R_{E} at which the targets will fully eclipse in all PRFs. In general, the maximum occlusion range is greater than the maximum detection range of the

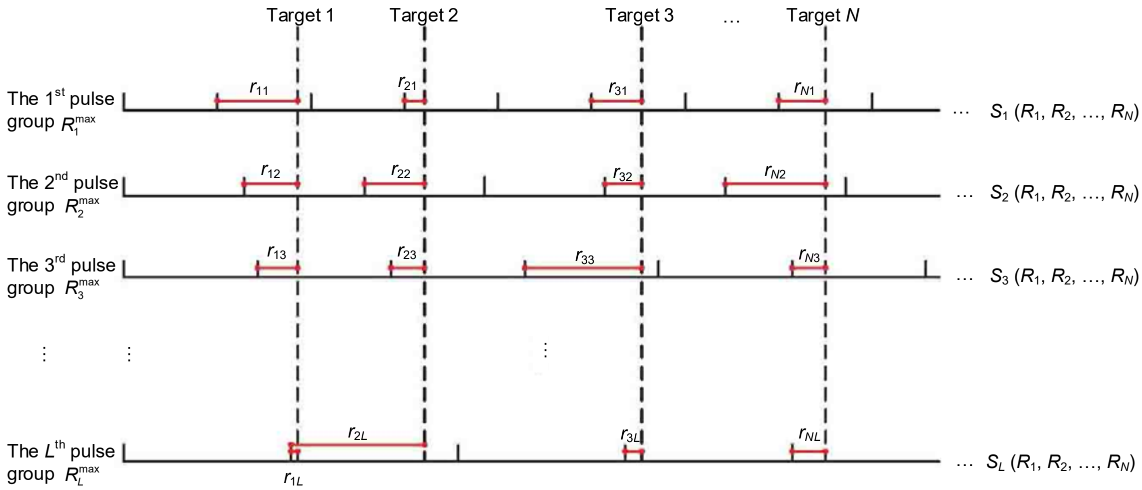

Fig. 2 Signal model for range estimation for multiple targets when there exists range ambiguity

radar, i.e., $R_{\mathrm{E}} > R_{\max}$. In the general framework for solving the ambiguity problem using the CRT, we are concerned mainly with the robustness and dynamic range. Here, the terminology "dynamic range" refers to the values of range for which the proposed algorithm functions, i.e., zero to the maximum decodeable range. The selection of the PRFs plays the decisive role for the performance of the CRT in robustness and dynamic range. The robustness is guaranteed by sharing the greatest common divisor M in the PRF combination. The folding numbers can be correctly recovered from the erroneous residues. In addition, the restriction on the PRF combination is that the value of PRFs divided by greatest common divisor is pairwise coprime. The dynamic range is proportional to the least common multiple of all the PRFs. Hence, more pairs of the PRF are required to obtain the maximum dynamic range. Of course, the range ambiguity problem can also be addressed using only two restricted PRFs. The corresponding occlusion repetition frequency is given by $f_{\mathrm{E}} = c/(2R_{\mathrm{E}})$. Hence, the staggered PRF should be selected according to the following principle:

$$\mathrm{PRF}_l = K_l f_{\mathrm{E}}, \quad 1 \le l \le L,\ K_l \in \mathbb{Z}^+. \tag{3}$$

A transmitted pulse width τ should be set so as to ensure the adequate energy in the waveform, and hence the adequate signal-to-noise ratio (SNR) and detection performance are necessary. According to Eq. (3), the maximum unambiguous range $R_l^{\max}$ related to PRF_l is given by

$$R_l^{\max} = \frac{R_{\mathrm{E}}}{K_l}, \quad 1 \le l \le L. \tag{4}$$

According to Eq. (4), given one R_{E}, the different maximum unambiguous range $R_l^{\max}$ can be derived by changing the value of K_l. We go back to the range estimation problem by the CRT algorithm. Assume that the modulus set $R^{\max} = \{R_1^{\max}, R_2^{\max}, \cdots, R_L^{\max}\}$ has a greatest common divisor M and that the remaining factors $\varGamma_l$ of the moduli divided by the greatest common divisor M are pairwise coprime; i.e., $\varGamma_i$ and $\varGamma_j$ for $1 \le i, j \le L$ $(i \ne j)$ are pairwise coprime.

According to Wang WJ and Xia (2010), the remainders of error-free $r_{n,l}$ $(1 \le l \le L)$ modulo GCD M are the same when all the range remainders $r_{n,l}$ $(1 \le l \le L)$ are error-free, i.e., $r_n^{\mathrm{c}} = r_{n,i} \mod M = r_{n,j} \mod M$ $(1 \le i, j \le L,\ i \ne j)$ for the n^{th} target. The estimation of common remainders r_n^{c} $(1 \le n \le N)$ for multiple targets is significant for the reconstruction of multiple target ranges. The common remainder r_n^{c} can be derived by $r_n^{\mathrm{c}} = R_n \mod M$.

Once the common remainder is determined, the unambiguous range R_n can be reconstructed by

$$R_n = q_n M + r_n^{\mathrm{c}}. \tag{5}$$

Similar to the definition of residues of $q_n = \lfloor R_n/M \rfloor$ in Wang WJ and Xia (2010), we can obtain the following important expression:

$$\begin{aligned}\langle q_n\rangle_{\Gamma_l} &= \left\langle \left\lfloor \frac{R_n}{M} \right\rfloor \right\rangle_{\Gamma_l} = \left\langle \frac{R_n - r_n^{\mathrm{c}}}{M} \right\rangle_{\Gamma_l} \\ &= \left\langle \frac{r_{n,l} - r_n^{\mathrm{c}}}{M} \right\rangle_{\Gamma_l},\end{aligned} \tag{6}$$

where $\langle q_n\rangle_{\Gamma_l}$ denotes the residue of q_n modulo Γ_l. However, it becomes difficult to robustly reconstruct the unambiguous ranges R_n $(1 \le n \le N)$ when there is noise error in the remainders. To estimate the unambiguous ranges from the erroneous residue set S_l $(1 \le l \le L)$, the common remainder r_n^{c} $(1 \le n \le N)$ should be first determined from the given residue sets. Intuitively, the determination procedure of common remainders from erroneous residue sets is not easy because the common remainders of the given remainder $\hat{r}_{n,l}$ modulo the greatest common divisor M are relatively different and not the same as those when the residue sets are error-free. Let $\tilde{r}_{n,k}^{\mathrm{c}} = \hat{r}_{n,k} \mod M$ be estimated common remainders. It should be noted that repeated residues may exist in set S_l, i.e., $\hat{r}_{i,l} = \hat{r}_{j,l}$ for $i \neq j$. Arrange the estimated common remainder set $\Lambda = \left\{\tilde{r}_{n,l}^{\mathrm{c}} | 1 \le n \le N, 1 \le l \le L\right\}$ in ascending order denoted by $\{\gamma_1, \gamma_2, \cdots, \gamma_\kappa\}$, where $\kappa \le NL$. The equality holds if there is no repeated remainder in set Λ. Before presenting the results, we give the definition of the cross remainder. For some R_n $(n = 1, 2, \cdots, N)$, if there exists γ_ξ such that some but not all of the remainder errors $\Delta r_{n,l}$ $(l = 1, 2, \cdots, L)$ satisfy $\gamma_\xi + \Delta r_{n,l} < 0$ or $\gamma_\xi + \Delta r_{i,j} \ge M$, then γ_ξ is called the cross remainder. Here, $\Delta r_{n,l} = r_{n,l} - \hat{r}_{n,l}$ is the error introduced correspondingly. The main obstacle to achieve robustness is from the cross remainder. Hence, we first give the following theorem to pave the way for verifying the existence of cross remainder:

Theorem 1 Denote the error bound $\delta = M/(4N)$. There must exist a cross remainder γ_ξ with $\xi \in \{1, 2, \cdots, \kappa\}$ in set Λ such that

$$\gamma_{\langle \xi+1\rangle_M} - \gamma_\xi + M \times \Delta(\xi = \kappa) > 2\delta,$$

where $\Delta(\xi = \kappa)$ is an indicator function with the mathematical form:

$$\Delta(\xi = \kappa) = \begin{cases} 1, & \xi = \kappa, \\ 0, & \text{otherwise}, \end{cases}$$

and $\langle \xi + 1\rangle_M$ denotes the remainder of $\xi + 1$ modulo M.

The proof of Theorem 1 is provided in Appendix A.

According to Theorem 1, the residue set Λ will be divided into two subcases:

(1) Case I, $\xi = \kappa$.

Define

$$\hat{r}_{n,l}^{\mathrm{c}} = \tilde{r}_{n,l}^{\mathrm{c}} \, (n = 1, 2, \cdots, N, l = 1, 2, \cdots, L).$$

(2) Case II, $\xi \neq \kappa$.

We divide this case into two subcases to discuss.

(a) If $\tilde{r}_{n,l}^{\mathrm{c}} \le \gamma_\xi$, define

$$\hat{r}_{n,l}^{\mathrm{c}} = \tilde{r}_{n,l}^{\mathrm{c}}.$$

(b) Otherwise, define

$$\hat{r}_{n,l}^{\mathrm{c}} = \tilde{r}_{n,l}^{\mathrm{c}} - M.$$

We give a straightforward illustration for the operations based on Theorem 1. In case I as depicted in Fig. 3a, there does not exist any cross residue since the distance between any pair of adjacent elements in set $\{\gamma_1, \gamma_2, \cdots, \gamma_\kappa\}$ is less than δ. In case II as depicted in Fig. 3b, we deal with this case to avoid tedious classification by employing the above trick. For case II, set $\{\gamma_1, \gamma_2, \cdots, \gamma_\kappa\}$ is divided into two subsets, i.e., $\{\gamma_1, \gamma_2, \cdots, \gamma_\xi\}$ and $\{\gamma_{\xi+1}, \gamma_{\xi+2}, \cdots, \gamma_\kappa\}$. After a shifting operation, the second subset turns to be $\{\gamma_{\xi+1} - M, \gamma_{\xi+2} - M, \cdots, \gamma_\kappa - M\}$, as illustrated in Fig. 3c. It is noted that such operation changes only the relative position of the common residues while not affecting the estimation $\langle R_n\rangle_M$ $(n = 1, 2, \cdots, N)$. Based on the above fact, we define

$$\tilde{q}_{n,l} = \left\langle \left\lfloor \frac{\hat{r}_{n,l} - \hat{r}_{n,l}^{\mathrm{c}}}{M} \right\rfloor \right\rangle_{\Gamma_l} = \left\langle \frac{\hat{r}_{n,l} - \hat{r}_{n,l}^{\mathrm{c}}}{M} \right\rangle_{\Gamma_l}. \tag{7}$$

For each Γ_l $(1 \le l \le L)$, we have

$$\left\langle \sum_{n=1}^{N} \tilde{q}_n \right\rangle_{\Gamma_l} = \left\langle \frac{\sum\limits_{n=1}^{N} \left(\hat{r}_{n,l} - \hat{r}_{n,l}^{\mathrm{c}}\right)}{M} \right\rangle_{\Gamma_l}. \tag{8}$$

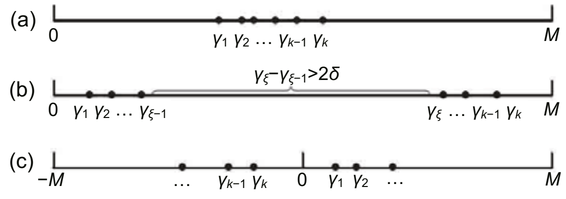

Fig. 3 Illustration of the definition of $\hat{r}_{n,l}^{\mathrm{c}}$: (a) small interval between common remainder; (b) big interval between common remainder; (c) negative-valued common remainder

It is explicitly shown from Eq. (8) that $\sum\limits_{n=1}^{N}\tilde{q}_n$ can be determined by the robust and closed-form CRT algorithm without knowing the correspondence from the fact that the order of $\hat{r}_{n,l}-\hat{r}^{\mathrm{c}}_{n,l}$ $(n=1,2,\cdots,N)$ does not affect the summation result in Eq. (8).

Denote $e_n=\tilde{q}_n-\bar{q}$ $(1\le n\le N)$, where $\bar{q}=\sum\limits_{n=1}^{N}\tilde{q}_n/N$. For each $\mathit{\Gamma}_l$, the residue sets of e_n modulo $\mathit{\Gamma}_l$ can be derived by

$$\langle e_n\rangle_{\mathit{\Gamma}_l}=\alpha_{n,l}=\tilde{q}_{n,l}-\langle\bar{q}\rangle_{\mathit{\Gamma}_l},\ 1\le n\le N. \tag{9}$$

It is implicitly hidden in Eq. (9) that the correspondence that target unambiguous range and the remainder $\alpha_{n,l}$ have is not known. In this study, we make full use of the symmetry polynomial to reconstruct the unambiguous ranges in the case where the correspondence between the unambiguous range and the remainder is not known in advance.

Theorem 2 Given an N-degree polynomial with the form

$$\begin{aligned}P(x)=&c_0x^N-c_1x^{N-1}+c_2x^{N-2}\\&+\cdots+(-1)^nc_nx^{N-n}+\cdots+c_N,\end{aligned}$$

where $c_0=1$ if the coefficient set $\{c_1,c_2,\cdots,c_N\}$ of polynomial $P(x)$ is constructed by

$$\begin{aligned}\langle c_1\rangle_{\mathit{\Gamma}_l}&=\sum_{n=1}^{N}\alpha_{n,l},\\\langle c_2\rangle_{\mathit{\Gamma}_l}&=\sum_{1\le i_1<i_2\le N}\alpha_{i_1,l}\alpha_{i_2,l},\\&\vdots\\\langle c_n\rangle_{\mathit{\Gamma}_l}&=\sum_{1\le i_1<\cdots<i_n\le N}\alpha_{i_1,l}\alpha_{i_2,l}\cdots\alpha_{i_n,l},\\&\vdots\\\langle c_N\rangle_{\mathit{\Gamma}_l}&=\prod_{n=1}^{N}\alpha_{n,l},\end{aligned}\tag{10}$$

where $\alpha_{n,l}$ $(n=1,2,\cdots,N, l=1,2,\cdots,L)$ can be calculated in advance according to Eq. (9), and the root of the polynomial equation $P(x)=0$ is the set $\{e_1,e_2,\cdots,e_N\}$.

The proof of Theorem 2 is provided in Appendix B.

It can be seen from Eq. (10) that $\langle c_n\rangle_{\mathit{\Gamma}_l}$ can be derived by summing $\alpha_{n,l}$ for n $(1\le n\le N)$ without knowing the correspondence due to the symmetry. It is noted that the above symmetry comes from the fact that the summation and product results in Eq. (10) can be derived without needing to know the correspondence. That is to say, the order of $\alpha_{n,l}$ does not affect the results of the summation and product in Eq. (10). Subsequently, the coefficients c_n $(n=1,2,\cdots,N)$ can be determined by the CRT algorithm since remainders $\langle c_n\rangle_{\mathit{\Gamma}_l}$ and the corresponding modulo $\mathit{\Gamma}_l$ have been known after implementing the computation of Eq. (10). Then the number e_n $(1\le n\le N)$ can be determined by solving the polynomial equation. After the above processes, the folding number $\tilde{q}_n$ $(1\le n\le N)$ can be calculated by $\tilde{q}_n=e_n+\bar{q}$ $(1\le n\le N)$. Finally, we can estimate unambiguous ranges with the following form:

$$\hat{R}_n=\tilde{q}_nM+\frac{1}{L}\sum_{l=1}^{L}\hat{r}^{\mathrm{c}}_{n,l},\quad n=1,2,\cdots,N. \tag{11}$$

To summarize, Algorithm 1 gives the detailed process of the range estimation for multiple targets via pseudo code.

In Algorithm 1, the CRT algorithm is exploited twice to recover single integers from the erroneous residue set. It is vitally important for Algorithm 1 whether the above CRT algorithm can robustly and correctly recover unambiguous integer with low computational cost. In the following, we establish a closed-form and analytical solution to solve the above problem at low cost. Algorithm 2 shows the pseudo code for the closed-form and robust CRT algorithm.

In addition, we have to stress the fact that the product $\tilde{q}_nM$ plays a key role to achieve robustness in the reconstruction of unambiguous range R_n; i.e., the reconstruction of the unambiguous range depends mainly on the calculation of the folding number. Fortunately, the roots of the above polynomial equation are all integer. Hence, it allows us to find the roots of the polynomial equation via a numerical solution. In this study, an efficient method called Aitken acceleration is adopted to determine the folding numbers because of its faster convergence.

As is well known, the Aitken acceleration method is a feasible numerical solution method for finding the zero point of a polynomial equation based on an iterative function. Before presenting the numerical method, we first give the iterative function

Algorithm 1 Range estimation for symmetry polynomial aided robust CRT algorithm

Input: Residue set $S_l(R_1, R_2, \cdots, R_N) = \bigcup_{n=1}^{N} \{\hat{r}_{n,l}\}$ for each PRF$_l$, the maximum unambiguous range $R_l^{\max}$, the greatest common divisor M, the number of targets N, and the number of staggered PRFs L

Output: Range estimate $\hat{R}_n$ $(n = 1, 2, \cdots, N)$

1: Calculate $\varGamma_l = \frac{R_l^{\max}}{M}$ $(1 \le l \le L)$ and error bound $\delta = \left\lceil \frac{M}{4N} \right\rceil$

2: Calculate the common residue set $\varLambda = \left\{\tilde{r}_{n,l}^{\mathrm{c}} | 1 \le n \le N, 1 \le l \le L\right\}$ by $\tilde{r}_{n,l}^{\mathrm{c}} = \langle \hat{r}_{n,l} \rangle_M$. The common residue set $\varLambda = \left\{\tilde{r}_{n,l}^{\mathrm{c}} | 1 \le n \le N, 1 \le l \le L\right\}$ turns out to be $\varPhi = \{\gamma_1, \gamma_2, \cdots, \gamma_\kappa\}$ by rearranging in ascending order

3: Obtain the difference set $\varPsi == \{\varphi_1, \varphi_2, \cdots, \varphi_{\kappa-1}\}$ with $\varphi_i = \gamma_{i+1} - \gamma_i$ $(1 \le i \le \kappa - 1)$. Then find the cross remainder γ_i $(1 \le i \le \kappa - 1)$ according to $\varphi_i > 2\delta$

4: Calculate the set $\left\{\hat{r}_{n,l}^{\mathrm{c}} | n = 1, 2, \cdots, N, l = 1, 2, \cdots, L\right\}$ according to Theorem 1

5: Calculate successively $\tilde{q}_{n,l}$ according to Eq. (7). Calculate $\sum_{n=1}^{N} \tilde{q}_n$ according to Eq. (8) and recover $\bar{q} = \sum_{n=1}^{N} \tilde{q}_n / N$

6: Calculate $\bigcup_{n=1}^{N} \alpha_{n,l}$ and $\langle \{c_1, c_2, \cdots, c_N\} \rangle_{\varGamma_l}$ for $l \in \{1, 2, \cdots, L\}$ according to Eqs. (9) and (10), respectively

7: Reconstruct coefficients $c_1, c_2, \cdots, c_N$ by Algorithm 2

8: Let $c_0 = 1$ and construct polynomial equation $P(x) = \sum_{i=0}^{N} (-1)^i c_i x^{N-i} = 0$

9: Solve the equation $P(x) = 0$ by Algorithm 3 and obtain N roots $e_1, e_2, \cdots, e_N$. Subsequently, the folding numbers $\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_N$ can be derived according to $\tilde{q}_n = e_n + \bar{q}$ $(1 \le n \le N)$, where the number of targets is known in advance

10: Calculate the unambiguous ranges $\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_N$ according to Eq. (11)

of the polynomial equation $P(x) = 0$, i.e.,

$$g(x) = \frac{c_N}{c_{N-1}} + \frac{1}{c_{N-1}} \sum_{n=0}^{N-2} c_n (-x)^{N-n}.$$

5 Numerical simulation

In this section, several simulations are performed to verify the efficacy of the proposed algorithm. Assume that the maximum occlusion range is set as $R_{\mathrm{E}} = 450.45$ km. Both range blindness and range ambiguity arise because of the choice of PRFs. Thus, PRFs have to be carefully chosen to ensure target visibility in four PRFs. Suppose that all the measured ranges in the four PRFs are obtained ignoring range blindness due to eclipsing.

Algorithm 2 CRT algorithm for single target

Input: Erroneous residue set $\{\hat{\nu}_1, \hat{\nu}_2, \cdots, \hat{\nu}_L\}$, the modulus set (the maximum unambiguous range) $\{R_1^{\max}, R_2^{\max}, \cdots, R_L^{\max}\}$, and the greatest common divisor M

Output: Unambiguous integer Q

1: Calculate $\varGamma_l = \frac{R_l^{\max}}{M}$ for $1 \le l \le L$

2: Calculate $\hat{\vartheta}_{l,1}$ for $2 \le l \le L$ according to $\hat{\vartheta}_{l,1} = \left[\frac{\hat{\nu}_l - \hat{\nu}_1}{M}\right]$

3: Calculate the remainder of $\hat{\vartheta}_{l,1} \bar{\varGamma}_{l,1}$ modulo $\varGamma_l$, i.e., $\hat{\zeta}_{l,1} = \left\langle \hat{\vartheta}_{l,1} \bar{\varGamma}_l \right\rangle_{\varGamma_l}$ for $2 \le l \le L$, where $\bar{\varGamma}_{i,1}$ denotes the modular multiplicative inverse of $\varGamma_1$ modulo $\varGamma_l$ $(2 \le l \le L)$

4: Estimate the folding number $\hat{n}_1 = \left\langle \sum_{l=2}^{L} \hat{\zeta}_{l,1} \theta_{l,1} \frac{\gamma_1}{\varGamma_l} \right\rangle_{\gamma_1}$, where $\gamma_1 = \prod_{l=2}^{L} \varGamma_l$ and $\theta_{l,1}$ is the modular multiplicative inverse of $\frac{\gamma_1}{\varGamma_l}$ modulo $\varGamma_l$

5: Again estimate the other folding numbers $\hat{n}_l = \frac{\hat{n}_1 \varGamma_1 - \hat{\vartheta}_{l,1}}{\varGamma_l}$ for $2 \le l \le L$

6: Reconstruct $Q = \frac{1}{L} \left(\sum_{l=1}^{L} \hat{n}_l \varGamma_l + \hat{\nu}_l \right)$

7: **if** $Q > \prod_{l=1}^{L} \varGamma_l / 2$ **then**

8: $\quad Q = Q - \prod_{l=1}^{L} \varGamma_l$

9: **else**

10: $\quad Q = Q$

11: **end if**

12: Return Q

Algorithm 3 Aitken acceleration iteration

Input: Iterative function $g(x)$ and precision ε

Output: N roots of polynomial equation

```
loop
    x_0 ← 1, 2, ..., 100
    for k ← 1, 2, ... do
        Iterate x̃_{k+1} = g(x_k), x̄_{k+1} = g(x̃_{k+1})
        Update x_{k+1} = x̄_{k+1} − (x̄_{k+1} − x̃_{k+1})^2 / (x̄_{k+1} − 2x̃_{k+1} + x_k)
        if |x_{k+1} − x_k| ≤ ε then
            break
        else
            continue
        end if
    end for
end loop
```

Choose $K_1 = 1287$, $K_2 = 1001$, $K_3 = 819$, and $K_4 = 693$, and the corresponding maximum unambiguous ranges are $R_1^{\max} = 350$ m, $R_2^{\max} = 450$ m, $R_3^{\max} = 550$ m, and $R_4^{\max} = 650$ m. It is noted that the number and range (range cell) of the target can be obtained by finding the peak of the echo signal after pulse compression. Assume that the SNR is fixed at 22 dB and that the probability of false alarm rate is set as 10^{-8}, i.e., SNR=22 dB and $P_{\mathrm{fa}} = 10^{-8}$.

As shown in Fig. 4, consider a set of multi-target trajectories on two-dimensional region $[-10,\ 25]$ km $\times$ $[-40,\ 60]$ km. There are four targets entering and exiting the surveillance region at different time and each target travels from different initial positions. Clearly, the simulations are performed in a time-varying digital scenario. The optimal sub-pattern assignment (OSPA) metric with cut-off parameter $c = 100$ m and order parameter $p = 1$ is adopted to measure the difference between the true range set and the estimated range set for multiple time-varying targets.

It can be seen from Fig. 5 that the true range in both X and Y directions of the target is larger than the maximum unambiguous range of PD radar. That will lead to range ambiguity. To solve the range problem, the proposed algorithm is used to estimate the range of the time-varying targets. Note that the estimated range using the proposed algorithm slightly deviates from the true range because of measurement noise, and the proposed algorithm achieves a satisfactory result in estimation precision over 40 Monte-Carlo trials. It is worth explaining the problem of range blindness using four PRFs in the simulation. On the assumption that the number of targets is known in advance, the proposed algorithm is feasible for up to two targets. Note that the proposed algorithm does not work when there exists range blindness due to the eclipsing given the PRFs used in this model.

To rigorously evaluate the performance of the proposed algorithm in the scenario with fixed SNR=22 dB, Fig. 6 exhibits the distance and localization error between the real range and the estimated range. The OSPA error has larger fluctuation due to the different error in measured ranges in a single Monte-Carlo trial. We further give the measured ranges and the true range for the targets at 20 s. As seen in Table 1, the proposed algorithm is

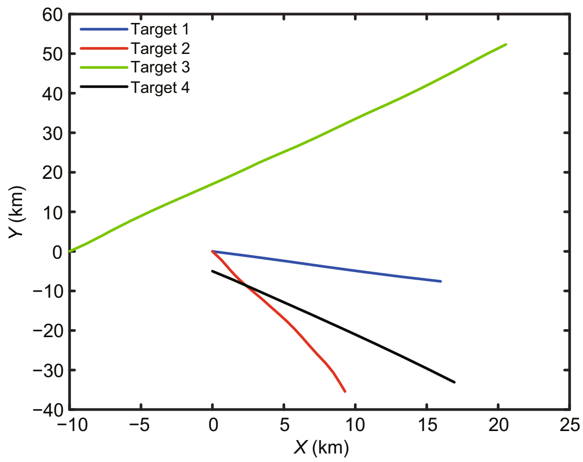

Fig. 4 Trajectories of the detected targets

References to color refer to the online version of this figure

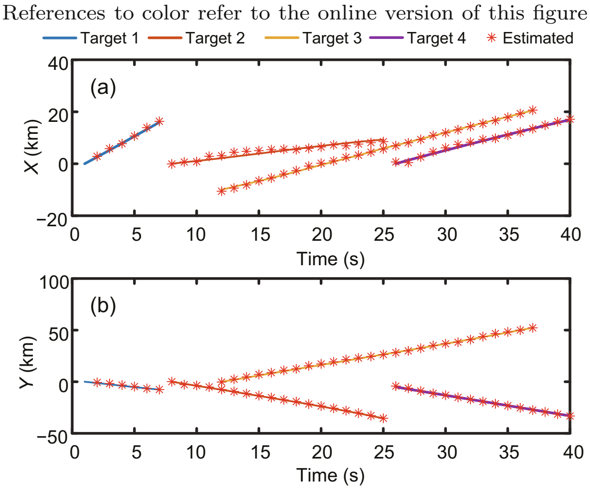

Fig. 5 X (a) and Y (b) directions of range estimation using the proposed algorithm

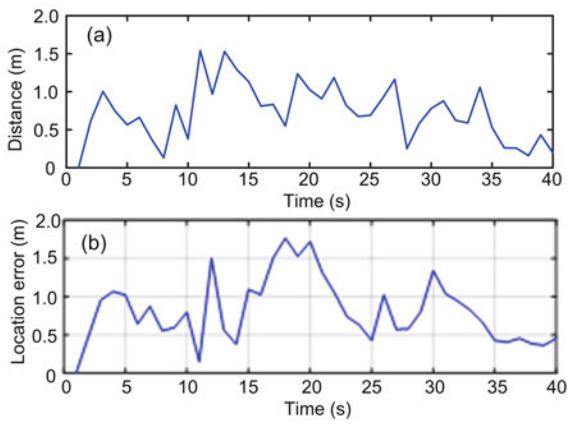

Fig. 6 OSPA distance (a) and location error (b) of range estimation of multiple targets

Table 1 Measured and estimated ranges in the X and Y directions of the targets at 20 s

Range	X (km)	Y (km)	Target
Measured range 1	0.1471	0.0698	1
	0.0871	0.2298	2
Measured range 2	0.0461	0.0198	1
	0.4371	0.1297	2
Measured range 3	0.1968	0.2199	1
	0.4373	0.3797	2
Measured range 4	0.4964	0.5596	1
	0.4372	0.5797	2
True range	6.7963	−23.8698	1
	−0.4372	16.3297	2
Estimated range	6.7964	−23.8697	1
	−0.4371	16.3297	2

able to reconstruct the unambiguous range from the erroneous measured ranges, where the visual ranges have deviated from the true value because of the noise. That is to say, the proposed algorithm is robust to the noise-induced range errors.

The next simulation is performed to verify the performance of the proposed algorithm for different SNRs. First, OSPA error is employed to measure the estimation accuracy for different SNRs. As shown in Fig. 7, the proposed algorithm can realize effective estimation of the target position in different SNRs. The higher the SNR is, the more precise position the proposed algorithm can achieve.

Second, the probability of target detection P_d is used to assess the performance of the proposed algorithm. P_d is defined as the probability of the trial statistic exceeding the detection threshold, where the estimated target position is within a certain distance cell of the actual value. As shown in Fig. 8, the detection probability can achieve 88% when the SNR is equivalent to 20 dB for $P_\mathrm{fa} = 10^{-5}$. The proposed algorithm is effective in accuracy performance for range estimation of multiple targets when there exists range ambiguity.

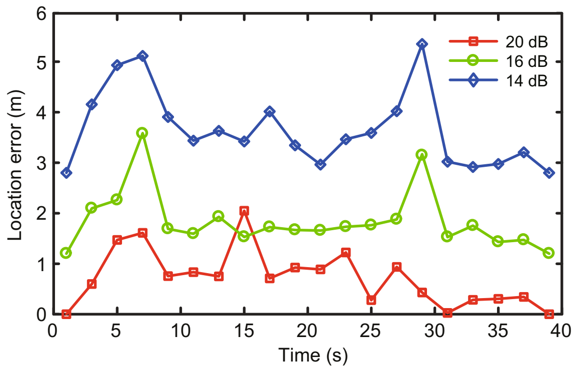

Fig. 7 OSPA error of the proposed algorithm with different SNRs

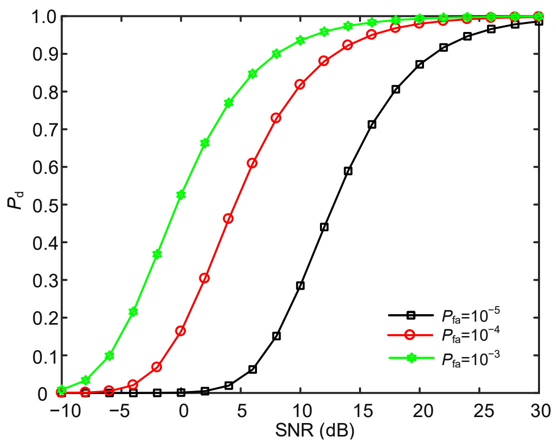

Fig. 8 Probability of target detection P_d with different probabilities of false alarm rate P_fa

Additionally, it is apparent from Fig. 9 that the robust CRT algorithm aided Aitken acceleration is able to improve the computational complexity. At the same time, the real time can be guaranteed at low time cost with a quantum level of 10^{-1} s for each trial.

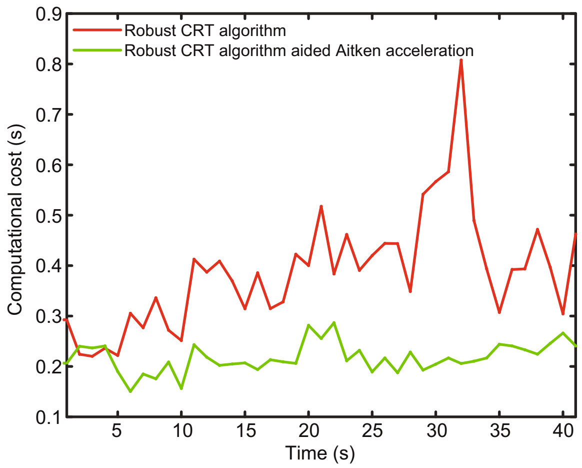

Fig. 9 Computational cost for two algorithms

6 Conclusions

This paper presents a symmetry polynomial aided CRT algorithm to estimate ranges of multiple targets in a PD radar where there exists range ambiguity. The proposed algorithm can correctly and robustly reconstruct the range of multiple targets without knowing the correspondence between the target and the remainder. Additionally, the closed-form and robust CRT algorithm for single target and the Aitken acceleration algorithm for finding roots of a polynomial equation are used to decrease the computational complexity of the proposed algorithm. The simulation results verify the efficacy of the proposed algorithm in reconstructing ranges for multiple targets on the assumption that all the measured ranges in the four PRFs are obtained ignoring range blindness due to eclipsing.

There are some shortcomings in our current work. First, in practical implementation, the proposed algorithm can be performed for three or more targets on the assumption that the number of targets is known in advance. Hence, the requirement to know in advance the number of targets detracts from a good and important method. Note that the range blindness has nothing to do with the number of targets since the range blindness occurs because of high eclipsing loss. Second, the proposed algorithm does not function if the returns from multiple targets (three or more) coincide. More importantly, the proposed algorithm may be invalid when the range blindness arises from eclipsing.

The dynamic range and error bound are two main performance indices for a robust CRT. On one hand, the measurement point information including Doppler, range, and bearing can provide important prior knowledge for target tracking (Zhou GT et al., 2014; Mertens et al., 2016; Tang et al., 2017; Xi et al., 2018; Zhang et al., 2019). Further study of the performance of the proposed algorithm may be the subject of the future work. On the other hand, the cross correlation of the ambiguous ranges of multi-target ghosting phenomenon is another venue for future work since it would happen with larger probability for the range reconstruction of the remote target.

Contributors

Chenghu CAO designed the research. Chenghu CAO and Yongbo ZHAO processed the data. Chenghu CAO drafted the paper. Yongbo ZHAO helped organize the paper. Chenghu CAO and Yongbo ZHAO revised and finalized the paper.

Compliance with ethics guidelines

Chenghu CAO and Yongbo ZHAO declare that they have no conflict of interest.

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Appendix A: Proof of Theorem 1

To further present our proof, we first define the following two metrics:

For any two common remainders, we have

$$D_i = r^{\mathrm{c}}_{\langle i+1\rangle_\kappa} - r^{\mathrm{c}}_i + M \times \varDelta\,(i=\kappa)\,,$$

and for any two integers, we have

$$d_M(X,Y) = \min_{k\in\mathbb{Z}} |X - Y + kM|\,.$$

Assume that there exists an index j_0 satisfying $D_{j_0} > 4\delta$. There must exist $\xi \in \{1, 2, \cdots, \kappa\}$ such that $d_M\left(\gamma_\xi, r^{\mathrm{c}}_{j_0}\right) \le \delta$ and $d_M\left(\gamma_{\langle\xi+1\rangle_\kappa}, r^{\mathrm{c}}_{\langle j_0+1\rangle_N}\right) \le \delta$. Considering the form of indicator function, we discuss the existence problem from the following two cases:

(1) Case I

When $j_0 \ne N$, i.e., $D_{j_0} = r^{\mathrm{c}}_{j_0+1} - r^{\mathrm{c}}_{j_0} > 4\delta$, we have two subcases:

(a) $\xi \ne \kappa$. In this subcase, $\gamma_{\langle\xi+1\rangle_\kappa} - \gamma_\xi = \gamma_{\xi+1} - \gamma_\xi \ge \left(r^{\mathrm{c}}_{j_0+1} - \delta\right) - \left(r^{\mathrm{c}}_{j_0} + \delta\right) > 2\delta$.

(b) $\xi = \kappa$, i.e., $\gamma_1 \ge r^{\mathrm{c}}_{j_0+1} - \delta$. In this subcase, $\gamma_1 - \gamma_\kappa + M \ge \gamma_1 \ge r^{\mathrm{c}}_{j_0+1} - \delta \ge r^{\mathrm{c}}_{j_0} + 4\delta - \delta > 2\delta$.

(2) Case II

When $j_0 = N$, i.e., $D_{j_0} = r^{\mathrm{c}}_1 - r^{\mathrm{c}}_N + M > 4\delta$, we also have two subcases:

(a) $\xi \neq \kappa$, i.e., $r_N^{\mathrm{c}} \geq M-\delta$ and $\gamma_\xi + (M - r_N^{\mathrm{c}}) \leq \delta$. In this subcase, $\gamma_{\xi+1} - \gamma_\xi \geq r_1^{\mathrm{c}} - \delta - (r_N^{\mathrm{c}} + \delta - M) > 2\delta$.

(b) $\xi = \kappa$. In this subcase, $\gamma_1 - \gamma_\xi + M \geq r_1^{\mathrm{c}} - \delta - (r_N^{\mathrm{c}} + \delta) + M > 2\delta$.

Theorem 1 has been proved.

Appendix B: Proof of Theorem 2

According to Eqs. (9) and (10), we can obtain the simultaneous congruence (B1):

$$\begin{aligned}
\langle c_1\rangle_{\Gamma_l} &= \sum_{n=1}^{N}\alpha_{n,l} = \sum_{n=1}^{N}\langle e_n\rangle_{\Gamma_l} = \left\langle \sum_{n=1}^{N} e_n \right\rangle_{\Gamma_l}, \\
\langle c_2\rangle_{\Gamma_l} &= \sum_{1\leq i_1<i_2\leq N}\alpha_{i_1,l}\alpha_{i_2,l} \\
&= \sum_{1\leq i_1<i_2\leq N}\langle e_{i_1}\rangle_{\Gamma_l}\langle e_{i_2}\rangle_{\Gamma_l} \\
&= \left\langle \sum_{1\leq i_1<i_2\leq N} e_{i_1}e_{i_2} \right\rangle_{\Gamma_l}, \\
&\vdots \\
\langle c_n\rangle_{\Gamma_l} &= \sum_{1\leq i_1<\cdots<i_n\leq N}\alpha_{i_1,l}\alpha_{i_2,l}\cdots\alpha_{i_n,l} \\
&= \left\langle \sum_{1\leq i_1<\cdots<i_n\leq N} e_{i_1}e_{i_2}\cdots e_{i_n} \right\rangle_{\Gamma_l}, \\
&\vdots \\
\langle c_N\rangle_{\Gamma_l} &= \prod_{n=1}^{N}\alpha_{n,l} = \left\langle \prod_{n=1}^{N} e_n \right\rangle_{\Gamma_l}.
\end{aligned} \tag{B1}$$

Since $c_i, e_i < \min\limits_{1\leq l\leq L} \Gamma_l$ for each i, we further obtain the following conclusion:

$$\begin{aligned}
c_1 &= \sum_{n=1}^{N} e_n, \\
c_2 &= \sum_{1\leq i_1<i_2\leq N} e_{i_1}e_{i_2}, \\
&\vdots \\
c_n &= \sum_{1\leq i_1<\cdots<i_n\leq N} e_{i_1}e_{i_2}\cdots e_{i_n}, \\
&\vdots \\
c_N &= \prod_{n=1}^{N} e_n.
\end{aligned} \tag{B2}$$

According to the formal generalized Viete theorem, we can know that the set $\{e_1, e_2, \cdots, e_N\}$ is the root of polynomial equation constructed by coefficient set $\{c_1, c_2, \cdots, c_N\}$.

Theorem 2 has been proved.

Appendix C: Robustness proof of the proposed algorithm

The estimated range reconstructed via the proposed algorithm is given by

$$\hat{R}_n = \tilde{q}_n M + \frac{1}{L}\sum_{l=1}^{L}\hat{r}_{n,l}^{\mathrm{c}}.$$

The deviation between the true range R_n and the estimated one $\hat{R}_n$ is given by

$$\left|R_n - \hat{R}_n\right| = \left|(q_n - \tilde{q}_n)M + \left(\widehat{r}_{nl}^{\mathrm{c}} - r_n^{\mathrm{c}}\right)\right|,$$

where $\widehat{r}_n^{\mathrm{c}}$ denotes the average of the common residue $\hat{r}_{nl}^{\mathrm{c}}$ for all l $(l = 1, 2, \cdots, L)$.

Based on the fact that the cross remainder γ_ξ $(\xi = 1, 2, \cdots, \kappa)$ must be within $[M-\delta, M]$ or $[0, \delta]$, there have to exist $\tilde{r}_{n\theta_1}^{\mathrm{c}} \in [\gamma_{\xi+1}, \gamma_\kappa]$ and $\tilde{r}_{n\theta_2}^{\mathrm{c}} \in [\gamma_1, \gamma_\kappa]$, where $\theta_1, \theta_2 \in \{1, 2, \cdots, L\}$. Then

$$\begin{aligned}
\frac{\tilde{r}_{n\theta_1} - \tilde{r}_{n\theta_1}^{\mathrm{c}}}{M} &= \left\langle \frac{R_n - kM\Gamma_{\theta_1} - r_n^{\mathrm{c}}}{M} \right\rangle_{\Gamma_{\theta_1}} \\
&= \left\langle \frac{R_n - r_n^{\mathrm{c}}}{M} - k\Gamma_{\theta_1} \right\rangle_{\Gamma_{\theta_1}} \\
&= \left\langle \left\lfloor \frac{R_n}{M} \right\rfloor \right\rangle_{\Gamma_{\theta_1}}.
\end{aligned}$$

Since $\tilde{r}_{n\theta_1}^{\mathrm{c}} \in [\gamma_{\xi+1}, \gamma_\kappa]$, $M - 2\delta \leq r_n^{\mathrm{c}} + \Delta r_{n\theta_1} < M$. Then

$$\tilde{q}_{n\theta_1} = \left\langle \frac{\tilde{r}_{n\theta_1} - \left(\tilde{r}_{n\theta_1}^{\mathrm{c}} - M\right)}{M} \right\rangle_{\Gamma_{\theta_1}} = \left\langle \left\lfloor \frac{R_n}{M} \right\rfloor + 1 \right\rangle_{\Gamma_{\theta_1}}.$$

Similarly, we can obtain

$$\begin{aligned}
\tilde{q}_{n\theta_2} &= \left\langle \frac{\tilde{r}_{n\theta_2} - (r_n^{\mathrm{c}} + \Delta r_{n\theta_2} - M)}{M} \right\rangle_{\Gamma_{\theta_2}} \\
&= \left\langle \left\lfloor \frac{R_n}{M} \right\rfloor + 1 \right\rangle_{\Gamma_{\theta_2}}.
\end{aligned}$$

Therefore, $\tilde{q}_n = \left\langle \left\lfloor \frac{R_n}{M} \right\rfloor + 1 \right\rangle_{\Gamma_l}$ for $n = 1, 2, \cdots, N$. That is to say, $0 < |q_n - \tilde{q}_n| < 1$ always holds for $n = 1, 2, \cdots, N$.

(1) Case I

For $M-2\delta \leq \tilde{r}^{\rm c}_{nl} = r^{\rm c}_{nl} + \Delta r_{n\theta_1} < M$ and $\hat{r}^{\rm c}_{nl} = \tilde{r}^{\rm c}_{nl} - M < 0$, then

$$\begin{aligned}\left|R_n - \hat{R}_n\right| &= \left|(q_n - \tilde{q}_n)\,M + \left(\widehat{r}^{\rm c}_n - r^{\rm c}_{nl}\right)\right| \\ &< \left|M + \tilde{r}^{\rm c}_{nl} - M - r^{\rm c}_{nl}\right| \\ &= \max_l \left|\Delta r_{nl}\right| < \delta.\end{aligned}$$

(2) Case II

For $0 < \hat{r}^{\rm c}_{nl} = \tilde{r}^{\rm c}_{nl} = \widehat{r}^{\rm c}_{nl} + \Delta r_{nl} - M < \delta$, then

$$\begin{aligned}\left|R_n - \hat{R}_n\right| &= \left|(q_n - \tilde{q}_n)\,M + \left(\widehat{r}^{\rm c}_n - r^{\rm c}_n\right)\right| \\ &< \left|M + \widehat{r}^{\rm c}_n + \Delta r_{nl} - M - \widehat{r}^{\rm c}_{nl}\right| \\ &= \max_l \left|\Delta r_{nl}\right| < \delta.\end{aligned}$$

The deviation between the estimated range $\hat{R}_n$ and the true range R_n follows

$$\left|\hat{R}_n - R_n\right| < \max r_{nl} = \delta.$$

Hence, the proposed algorithm is robust to erroneous residues. The robustness of the proposed algorithm has been proved.