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Controllability of Boolean control networks with multiple time delays in both states and controls*

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Abstract: In this paper, the problem of controllability of Boolean control networks (BCNs) with multiple time delays in both states and controls is investigated. First, the controllability problem of BCNs with multiple time delays in controls is considered. For this controllability problem, a controllability matrix is constructed by defining a new product of matrices, based on which a necessary and sufficient controllability condition is obtained. Then, the controllability of BCNs with multiple time delays in states is studied by giving a necessary and sufficient condition. Subsequently, based on these results, a controllability matrix for BCNs with multiple time delays in both states and controls is proposed that provides a concise controllability condition. Finally, two examples are given to illustrate the main results.

Key words: Boolean control networks; Semi-tensor product of matrices; Controllability; Time delay
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1 Introduction

In system biology, genetic regulatory networks (GRNs) are essential networks. Boolean networks (BNs), first proposed by Kauffman (1969), are an effective tool in modeling, analyzing, and simulating GRNs, wherein each gene is characterized by a Boolean variable (active (1) or inactive (0)), and interactions between the states of each gene are determined by logical functions composed of logical operators. BNs with external inputs are called Boolean control networks (BCNs). BNs and BCNs have attracted much attention from biologists, physicists, and systems scientists (Albert and Othmer, 2003; Chaves et al., 2005; Klmat et al., 2006; Cheng and Qi, 2009). In particular, Cheng and Qi (2010) proposed a generalized matrix product, called the semi-tensor product (STP), based on which an algebraic state-space representation framework has been established for the analysis and control of BNs or BCNs (Cheng et al., 2011). The framework makes it relatively easy to formulate and solve classical control-theoretic problems for BNs or BCNs, and thereby many fundamental results of BNs or BCNs have been obtained, such as controllability and observability (Zhao et al., 2010; Fornasini and Valcher, 2013; Liang et al., 2017; Weiss and Margaliot, 2019; Zhou et al., 2019; Zhang X et al., 2021; Zhu et al., 2021), optimal control (Fornasini and Valcher, 2014; Wu et al., 2021; Gao et al., 2022), stability and stabilization (Li R et al., 2013; Zhong et al., 2020; Acernese et al., 2021; Guo et al., 2021; Li HT et al., 2021; Shen et al., 2021), system decomposition and decoupling (Zou and Zhu, 2015; Li YF and Zhu, 2020, 2022, 2023; Li YF et al., 2021;

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Feng et al., 2022), output regulation (Li HT et al., 2017), and synchronization (Zhong et al., 2014; Chen HW et al., 2018).

As is commonly known, the time delay phenomenon encountered in the real world should be taken into account to reduce its impact on the dynamic behavior of models (Wang WQ and Zhong, 2012; Wang ZD et al., 2018). For BNs, based on the STP, many results concerning BCNs with time delays in states have been presented (Li FF and Sun, 2011; Li FF et al., 2011; Li R et al., 2012; Zheng and Feng, 2020), and the controllability of BCNs with delays in states has been investigated widely. In several works (Li FF and Sun, 2011; Li R et al., 2012; Han et al., 2014; Lu et al., 2016), the controllability problem of BCNs with a constant time delay in states has been investigated. For BCNs with time-varying delays in states, the controllability problem has been investigated (Zhang LJ and Zhang, 2013; Ding et al., 2018). Because the information transmissions between different pairs of nodes in a complex network are, in general, unsynchronized, BCNs with multiple time delays (high-order BCNs) have been proposed as a better model for GRNs (Chen H and Sun, 2013). Moreover, necessary and sufficient conditions for the controllability of BCNs with multiple constant delays (i.e., high-order BCNs) have been obtained (Li FF and Sun, 2012; Chen H and Sun, 2013; Ding et al., 2018).

System performance is greatly impacted by time delays in controls, and systems with such delays have been studied widely in economic, biological, and physiological industry fields (Cui et al., 2009; Liu and Zhao, 2011; Klamka, 2019). The controllability of systems with time delays in both states and controls has been investigated (Dauer and Gahl, 1977; Yang et al., 2009). The controllability of BCNs with time delays in both states and controls has been studied in Han et al. (2014), in which the considered system has a constant delay. A natural question is, what is the controllability condition for BCNs with multiple time delays in both states and controls? In addition, controllability matrices play an essential role in controllability analysis of delay-free BCNs, and provide a concise criterion for controllability; i.e., a delay-free BCN is controllable if and only if all elements of its controllability matrix are nonzero (Zhao et al., 2010). An interesting issue to be considered is how to construct controllability matrices for BCNs with multiple time delays in both states and controls. Furthermore, is there a similar concise criterion for the controllability of BCNs with multiple time delays in both states and controls with the help of controllability matrices? These questions motivated us to study the controllability of BCNs with multiple time delays in both states and controls. The main contributions of this paper are summarized as follows:

1. To our knowledge, we are investigating controllability of BCNs with multiple time delays in both states and controls for the first time. In our study, this problem is transformed into two relatively simple problems to be solved: BCN controllability with multiple time delays in controls and with multiple time delays in states.

2. To solve the controllability problem of BCNs with multiple time delays in controls, a relatively simple and clear controllability matrix is constructed by defining a new product of matrices. A necessary and sufficient condition is then obtained with the help of the controllability matrix for the controllability problem.

3. A necessary and sufficient condition is proposed for the controllability problem of BCNs with multiple time delays in states. Based on these results of controllability of BCNs with multiple time delays in states and BCNs with multiple time delays in controls, the controllability matrix for BCNs with multiple time delays in both states and controls is constructed to provide a concise controllability criterion similar to that for delay-free BCNs.

2 Preliminaries and problem setting

Throughout the paper, we use the following notations:

(1) $\mathcal{Z}$: set of all integers;

(2) $[a, b] := \{a, a+1, \cdots, b\}$, where $a, b \in \mathcal{Z}$;

(3) $\mathbb{R}_{m\times n}$: set of real matrices of $m \times n$;

(4) $\mathcal{B}_{m\times n}$: set of $m \times n$ matrices with each entry in $\mathcal{B}$, where $\mathcal{B} := \{1,\ 0\}$;

(5) $\mathcal{B}^n$: set of n-dimensional Boolean vectors;

(6) $\boldsymbol{A} > 0$ means that each entry of matrix $\boldsymbol{A}$ is positive;

(7) $\mathrm{Col}_i(\boldsymbol{A})$: the i^{th} column of matrix $\boldsymbol{A}$;

(8) $\boldsymbol{A}^{\mathrm{T}}$: transpose of matrix $\boldsymbol{A}$;

(9) $\mathbf{1}_n$: $[\underbrace{1, 1, \ldots, 1}_{n}]^{\mathrm{T}}$;

(10) $\boldsymbol{\delta}_k^i$: the i^{th} column of identity matrix $\boldsymbol{I}_k$;

(11) $\Delta_k := \{\boldsymbol{\delta}_k^i : i = 1, 2, \cdots, k\}$, specially denote $\Delta_2 = \{\boldsymbol{\delta}_2^1, \boldsymbol{\delta}_2^2\}$ by Δ;

(12) $\mathcal{L}_{m\times n} := \{\boldsymbol{L} \in \mathbb{R}_{m\times n} \,|\, \text{Col}(\boldsymbol{L}) \subset \Delta_m\}$, and $\boldsymbol{L} \in \mathcal{L}_{m\times n}$ is called a logical matrix;

(13) $\boldsymbol{\delta}_m[i_1, i_2, \cdots, i_r] := [\boldsymbol{\delta}_m^{i_1}, \boldsymbol{\delta}_m^{i_2}, \cdots, \boldsymbol{\delta}_m^{i_r}]$.

Definition 1 (Cheng and Qi, 2010) Set $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}_{m\times n}$ and $\boldsymbol{B} = (b_{ij}) \in \mathbb{R}_{p\times q}$. Let $\alpha = \text{lcm}(n, p)$ be the least common multiple of n and p. Then the STP of $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as

$$\boldsymbol{A} \ltimes \boldsymbol{B} = (\boldsymbol{A} \otimes \boldsymbol{I}_{\frac{\alpha}{n}})(\boldsymbol{B} \otimes \boldsymbol{I}_{\frac{\alpha}{p}}). \tag{1}$$

In Definition 1, when $n = p$, the STP degenerates to the conventional matrix product. Moreover, the STP keeps most properties of the conventional product, e.g., associativity, distributivity, and the transpose and inverse of products (Cheng et al., 2011). Hence, in the following discussion, $\boldsymbol{A} \ltimes \boldsymbol{B}$ is denoted by $\boldsymbol{AB}$ for simplicity of presentation.

Definition 2 (Cheng and Qi, 2010) The k-dimensional power-reducing matrix is defined as

$$\boldsymbol{M}_{\text{r},k} = [\boldsymbol{\delta}_k^1 \otimes \boldsymbol{\delta}_k^1,\ \boldsymbol{\delta}_k^2 \otimes \boldsymbol{\delta}_k^2,\ \cdots,\ \boldsymbol{\delta}_k^k \otimes \boldsymbol{\delta}_k^k]. \tag{2}$$

Definition 3 (Cheng and Qi, 2010) The swap matrix $\boldsymbol{W}_{[m,n]}$ is defined as

$$\begin{aligned}\boldsymbol{W}_{[m,n]} = [&\boldsymbol{\delta}_n^1 \otimes \boldsymbol{\delta}_m^1,\ \boldsymbol{\delta}_n^2 \otimes \boldsymbol{\delta}_m^1,\ \cdots,\ \boldsymbol{\delta}_n^n \otimes \boldsymbol{\delta}_m^1,\ \cdots,\\ &\boldsymbol{\delta}_n^1 \otimes \boldsymbol{\delta}_m^m,\ \boldsymbol{\delta}_n^2 \otimes \boldsymbol{\delta}_m^m,\ \cdots,\ \boldsymbol{\delta}_n^n \otimes \boldsymbol{\delta}_m^m].\end{aligned}$$

Let $\boldsymbol{x} \in \Delta_m$ and $\boldsymbol{y} \in \Delta_n$. Then $\boldsymbol{xx} = \boldsymbol{M}_{\text{r},m}\boldsymbol{x}$ and $\boldsymbol{W}_{[m,n]}\boldsymbol{xy} = \boldsymbol{yx}$. In addition, for any matrix $\boldsymbol{M}$ and column vector $\boldsymbol{x} \in \mathbb{R}_t$, it holds that $\boldsymbol{xM} = (\boldsymbol{I}_t \otimes \boldsymbol{M})\boldsymbol{x}$ (Cheng et al., 2011).

Identify Boolean variable $\boldsymbol{X} \in \mathcal{B}$ with logical vector $\boldsymbol{x} \in \Delta_2$ as $\boldsymbol{X} = 1 \sim \boldsymbol{x} = \boldsymbol{\delta}_2^1$ and $\boldsymbol{X} = 0 \sim \boldsymbol{x} = \boldsymbol{\delta}_2^2$. Let $\boldsymbol{x}_i$ be the vector form of logical variable $\boldsymbol{X}_i$; i.e., $\boldsymbol{X}_i \sim \boldsymbol{x}_i$, $i = 1, 2, \cdots, n$. Then there is a one-to-one correspondence between $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n)^{\text{T}} \in \mathcal{B}^n$ and $\boldsymbol{x} = \ltimes_{i=1}^n \boldsymbol{x}_i \in \Delta_{2^n}$.

Consider a BCN described by the logical equations:

$$\begin{cases}\begin{aligned}\boldsymbol{X}_1(t+1) =& f_1\Big(\boldsymbol{X}_1(t), \boldsymbol{X}_1(t-1), \cdots, \boldsymbol{X}_1(t-\lambda),\\ &\cdots, \boldsymbol{X}_n(t), \boldsymbol{X}_n(t-1), \cdots, \boldsymbol{X}_n(t-\lambda),\\ &\boldsymbol{U}_1(t), \boldsymbol{U}_1(t-1), \cdots, \boldsymbol{U}_1(t-\mu), \cdots,\\ &\boldsymbol{U}_m(t), \boldsymbol{U}_m(t-1), \cdots, \boldsymbol{U}_m(t-\mu)\Big),\\ &\vdots\\ \boldsymbol{X}_n(t+1) =& f_n\Big(\boldsymbol{X}_1(t), \boldsymbol{X}_1(t-1), \cdots, \boldsymbol{X}_1(t-\lambda),\\ &\cdots, \boldsymbol{X}_n(t), \boldsymbol{X}_n(t-1), \cdots, \boldsymbol{X}_n(t-\lambda),\\ &\boldsymbol{U}_1(t), \boldsymbol{U}_1(t-1), \cdots, \boldsymbol{U}_1(t-\mu), \cdots,\\ &\boldsymbol{U}_m(t), \boldsymbol{U}_m(t-1), \cdots, \boldsymbol{U}_m(t-\mu)\Big),\end{aligned}\end{cases} \tag{3}$$

where λ and μ are positive integer delays for state and control respectively, $\boldsymbol{X}_i, \boldsymbol{U}_j \in \mathcal{B}$, $i = 1, 2, \cdots, n$, and $j = 1, 2, \cdots, m$. In addition, $f_i : \mathcal{B}^{n+m} \to \mathcal{B}^n$ is the system mapping. Let $\boldsymbol{x} = \ltimes_{i=1}^n \boldsymbol{x}_i$, $\boldsymbol{u} = \ltimes_{i=1}^m \boldsymbol{u}_i$, where $\boldsymbol{X}_i \sim \boldsymbol{x}_i$, $\boldsymbol{U}_i \sim \boldsymbol{u}_i$. Based on the properties of the STP, BCN (3) can be converted into the following algebraic form:

$$\begin{aligned}\boldsymbol{x}(t+1) =& \boldsymbol{L}\boldsymbol{u}(t)\boldsymbol{u}(t-1)\cdots\boldsymbol{u}(t-\mu)\\ &\cdot \boldsymbol{x}(t)\boldsymbol{x}(t-1)\cdots\boldsymbol{x}(t-\lambda),\end{aligned} \tag{4}$$

where $\boldsymbol{L} \in \mathcal{L}_{2^n \times 2^{(\mu+1)m+(\lambda+1)n}}$.

Remark 1 The conversion process between logical form (3) and algebraic form (4) can be found in Cheng and Qi (2010).

From algebraic form (4), the initial state space of BCN (4) is $\mathcal{I}_0 := \underbrace{\Delta_{2^n} \times \Delta_{2^n} \times \cdots \times \Delta_{2^n}}_{\lambda+1}$. There is a natural one-to-one mapping from $\mathcal{I}_0$ to $\Delta_{2^{n(\lambda+1)}}$ as $(\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_\lambda) \longmapsto \boldsymbol{x}_0 \ltimes \boldsymbol{x}_1 \ltimes \cdots \ltimes \boldsymbol{x}_\lambda$. Then, $\Delta_{2^{n(\lambda+1)}}$ can be seen as the space of initial states. For convenience of presentation, we denote $\boldsymbol{x}_t = \boldsymbol{x}(t) \ltimes \boldsymbol{x}(t-1) \ltimes \cdots \ltimes \boldsymbol{x}(t-\lambda)$. For a given initial state $\boldsymbol{x}_0 = \boldsymbol{x}(0) \ltimes \boldsymbol{x}(-1) \ltimes \cdots \ltimes \boldsymbol{x}(-\lambda) \in \Delta_{2^{n(\lambda+1)}}$ and a control sequence $\{\boldsymbol{u}(t)\}_{t=-\mu}^{t-1}$, the solution to BCN (4) is denoted by $\boldsymbol{x}(t; \boldsymbol{x}_0, \{\boldsymbol{u}(t)\}_{t=-\mu}^{t-1})$.

Definition 4 Consider BCN (4). A state $\boldsymbol{x}_{\text{d}} \in \Delta_{2^n}$ is said to be s-step reachable from $\boldsymbol{x}_0 \in \Delta_{2^{n(\lambda+1)}}$, if there exists a control sequence $\{\boldsymbol{u}(t)\}_{t=-\mu}^{s-1}$, such that $\boldsymbol{x}(s; \boldsymbol{x}_0, \{\boldsymbol{u}(t)\}_{t=-\mu}^{s-1}) = \boldsymbol{x}_{\text{d}}$.

Denote the set of all s-step reachable states from $\boldsymbol{x} \in \Delta_{2^{n(\lambda+1)}}$ and the set of all reachable states from $\boldsymbol{x} \in \Delta_{2^{n(\lambda+1)}}$ by $\mathcal{R}_s(\boldsymbol{x})$ and $\mathcal{R}(\boldsymbol{x})$, respectively. Obviously, $\mathcal{R}(\boldsymbol{x}) = \cup_{s=1}^{+\infty}\mathcal{R}_s(\boldsymbol{x})$.

Definition 5 BCN (4) is said to be controllable at $\boldsymbol{x}_0$ if $\mathcal{R}(\boldsymbol{x}_0) = \Delta_{2^n}$. BCN (4) is said to be controllable, if for any $\boldsymbol{x}_0 \in \Delta_{2^{n(\lambda+1)}}$, BCN (4) is controllable at $\boldsymbol{x}_0$.

When $\mu = 0, \lambda = 0$, BCN (4) degenerates to

$$\boldsymbol{x}(t+1) = \boldsymbol{L}\boldsymbol{u}(t)\boldsymbol{x}(t), \tag{5}$$

where $\boldsymbol{L} \in \mathcal{L}_{2^n \times 2^{n+m}}$. Let $\boldsymbol{L} = [\boldsymbol{L}_1, \boldsymbol{L}_2, \cdots, \boldsymbol{L}_{2^m}]$. In Zhao et al. (2010), the controllability matrix for BCN (5) was defined as $\boldsymbol{\mathcal{C}} = \sum_{i=1}^{2^n}(\sum_{j=1}^{2^m} \boldsymbol{L}_j)^i$.

Lemma 1 (Zhao et al., 2010) BCN (5) is controllable if and only if $\boldsymbol{\mathcal{C}} > 0$.

A natural question is whether we can construct a similar controllability matrix to obtain a concise criterion for the controllability of BCN (4). This is exactly the problem to be discussed in this study.

3 Main results

3.1 Controllability of BCNs with multiple time delays in controls

In this subsection, we consider the case of time delays only in controls; i.e., $\lambda = 0$ in BCN (4). Then the considered system becomes

$$\boldsymbol{x}(t+1) = \boldsymbol{L}\boldsymbol{u}(t)\boldsymbol{u}(t-1)\cdots\boldsymbol{u}(t-\mu)\boldsymbol{x}(t). \tag{6}$$

We divide $\boldsymbol{L}$ into $2^{(\mu+1)m}$ blocks as follows:

$$\begin{aligned}\boldsymbol{L} = [&\boldsymbol{L}_{1\cdots11}, \boldsymbol{L}_{1\cdots12}, \cdots, \boldsymbol{L}_{1\cdots12^m}, \cdots,\\ &\boldsymbol{L}_{2^m\cdots2^m1}, \boldsymbol{L}_{2^m\cdots2^m2}, \cdots, \boldsymbol{L}_{2^m\cdots2^m2^m}],\end{aligned}$$

where $\boldsymbol{L}_{i_1i_2\cdots i_{\mu+1}} \in \mathcal{L}_{2^n\times2^n}, i_1, i_2, \cdots, i_{\mu+1} \in [1, 2^m]$.

Definition 6 BCN (6) is said to be controllable if for any given initial state $\boldsymbol{x}_0 \in \Delta_{2^n}$ and destination state $\boldsymbol{x}_\mathrm{d} \in \Delta_{2^n}$, there exists an s and a control sequence $\{\boldsymbol{u}(t)\}_{t=-\mu}^{s-1}$, such that $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\boldsymbol{x}(s) = \boldsymbol{x}_\mathrm{d}$.

From Eq. (6), letting $\boldsymbol{x}(0) = \boldsymbol{x}_0$, $\boldsymbol{u}(t) = \boldsymbol{\delta}_{2^m}^{h_t}$, $t \in [-\mu, s-1]$, by a simple calculation, we have

$$\begin{aligned}\boldsymbol{x}(s) &= \boldsymbol{L}\boldsymbol{u}(s-1)\boldsymbol{u}(s-2)\cdots\boldsymbol{u}(s-\mu-1)\cdots\\ &\quad\cdot\boldsymbol{L}\boldsymbol{u}(0)\boldsymbol{u}(-1)\cdots\boldsymbol{u}(-\mu)\boldsymbol{x}(0)\\ &= \boldsymbol{L}_{h_{s-1}h_{s-2}\cdots h_{s-\mu-1}}\boldsymbol{L}_{h_{s-2}h_{s-3}\cdots h_{s-\mu-2}}\cdots\\ &\quad\cdot\boldsymbol{L}_{h_0h_{-1}\cdots h_{-\mu}}\boldsymbol{x}_0.\end{aligned} \tag{7}$$

Let

$$\begin{aligned}\boldsymbol{\mathcal{M}}^s := &\sum_{h_{s-1}=1}^{2^m}\sum_{h_{s-2}=1}^{2^m}\cdots\sum_{h_{-\mu}=1}^{2^m}\boldsymbol{L}_{h_{s-1}h_{s-2}\cdots h_{s-\mu-1}}\\ &\cdot\boldsymbol{L}_{h_{s-2}h_{s-3}\cdots h_{s-\mu-2}}\cdots\boldsymbol{L}_{h_0h_{-1}\cdots h_{-\mu}}.\end{aligned} \tag{8}$$

To simplify $\boldsymbol{\mathcal{M}}^s$, we rearrange $\boldsymbol{L}$ as

$$\begin{aligned}\tilde{\boldsymbol{L}} &= (\tilde{\boldsymbol{L}}_{i_1i_2\cdots i_{\mu+1}})\\ &= \begin{bmatrix}\boldsymbol{L}_{1\cdots11} & \boldsymbol{L}_{1\cdots12} & \cdots & \boldsymbol{L}_{1\cdots12^m}\\ \boldsymbol{L}_{1\cdots21} & \boldsymbol{L}_{1\cdots22} & \cdots & \boldsymbol{L}_{1\cdots22^m}\\ \vdots & \vdots & & \vdots\\ \boldsymbol{L}_{2^m\cdots2^m1} & \boldsymbol{L}_{2^m\cdots2^m2} & \cdots & \boldsymbol{L}_{2^m\cdots2^m2^m}\end{bmatrix}\\ &\in \mathcal{B}_{2^{n+\mu m}\times2^{m+n}},\end{aligned} \tag{9}$$

where $\boldsymbol{L}_{i_1i_2\cdots i_{\mu+1}} \in \mathcal{L}_{2^n\times2^n}, i_1, i_2, \cdots, i_{\mu+1} \in [1, 2^m]$.

Definition 7 Given a matrix $\boldsymbol{A} \in \mathbb{R}_{p\times q}$, let $s|p$ and $t|q$ with $sm = p$ and $tn = q$. Then $\boldsymbol{A}$ is expressed as a block decomposed form as $\boldsymbol{A} = (\boldsymbol{A}_{ij})_{m\times n}$, where $\boldsymbol{A}_{ij} \in \mathbb{R}_{s\times t}$. We define

$$\mathrm{diag}_{s\times t}(\boldsymbol{A}) := \begin{bmatrix}\boldsymbol{A}_{11}\boldsymbol{A}_{12}\cdots\boldsymbol{A}_{1n} & & & \\ & \boldsymbol{A}_{21}\boldsymbol{A}_{22}\cdots\boldsymbol{A}_{2n} & & \\ & & \ddots & \\ & & & \boldsymbol{A}_{m1}\boldsymbol{A}_{m2}\cdots\boldsymbol{A}_{mn}\end{bmatrix}.$$

It is obvious that $\mathrm{diag}_{s\times t}(\boldsymbol{A}) \in \mathbb{R}_{ms\times mnt}$ in Definition 7.

Definition 8 Given $\boldsymbol{A} \in \mathbb{R}_{u\times v}$ and $\boldsymbol{B} \in \mathbb{R}_{x\times y}$, let $a|u$ and $b|v$ with $mpa = u$ and $nb = v$, and $x = npb$. Then $\boldsymbol{A}$ is expressed as a block decomposed form as $\boldsymbol{A} = (\boldsymbol{A}_{ij})_{mp\times n}$, where $\boldsymbol{A}_{ij} \in \mathbb{R}_{a\times b}$. We define a new product of $\boldsymbol{A}$ and $\boldsymbol{B}$ denoted by $\boldsymbol{A}\circ\boldsymbol{B}$ as

$$\boldsymbol{A}\circ\boldsymbol{B} := \mathrm{diag}_{a\times b}(\boldsymbol{A})\,(\boldsymbol{1}_m\otimes\boldsymbol{B})\,.$$

Specifically, denote $\boldsymbol{A}^{(2)} := \boldsymbol{A}\circ\boldsymbol{A}$ and $\boldsymbol{A}^{(s)} := \boldsymbol{A}\circ\boldsymbol{A}^{(s-1)}$.

Proposition 1 Consider a matrix $\boldsymbol{A} \in \mathbb{R}_{pnk\times nk}$, which is expressed as a block decomposed form as $\boldsymbol{A} = (\boldsymbol{A}_{ij})_{pn\times n}$ with $\boldsymbol{A}_{ij} \in \mathbb{R}_{k\times k}$. Let

$$\boldsymbol{A} = \begin{pmatrix}\boldsymbol{A}_1\\ \boldsymbol{A}_2\\ \vdots\\ \boldsymbol{A}_n\end{pmatrix}.$$

Then

$$\boldsymbol{A}^{(s)} = \begin{pmatrix}\mathrm{diag}_{k\times k}(\boldsymbol{A}_1)\\ \mathrm{diag}_{k\times k}(\boldsymbol{A}_2)\\ \vdots\\ \mathrm{diag}_{k\times k}(\boldsymbol{A}_n)\end{pmatrix}^{s-1}\boldsymbol{A},$$

where $s = 2, 3, \cdots$.

Proof According to the definition of "$\circ$," with a straightforward computation, we have

$$\boldsymbol{A}^{(2)} = \boldsymbol{A} \circ \boldsymbol{A} = \mathrm{diag}_{k\times k}(\boldsymbol{A})\,(\boldsymbol{1}_n \otimes \boldsymbol{A}) =$$
$$\begin{pmatrix} \mathrm{diag}_{k\times k}(\boldsymbol{A}_1) & & & \\ & \mathrm{diag}_{k\times k}(\boldsymbol{A}_2) & & \\ & & \ddots & \\ & & & \mathrm{diag}_{k\times k}(\boldsymbol{A}_n) \end{pmatrix} \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{A} \\ \vdots \\ \boldsymbol{A} \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{diag}_{k\times k}(\boldsymbol{A}_1) \\ \mathrm{diag}_{k\times k}(\boldsymbol{A}_2) \\ \vdots \\ \mathrm{diag}_{k\times k}(\boldsymbol{A}_n) \end{pmatrix} \boldsymbol{A}$$

and

$$\boldsymbol{A}^{(3)} = \boldsymbol{A} \circ \boldsymbol{A}^{(2)} = \mathrm{diag}_{k\times k}(\boldsymbol{A}) \left(\boldsymbol{1}_n \otimes \boldsymbol{A}^{(2)}\right)$$
$$= \begin{pmatrix} \mathrm{diag}_{k\times k}(\boldsymbol{A}_1) & & & \\ & \mathrm{diag}_{k\times k}(\boldsymbol{A}_2) & & \\ & & \ddots & \\ & & & \mathrm{diag}_{k\times k}(\boldsymbol{A}_n) \end{pmatrix} \begin{pmatrix} \boldsymbol{A}^{(2)} \\ \boldsymbol{A}^{(2)} \\ \vdots \\ \boldsymbol{A}^{(2)} \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{diag}_{k\times k}(\boldsymbol{A}_1) \\ \mathrm{diag}_{k\times k}(\boldsymbol{A}_2) \\ \vdots \\ \mathrm{diag}_{k\times k}(\boldsymbol{A}_n) \end{pmatrix} \boldsymbol{A}^{(2)}$$
$$= \begin{pmatrix} \mathrm{diag}_{k\times k}(\boldsymbol{A}_1) \\ \mathrm{diag}_{k\times k}(\boldsymbol{A}_2) \\ \vdots \\ \mathrm{diag}_{k\times k}(\boldsymbol{A}_n) \end{pmatrix}^2 \boldsymbol{A}.$$

Similarly,

$$\boldsymbol{A}^{(s)} = \begin{pmatrix} \mathrm{diag}_{k\times k}(\boldsymbol{A}_1) \\ \mathrm{diag}_{k\times k}(\boldsymbol{A}_2) \\ \vdots \\ \mathrm{diag}_{k\times k}(\boldsymbol{A}_n) \end{pmatrix}^{s-1} \boldsymbol{A}.$$

Remark 2 Obviously, if matrices $\boldsymbol{A}_{ij}$ and $\boldsymbol{B}_{st}$ are replaced by $a_{ij} \in \mathbb{R}$ and $b_{st} \in \mathbb{R}$ respectively, in Definitions 7 and 8 and Proposition 1, Proposition 1 still holds.

Lemma 2

$$\begin{aligned} &(\tilde{\boldsymbol{L}}^{(s)})_{i_1 i_2 \cdots i_{\mu+1}} \\ &= \sum_{h_1=1}^{2^m} \sum_{h_2=1}^{2^m} \cdots \sum_{h_{s-1}=1}^{2^m} \boldsymbol{L}_{i_1 i_2 \cdots i_\mu h_1} \\ &\quad \cdot \boldsymbol{L}_{i_2 i_3 \cdots i_\mu h_1 h_2} \cdots \boldsymbol{L}_{h_{s-\mu} h_{s-\mu+1} \cdots h_{s-1} i_{\mu+1}}, \end{aligned} \tag{10}$$

where $s = 2, 3, \cdots$.

Proof We first prove that when $s = 2$, Eq. (10) holds. From Eq. (9), one has that $\tilde{\boldsymbol{L}}$ consists of $2^{\mu m} \times 2^m$ blocks, where each block $\boldsymbol{L}_{i_1 i_2 \cdots i_{\mu+1}} \in \mathcal{L}_{2^n \times 2^n}$, $i_1, i_2, \cdots, i_{\mu+1} \in [1, 2^m]$. Let

$$\tilde{\boldsymbol{L}} := \begin{pmatrix} \tilde{\boldsymbol{L}}_1 \\ \tilde{\boldsymbol{L}}_2 \\ \vdots \\ \tilde{\boldsymbol{L}}_{2^m} \end{pmatrix} \text{ and } \tilde{\boldsymbol{L}}_i := \begin{pmatrix} \boldsymbol{\alpha}_{i_1} \\ \boldsymbol{\alpha}_{i_2} \\ \vdots \\ \boldsymbol{\alpha}_{i_{2^{(\mu-1)m}}} \end{pmatrix},$$

where $\boldsymbol{\alpha}_{i_j} \in \mathcal{L}_{2^n \times 2^{n+m}}$, $i = 1, 2, \cdots, 2^m$, $j = 1, 2, \cdots, 2^{(\mu-1)m}$. Let $\bar{\boldsymbol{L}} := \begin{pmatrix} \bar{\boldsymbol{L}}_1 \\ \bar{\boldsymbol{L}}_2 \\ \vdots \\ \bar{\boldsymbol{L}}_{2^{(\mu-1)m}} \end{pmatrix}$, where $\bar{\boldsymbol{L}}_i \in \mathcal{B}_{2^{n+m} \times 2^{n+m}}$, $i = 1, 2, \cdots, 2^{(\mu-1)m}$. According to Proposition 1, a straightforward computation is shown at the bottom of this page.

Thus,

$$(\tilde{\boldsymbol{L}}^{(2)})_{i_1 i_2 \cdots i_{\mu+1}} = \sum_{h_1=1}^{2^m} \boldsymbol{L}_{i_1 i_2 \cdots i_\mu h_1} \boldsymbol{L}_{i_2 i_3 \cdots i_\mu h_1 i_{\mu+1}}.$$

$$\tilde{\boldsymbol{L}}^{(2)} = \begin{pmatrix} \mathrm{diag}_{2^n \times 2^n}(\tilde{\boldsymbol{L}}_1) \\ \vdots \\ \mathrm{diag}_{2^n \times 2^n}(\tilde{\boldsymbol{L}}_{2^m}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}_{1_1}(\bar{\boldsymbol{L}}_1) \\ \vdots \\ \boldsymbol{\alpha}_{1_{2^{(\mu-1)m}}}(\bar{\boldsymbol{L}}_{2^{(\mu-1)m}}) \\ \vdots \\ \boldsymbol{\alpha}_{2^m{}_1}(\bar{\boldsymbol{L}}_1) \\ \vdots \\ \boldsymbol{\alpha}_{2^m{}_{2^{(\mu-1)m}}}\left(\bar{\boldsymbol{L}}_{2^{(\mu-1)m}}\right) \end{pmatrix} = \begin{bmatrix} \sum\limits_{h_1=1}^{2^m} \boldsymbol{L}_{11\cdots 1h_1} \boldsymbol{L}_{11\cdots 1h_1 1} & \cdots & \sum\limits_{h_1=1}^{2^m} \boldsymbol{L}_{11\cdots 1h_1} \boldsymbol{L}_{11\cdots 1h_1 2^m} \\ \vdots & \cdots & \vdots \\ \sum\limits_{h_1=1}^{2^m} \boldsymbol{L}_{12^m\cdots 2^m h_1} \boldsymbol{L}_{12^m\cdots 2^m h_1 1} & \cdots & \sum\limits_{h_1=1}^{2^m} \boldsymbol{L}_{12^m\cdots 2^m h_1} \boldsymbol{L}_{12^m\cdots 2^m h_1 2^m} \\ \vdots & \cdots & \vdots \\ \sum\limits_{h_1=1}^{2^m} \boldsymbol{L}_{2^m 1\cdots 1h_1} \boldsymbol{L}_{2^m 1\cdots 1h_1 1} & \cdots & \sum\limits_{h_1=1}^{2^m} \boldsymbol{L}_{2^m 1\cdots 1h_1} \boldsymbol{L}_{2^m 1\cdots 1h_1 2^m} \\ \vdots & \cdots & \vdots \\ \sum\limits_{h_1=1}^{2^m} \boldsymbol{L}_{2^m 2^m\cdots 2^m h_1} \boldsymbol{L}_{2^m 2^m\cdots 2^m h_1 1} & \cdots & \sum\limits_{h_1=1}^{2^m} \boldsymbol{L}_{2^m 2^m\cdots 2^m h_1} L_{2^m 2^m\cdots 2^m h_1 2^m} \end{bmatrix}.$$

Suppose that when $s = k$, Eq. (10) holds. In the following, we consider $s = k + 1$.

$$\begin{aligned}&\left(\tilde{\boldsymbol{L}}^{(k+1)}\right)_{i_1 i_2 \cdots i_{\mu+1}}\\ =&\sum_{h=1}^{2^m} \boldsymbol{L}_{i_1 i_2 \cdots i_\mu h} \left(\tilde{\boldsymbol{L}}^{(k)}\right)_{i_2 i_3 \cdots i_\mu h i_{\mu+1}}\\ =&\sum_{h=1}^{2^m} \boldsymbol{L}_{i_1 i_2 \cdots i_\mu h} \sum_{h_1=1}^{2^m}\sum_{h_2=1}^{2^m}\cdots\sum_{h_{k-1}=1}^{2^m} \boldsymbol{L}_{i_2 i_3 \cdots i_\mu h h_1}\\ &\cdot \boldsymbol{L}_{i_3 i_4 \cdots i_\mu h h_1 h_2}\cdots \boldsymbol{L}_{h_{k-\mu} h_{k-\mu+1} \cdots h_{k-1} i_{\mu+1}}\\ =&\sum_{h=1}^{2^m}\sum_{h_1=1}^{2^m}\sum_{h_2=1}^{2^m}\cdots\sum_{h_{k-1}=1}^{2^m} \boldsymbol{L}_{i_1 i_2 \cdots i_\mu h}\boldsymbol{L}_{i_2 i_3 \cdots i_\mu h h_1}\\ &\cdot \boldsymbol{L}_{i_3 i_4 \cdots i_\mu h h_1 h_2}\cdots \boldsymbol{L}_{h_{k-\mu} h_{k-\mu+1} \cdots h_{k-1} i_{\mu+1}}.\end{aligned}$$

Therefore, for any $s = 2, 3, \cdots$, Eq. (10) holds.

From Lemma 2 and the meaning of $\boldsymbol{\mathcal{M}}^s$, it follows that

$$\boldsymbol{\mathcal{M}}^s = \mathbf{1}^{\mathrm{T}}_{2^{\mu m}} \tilde{\boldsymbol{L}}^{(s)} \mathbf{1}_{2^m}. \quad (11)$$

Proposition 2 There exists $\{\boldsymbol{u}(t)\}_{t=-\mu}^{s-1}$, such that system (6) reaches $\boldsymbol{x}(s) = \boldsymbol{\delta}^i_{2^n}$ from $\boldsymbol{x}_0 = \boldsymbol{\delta}^j_{2^n}$ if and only if $(\boldsymbol{\mathcal{M}}^s)_{ij} > 0$.

Proof If there exists $\{\boldsymbol{u}(t)\}_{t=-\mu}^{s-1}$, where $\boldsymbol{u}(t)=\boldsymbol{\delta}^{h_t}_{2^m}$, $t \in [-\mu, s-1]$, such that system (6) reaches $\boldsymbol{x}(s) = \boldsymbol{\delta}^i_{2^n}$ from $\boldsymbol{x}_0 = \boldsymbol{\delta}^j_{2^n}$, then by Eq. (7), it follows that $(\boldsymbol{L}_{h_{s-1}h_{s-2}\cdots h_{s-\mu-1}}\boldsymbol{L}_{h_{s-2}h_{s-3}\cdots h_{s-\mu-2}}\cdots\boldsymbol{L}_{h_0 h_{-1}\cdots h_{-\mu}})_{ij} > 0$. Given Eq. (8), we have $(\boldsymbol{\mathcal{M}}^s)_{ij} > 0$. The reverse is also true.

According to the Hamilton–Cayley theory, if $(\boldsymbol{\mathcal{M}}^s)_{ij} = 0$ for $s = 0, 1, \cdots, 2^n$, then $(\boldsymbol{\mathcal{M}}^s)_{ij} = 0$ for $s = 2^n+1, 2^n+2, \cdots$. Let

$$\boldsymbol{\mathcal{M}} := \sum_{s=1}^{2^n} \boldsymbol{\mathcal{M}}^s = \mathbf{1}^{\mathrm{T}}_{2^{\mu m}} \tilde{\boldsymbol{L}}^{(s)} \mathbf{1}_{2^m}, \quad (12)$$

and we call it the controllability matrix of BCN (6).

Theorem 1 BCN (6) is controllable if and only if $\boldsymbol{\mathcal{M}} > 0$.

Proof For any given $i, j \in [1, 2^n]$, let the initial state $\boldsymbol{x}_0 = \boldsymbol{\delta}^j_{2^n}$ and the destination state $\boldsymbol{x}_{\mathrm{d}} = \boldsymbol{\delta}^i_{2^n}$.

(Necessity) If BCN (6) is controllable, then there exists an $s > 0$ and a control sequence $\{\boldsymbol{u}(t)\}_{t=-\mu}^{s-1}$, such that $\boldsymbol{x}(0) = \boldsymbol{\delta}^j_{2^n}$ and $\boldsymbol{x}(s) = \boldsymbol{\delta}^i_{2^n}$. According to Proposition 2, $(\boldsymbol{\mathcal{M}}^s)_{ij} > 0$. Given $\boldsymbol{\mathcal{M}}_{ij} = (\sum_{s=1}^{2^n}\boldsymbol{\mathcal{M}}^s)_{ij}$, we have $\boldsymbol{\mathcal{M}}_{ij} > 0$. By the arbitrariness of i, j, we have $\boldsymbol{\mathcal{M}} > 0$.

(Sufficiency) Considering $\boldsymbol{\mathcal{M}}_{ij} = (\sum_{s=1}^{2^n}\boldsymbol{\mathcal{M}}^s)_{ij}$, from $\boldsymbol{\mathcal{M}}_{ij} > 0$, it follows that there exists an $s > 0$ such that $(\boldsymbol{\mathcal{M}}^s)_{ij} > 0$. From Proposition 2, $\boldsymbol{x}_{\mathrm{d}} = \boldsymbol{\delta}^i_{2^n}$ can be reachable from $\boldsymbol{x}_0 = \boldsymbol{\delta}^j_{2^n}$ at the s^{th} step. By the arbitrariness of i, j, we have that the system is controllable.

3.2 Controllability of BCNs with multiple time delays in states

In this subsection, we consider the case of time delays only in states; i.e., $\mu = 0$ in BCN (4). Then, BCN (4) degenerates to

$$\boldsymbol{x}(t+1) = \boldsymbol{L}\boldsymbol{u}(t)\boldsymbol{x}(t)\boldsymbol{x}(t-1)\cdots\boldsymbol{x}(t-\lambda). \quad (13)$$

Let

$$\boldsymbol{z}(t) = \boldsymbol{x}(t)\boldsymbol{x}(t-1)\cdots\boldsymbol{x}(t-\lambda). \quad (14)$$

Then

$$\begin{aligned}&\boldsymbol{z}(t+1)\\ =&\boldsymbol{x}(t+1)\boldsymbol{x}(t)\cdots\boldsymbol{x}(t-\lambda+1)\\ =&\boldsymbol{L}\boldsymbol{u}(t)\boldsymbol{x}(t)\boldsymbol{x}(t-1)\cdots\boldsymbol{x}(t-\lambda)\boldsymbol{x}(t)\cdots\boldsymbol{x}(t-\lambda+1)\\ =&\boldsymbol{L}\boldsymbol{u}(t)\boldsymbol{W}_{[2^{\lambda n},2^{(\lambda+1)n}]}[\boldsymbol{x}(t)\boldsymbol{x}(t-1)\cdots\boldsymbol{x}(t-\lambda+1)]^2\\ &\cdot\boldsymbol{x}(t-\lambda)\\ =&\boldsymbol{L}\boldsymbol{u}(t)\boldsymbol{W}_{[2^{\lambda n},2^{(\lambda+1)n}]}\boldsymbol{M}_{\mathrm{r},2^{\lambda n}}\boldsymbol{x}(t)\boldsymbol{x}(t-1)\cdots\boldsymbol{x}(t-\lambda)\\ =&\boldsymbol{L}\left(\boldsymbol{I}_{2^m}\otimes(\boldsymbol{W}_{[2^{\lambda n},2^{(\lambda+1)n}]}\boldsymbol{M}_{\mathrm{r},2^{2n}})\right)\boldsymbol{u}(t)\boldsymbol{z}(t).\end{aligned}$$

Let $\hat{\boldsymbol{L}} := \boldsymbol{L}\left(\boldsymbol{I}_{2^m}\otimes(\boldsymbol{W}_{[2^{\lambda n},2^{(\lambda+1)n}]}\boldsymbol{M}_{\mathrm{r},2^{\lambda n}})\right)$. Then we have

$$\boldsymbol{z}(t+1) = \hat{\boldsymbol{L}}\boldsymbol{u}(t)\boldsymbol{z}(t), \quad (15)$$

which is a delay-free BCN. From Lemma 1, BCN (15) is controllable if and only if $\hat{\boldsymbol{\mathcal{C}}} > 0$, where

$$\hat{\boldsymbol{\mathcal{C}}} = \sum_{i=1}^{2^{(\lambda+1)n}} (\hat{\boldsymbol{L}} \ltimes \mathbf{1}_{2^m})^i.$$

Denote the initial state of Eq. (13) $\boldsymbol{x}(0)\boldsymbol{x}(-1)\cdots\boldsymbol{x}(-\lambda)$ by $\boldsymbol{x}_0$. Considering Eq. (14), we have $\boldsymbol{x}_0 = \boldsymbol{z}(0)$ and

$$\boldsymbol{x}(t) = (\boldsymbol{I}_{2^n}\otimes\mathbf{1}^{\mathrm{T}}_{2^{\lambda n}})\boldsymbol{z}(t). \quad (16)$$

Theorem 2 Consider BCN (13). Given the initial state $\boldsymbol{x}_0 = \boldsymbol{\delta}^{\beta}_{2^{(\lambda+1)n}}$ and destination state $\boldsymbol{x}_{\mathrm{d}} = \boldsymbol{\delta}^{\alpha}_{2^n}$, there exists a control sequence $\{\boldsymbol{u}(t)\}_{t=0}^{s-1}$ such that $\boldsymbol{x}(s) = \boldsymbol{x}_{\mathrm{d}}$ can be reached from $\boldsymbol{x}_0$ if and only if $[(\boldsymbol{I}_{2^n}\otimes\mathbf{1}^{\mathrm{T}}_{2^{\lambda n}})\hat{\boldsymbol{\mathcal{C}}}]_{\alpha\beta} > 0$.

Proof (Necessity) Suppose that there exists a control sequence $\{\boldsymbol{u}(t)\}_{t=0}^{s-1}$ such that system (13) reaches $\boldsymbol{x}_{\rm d}$ from $\boldsymbol{x}_0$ at the s instant; i.e., $\boldsymbol{x}(s) = \boldsymbol{\delta}_{2^n}^{a}$. Under the same control sequence, assume that BCN (15) reaches $\boldsymbol{\delta}_{2^{(\lambda+1)n}}^{\gamma}$ from $\boldsymbol{z}_0$ at the s instant (note that $\boldsymbol{x}_0 = \boldsymbol{z}_0$); i.e., $\boldsymbol{z}(s) = \boldsymbol{\delta}_{2^{(\lambda+1)n}}^{\gamma}$. Then

$$\hat{\boldsymbol{\mathcal{C}}}_{\gamma\beta} > 0. \tag{17}$$

Considering $\boldsymbol{z}(s)$ and $\boldsymbol{x}(s)$, from Eq. (16), we have

$$[(\boldsymbol{I}_{2^n} \otimes \boldsymbol{1}_{2^{\lambda n}}^{\rm T})]_{\alpha\gamma} > 0. \tag{18}$$

Combining inequalities (17) and (18), we have

$$[(\boldsymbol{I}_{2^n} \otimes \boldsymbol{1}_{2^{\lambda n}}^{\rm T})\hat{\boldsymbol{\mathcal{C}}}]_{\alpha\beta} = \sum_{i=1}^{2^{(\lambda+1)n}} (\boldsymbol{I}_{2^n} \otimes \boldsymbol{1}_{2^{\lambda n}}^{\rm T})_{\alpha i}\hat{\boldsymbol{\mathcal{C}}}_{i\beta} > 0.$$

(Sufficiency) Given that $[(\boldsymbol{I}_{2^n} \otimes \boldsymbol{1}_{2^{\lambda n}}^{\rm T})\hat{\boldsymbol{\mathcal{C}}}]_{\alpha\beta} = \sum_{i=1}^{2^{(\lambda+1)n}} (\boldsymbol{I}_{2^n} \otimes \boldsymbol{1}_{2^{\lambda n}}^{\rm T})_{\alpha i}\hat{\boldsymbol{\mathcal{C}}}_{i\beta}$, if $\left[\left(\boldsymbol{I}_{2^n} \otimes \boldsymbol{1}_{2^{\lambda n}}^{\rm T}\right)\hat{\boldsymbol{\mathcal{C}}}\right]_{\alpha\beta} > 0$, then there exists ξ such that

$$\hat{\boldsymbol{\mathcal{C}}}_{\xi\beta} > 0, \tag{19}$$

$$\left(\boldsymbol{I}_{2^n} \otimes \boldsymbol{1}_{2^{\lambda n}}^{\rm T}\right)_{\alpha\xi} > 0. \tag{20}$$

Inequality (19) implies that there exists a control sequence $\{\boldsymbol{u}(t)\}_{t=0}^{s-1}$, such that Eq. (15) reaches $\boldsymbol{\delta}_{2^{(\lambda+1)n}}^{\xi}$ from $\boldsymbol{z}(0) = \boldsymbol{\delta}_{2^{(\lambda+1)n}}^{\beta}$ at the s instant. Combining inequality (20) and Eq. (16), under the same control sequence $\{\boldsymbol{u}(t)\}_{t=0}^{s-1}$, system (13) reaches $\boldsymbol{x}_{\rm d} = \boldsymbol{\delta}_{2^n}^{\alpha}$ from $\boldsymbol{x}_0 = \boldsymbol{\delta}_{2^{(\lambda+1)n}}^{\beta}$ at the s instant.

BCN (13) is said to be controllable if for any given initial state $\boldsymbol{x}_0 \in \Delta_{2^{(\lambda+1)n}}$ and destination state $\boldsymbol{x}_{\rm d} \in \Delta_{2^n}$, there exists an s and a control sequence $\{\boldsymbol{u}(t)\}_{t=0}^{s-1}$, such that $\boldsymbol{x}(s) = \boldsymbol{x}_{\rm d}$. Based on Theorem 2, the following theorem holds:

Theorem 3 BCN (13) is controllable if and only if $(\boldsymbol{I}_{2^n} \otimes \boldsymbol{1}_{2^{\lambda n}}^{\rm T})\hat{\boldsymbol{\mathcal{C}}} > 0$.

3.3 Controllability of BCNs with multiple time delays in both states and controls

Based on the results obtained in Sections 3.1 and 3.2, we investigate the controllability of BCNs with multiple time delays in both states and controls in this subsection.

Consider BCN (4). Let

$$\boldsymbol{z}(t) = \boldsymbol{x}(t)\boldsymbol{x}(t-1)\cdots\boldsymbol{x}(t-\lambda). \tag{21}$$

Omitting the same process as that in Section 3.2, one can obtain

$$\boldsymbol{z}(t+1) = \hat{\boldsymbol{L}}\boldsymbol{u}(t)\boldsymbol{u}(t-1)\cdots\boldsymbol{u}(t-\mu)\boldsymbol{z}(t), \tag{22}$$

where $\hat{\boldsymbol{L}} = \boldsymbol{L}\left(\boldsymbol{I}_{2^{(\mu+1)m}} \otimes (\boldsymbol{W}_{[2^{\lambda n}, 2^{(\lambda+1)n}]}\boldsymbol{M}_{{\rm r},2^{\lambda n}})\right)$. Let

$$\begin{aligned}\hat{\boldsymbol{L}} = [&\hat{\boldsymbol{L}}_{1\cdots 11}, \hat{\boldsymbol{L}}_{1\cdots 12}, \cdots, \hat{\boldsymbol{L}}_{1\cdots 12^m}, \cdots, \\ &\hat{\boldsymbol{L}}_{2^m\cdots 2^m 1}, \hat{\boldsymbol{L}}_{2^m\cdots 2^m 2}, \cdots, \hat{\boldsymbol{L}}_{2^m\cdots 2^m 2^m}],\end{aligned}$$

where $\boldsymbol{L}_{i_1 i_2\cdots i_{\mu+1}} \in \mathcal{L}_{2^{(\lambda+1)n}\times 2^{(\lambda+1)n}}$.

According to Theorem 1, BCN (22) is controllable if and only if

$$\hat{\boldsymbol{\mathcal{M}}} := \sum_{s=1}^{2^{(\lambda+1)n}} \boldsymbol{1}_{2^{\mu m}}^{\rm T}\tilde{\hat{\boldsymbol{L}}}^{(s)}\boldsymbol{1}_{2^m} > 0,$$

where

$$\begin{aligned}\tilde{\hat{\boldsymbol{L}}} &= \left(\tilde{\hat{\boldsymbol{L}}}_{i_1 i_2\cdots i_{\mu+1}}\right) \\ &= \begin{bmatrix} \hat{\boldsymbol{L}}_{1\cdots 11} & \hat{\boldsymbol{L}}_{1\cdots 12} & \cdots & \hat{\boldsymbol{L}}_{1\cdots 12^m} \\ \hat{\boldsymbol{L}}_{1\cdots 21} & \hat{\boldsymbol{L}}_{1\cdots 22} & \cdots & \hat{\boldsymbol{L}}_{1\cdots 22^m} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{\boldsymbol{L}}_{2^m\cdots 2^m 1} & \hat{\boldsymbol{L}}_{2^m\cdots 2^m 2} & \cdots & \hat{\boldsymbol{L}}_{2^m\cdots 2^m 2^m} \end{bmatrix} \\ &\in \mathcal{B}_{2^{(\lambda+1)n+\mu m}\times 2^{m+(\lambda+1)n}}.\end{aligned}$$

By Eq. (21), we have

$$\boldsymbol{x}(t) = (\boldsymbol{I}_{2^n} \otimes \boldsymbol{1}_{2^{\lambda n}}^{\rm T})\boldsymbol{z}(t). \tag{23}$$

We call $\left(\boldsymbol{I}_{2^n} \otimes \boldsymbol{1}_{2^{\lambda n}}^{\rm T}\right)\hat{\boldsymbol{\mathcal{M}}}$ the controllability matrix of BCN (4). Considering Eq. (23), according to Theorem 3, we have the following theorem:

Theorem 4 BCN (4) is controllable if and only if $\left(\boldsymbol{I}_{2^n} \otimes \boldsymbol{1}_{2^{\lambda n}}^{\rm T}\right)\hat{\boldsymbol{\mathcal{M}}} > 0$.

4 Examples

Example 1 Consider the following BCN with time delays in controls:

$$\boldsymbol{x}(t+1) = \boldsymbol{L}\boldsymbol{u}(t)\boldsymbol{u}(t-1)\boldsymbol{u}(t-2)\boldsymbol{x}(t), \tag{24}$$

where $\boldsymbol{u} \in \Delta_2$, $\boldsymbol{x}(t) \in \Delta_4$,

$$\begin{aligned}\boldsymbol{L}=\boldsymbol{\delta}_4[&3,1,1,3,1,2,2,1,2,2,4,1,2,3,4,1,\\ &3,1,3,2,3,2,3,2,2,3,3,2,1,4,1,1].\end{aligned}$$

Let $\boldsymbol{L} = [\boldsymbol{L}_{111}, \boldsymbol{L}_{112}, \boldsymbol{L}_{121}, \boldsymbol{L}_{122}, \boldsymbol{L}_{211}, \boldsymbol{L}_{212}, \boldsymbol{L}_{221}, \boldsymbol{L}_{222}]$, where $\boldsymbol{L}_{i_1 i_2 i_3} \in \mathcal{L}_{4\times 4}$, $i_1, i_2, i_3 \in [1, 2]$, and

$$\tilde{\boldsymbol{L}} = \begin{bmatrix} \boldsymbol{L}_{111} & \boldsymbol{L}_{112} \\ \boldsymbol{L}_{121} & \boldsymbol{L}_{122} \\ \boldsymbol{L}_{211} & \boldsymbol{L}_{212} \\ \boldsymbol{L}_{221} & \boldsymbol{L}_{222} \end{bmatrix}.$$

A straightforward calculation shows that

$$\boldsymbol{\mathcal{M}}^1 = \mathbf{1}_4^{\mathrm{T}} \tilde{\boldsymbol{L}} \mathbf{1}_2 = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 3 & 3 & 1 & 3 \\ 3 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix},$$

$$\boldsymbol{\mathcal{M}}^2 = \mathbf{1}_4^{\mathrm{T}} \tilde{\boldsymbol{L}}^{(2)} \mathbf{1}_2 = \begin{bmatrix} 2 & 6 & 6 & 4 \\ 5 & 6 & 3 & 3 \\ 6 & 3 & 4 & 8 \\ 3 & 1 & 3 & 1 \end{bmatrix} > 0.$$

Thus, $\boldsymbol{\mathcal{M}} = \sum_{s=1}^{4} \boldsymbol{\mathcal{M}}^s > 0$. According to Theorem 1, system (24) is controllable.

Example 2 Consider a BCN with time delays in both states and controls:

$$\boldsymbol{x}(t+1) = \boldsymbol{L}\boldsymbol{u}(t)\boldsymbol{u}(t-1)\boldsymbol{x}(t)\boldsymbol{x}(t-1), \tag{25}$$

where $\boldsymbol{L} = \boldsymbol{\delta}_2[1,1,1,1,1,1,1,1,1,1,1,2,1,1,1]$, $\boldsymbol{u}, \boldsymbol{x} \in \Delta_2$. Let $\boldsymbol{z}(t) = \boldsymbol{x}(t)\boldsymbol{x}(t-1)$. Then we have

$$\begin{aligned} &\boldsymbol{z}(t+1) \\ =&\boldsymbol{x}(t+1)\boldsymbol{x}(t) \\ =&\boldsymbol{L}\boldsymbol{u}(t)\boldsymbol{u}(t-1)\boldsymbol{x}(t)\boldsymbol{x}(t-1)\boldsymbol{x}(t) \\ =&\boldsymbol{L}\left(\boldsymbol{I}_{2^2} \otimes \left(\boldsymbol{W}_{[2,2^2]}\boldsymbol{M}_{\mathrm{r},2}\right)\right)\boldsymbol{u}(t)\boldsymbol{u}(t-1)\boldsymbol{z}(t). \end{aligned}$$

Denote $\hat{\boldsymbol{L}} := \boldsymbol{L}\left(\boldsymbol{I}_{2^2} \otimes \boldsymbol{W}_{[2,2^2]}\boldsymbol{M}_{\mathrm{r},2^2}\right)$. By a straightforward calculation, we have

$$\hat{\boldsymbol{L}} = \boldsymbol{\delta}_4[1,1,2,2,1,1,2,2,1,1,2,2,3,1,2,2].$$

Then we have

$$\boldsymbol{z}(t+1) = \hat{\boldsymbol{L}}\boldsymbol{u}(t)\boldsymbol{u}(t-1)\boldsymbol{z}(t). \tag{26}$$

Let $\hat{\boldsymbol{L}} = \left[\hat{\boldsymbol{L}}_{11}, \hat{\boldsymbol{L}}_{12}, \hat{\boldsymbol{L}}_{21}, \hat{\boldsymbol{L}}_{22}\right]$, where $\hat{\boldsymbol{L}} \in \mathcal{L}_{4\times 16}$, and

$$\tilde{\hat{\boldsymbol{L}}} = \begin{bmatrix} \hat{\boldsymbol{L}}_{11} & \hat{\boldsymbol{L}}_{12} \\ \hat{\boldsymbol{L}}_{21} & \hat{\boldsymbol{L}}_{22} \end{bmatrix} \in \mathcal{B}_{2^3\times 2^3}.$$

A straightforward calculation yields that

$$\hat{\boldsymbol{\mathcal{M}}} = \sum_{s=1}^{4} \mathbf{1}_2^{\mathrm{T}} \tilde{\hat{\boldsymbol{L}}}^{(s)} \mathbf{1}_2 = \begin{bmatrix} 43 & 44 & 40 & 40 \\ 8 & 8 & 8 & 12 \\ 9 & 8 & 12 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\boldsymbol{I}_2 \otimes \mathbf{1}_2^{\mathrm{T}}\right)\hat{\boldsymbol{\mathcal{M}}} = \begin{bmatrix} 51 & 52 & 48 & 52 \\ 9 & 8 & 12 & 8 \end{bmatrix} > 0.$$

It follows from Theorem 4 that system (25) is controllable.

5 Conclusions

The controllability problem of BCNs with multiple time delays in both states and controls has been studied using the STP of matrices, which is transformed into two problems: the controllability problem of BCNs with multiple time delays in controls and the controllability problem of BCNs with multiple time delays in states. For these two controllability problems, necessary and sufficient conditions have been given, and subsequently, based on them, a controllability matrix and a necessary and sufficient condition have been proposed for the controllability of BCNs with multiple time delays in both states and controls. In future work, we will investigate some observability problems for BCNs with multiple time delays in both states and controls.

Contributors

Yifeng LI designed the research and drafted the paper. Lan WANG helped organize the paper. Yifeng LI and Lan WANG revised and finalized the paper.

Compliance with ethics guidelines

Yifeng LI and Lan WANG declare that they have no conflict of interest.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

Acernese A, Yerudkar A, Glielmo L, et al., 2021. Reinforcement learning approach to feedback stabilization problem of probabilistic Boolean control networks. *IEEE Contr Syst Lett*, 5(1):337-342. https://doi.org/10.1109/LCSYS.2020.3001993

Albert R, Othmer HG, 2003. The topology of the regulatory interactions predicts the expression pattern of the segment polarity genes in drosophila melanogaster. *J Theor Biol*, 223(1):1-18. https://doi.org/10.1016/s0022-5193(03)00035-3

Chaves M, Albert R, Sontag ED, 2005. Robustness and fragility of Boolean models for genetic regulatory networks. *J Theor Biol*, 235(3):431-449. https://doi.org/10.1016/j.jtbi.2005.01.023

Chen H, Sun JT, 2013. A new approach for global controllability of higher order Boolean control network. *Neur Netw*, 39:12-17. https://doi.org/10.1016/j.neunet.2012.12.004

Chen HW, Liang JL, Lu JQ, et al., 2018. Synchronization for the realization-dependent probabilistic Boolean networks. *IEEE Trans Neur Netw Learn Syst*, 29(4):819-831. https://doi.org/10.1109/TNNLS.2017.2647989

Cheng DZ, Qi HS, 2009. Controllability and observability of Boolean control networks. *Automatica*, 45(7):1659-1667.
https://doi.org/10.1016/j.automatica.2009.03.006

Cheng DZ, Qi HS, 2010. A linear representation of dynamics of Boolean networks. *IEEE Trans Autom Contr*, 55(10):2251-2258.
https://doi.org/10.1109/TAC.2010.2043294

Cheng DZ, Qi HS, Li ZQ, 2011. Analysis and Control of Boolean Networks: a Semi-Tensor Product Approach. Springer, London, UK.
https://doi.org/10.1007/978-0-85729-097-7

Cui P, Zhang CH, Zhang HS, et al., 2009. Indefinite linear quadratic optimal control problem for singular discrete-time system with multiple input delays. *Automatica*, 45(10):2458-2461.
https://doi.org/10.1016/j.automatica.2009.06.018

Dauer JP, Gahl RD, 1977. Controllability of nonlinear delay systems. *J Optim Theory Appl*, 21(1):59-70.
https://doi.org/10.1007/BF00932544

Ding Y, Xie D, Guo YQ, 2018. Controllability of Boolean control networks with multiple time delays. *IEEE Trans Contr Netw Syst*, 5(4):1787-1795.
https://doi.org/10.1109/TCNS.2017.2763744

Feng JE, Li YL, Fu SH, et al., 2022. New method for disturbance decoupling of Boolean networks. *IEEE Trans Autom Contr*, 67(9):4794-4800.
https://doi.org/10.1109/TAC.2022.3161609

Fornasini E, Valcher ME, 2013. Observability, reconstructibility and state observers of Boolean control networks. *IEEE Trans Autom Contr*, 58(6):1390-1401.
https://doi.org/10.1109/TAC.2012.2231592

Fornasini E, Valcher ME, 2014. Optimal control of Boolean control networks. *IEEE Trans Autom Contr*, 59(5):1258-1270.
https://doi.org/10.1109/TAC.2013.2294821

Gao SH, Sun CK, Xiang C, et al., 2022. Finite-horizon optimal control of Boolean control networks: a unified graph-theoretical approach. *IEEE Trans Neur Netw Learn Syst*, 33(1):157-171.
https://doi.org/10.1109/TNNLS.2020.3027599

Guo YQ, Shen YW, Gui WH, 2021. Asymptotical stability of logic dynamical systems with random impulsive disturbances. *IEEE Trans Autom Contr*, 66(2):513-525.
https://doi.org/10.1109/TAC.2020.2985302

Han M, Liu Y, Tu YS, 2014. Controllability of Boolean control networks with time delays both in states and inputs. *Neurocomputing*, 129:467-475.
https://doi.org/10.1016/j.neucom.2013.09.012

Kauffman SA, 1969. Metabolic stability and epigenesis in randomly constructed genetic nets. *J Theor Biol*, 22(3):437-467.
https://doi.org/10.1016/0022-5193(69)90015-0

Klamka J, 2019. Stochastic controllability of systems with multiple delays in control. *Int J Appl Math Comput Sci*, 19(1):39-48.
https://doi.org/10.2478/v10006-009-0003-9

Klmat S, Saez-Rodriguez J, Lindquist JA, 2006. A methodology for the structural and functional analysis of signaling and regulatory networks. *BMC Bioinform*, 7:56.
https://doi.org/10.1186/1471-2105-7-56

Li FF, Sun JT, 2011. Controllability of Boolean control networks with time delays in states. *Automatica*, 47(3):603-607.
https://doi.org/10.1016/j.automatica.2011.01.040

Li FF, Sun JT, 2012. Controllability of higher order Boolean control networks. *Appl Math Comput*, 219(1):158-169.
https://doi.org/10.1016/j.amc.2012.05.059

Li FF, Sun JT, Wu QD, 2011. Observability of Boolean control networks with state time delays. *IEEE Trans Neur Netw*, 22(6):948-954.
https://doi.org/10.1109/TNN.2011.2126594

Li HT, Xie LH, Wang YZ, 2017. Output regulation of Boolean control networks. *IEEE Trans Autom Contr*, 62(6):2993-2998.
https://doi.org/10.1109/TAC.2016.2606600

Li HT, Yang XR, Wang SL, 2021. Robustness for stability and stabilization of Boolean networks with stochastic function perturbations. *IEEE Trans Autom Contr*, 66(3):1231-1237.
https://doi.org/10.1109/TAC.2020.2997282

Li R, Yang M, Chu TG, 2012. Synchronization of Boolean networks with time delays. *Appl Math Comput*, 219(3):917-927.
https://doi.org/10.1016/j.amc.2012.06.071

Li R, Yang M, Chu TG, 2013. State feedback stabilization for Boolean control networks. *IEEE Trans Autom Contr*, 58(7):1853-1857.
https://doi.org/10.1109/TAC.2013.2238092

Li YF, Zhu JD, 2020. Cascading decomposition of Boolean control networks: a graph-theoretical method. *Front Inform Technol Electron Eng*, 21(2):304-315.
https://doi.org/10.1631/FITEE.1900422

Li YF, Zhu JD, 2022. Necessary and sufficient vertex partition conditions for input-output decoupling of Boolean control networks. *Automatica*, 137:110097.
https://doi.org/10.1016/j.automatica.2021.110097

Li YF, Zhu JD, 2023. Observability decomposition of Boolean control networks. *IEEE Trans Autom Contr*, 68(2):1267-1274.
https://doi.org/10.1109/TAC.2022.3149970

Li YF, Zhu JD, Li BW, et al., 2021. A necessary and sufficient graphic condition for the original disturbance decoupling of Boolean networks. *IEEE Trans Autom Contr*, 66(8):3765-3772.
https://doi.org/10.1109/TAC.2020.3025507

Liang JL, Chen HW, Lam J, 2017. An improved criterion for controllability of Boolean control networks. *IEEE Trans Autom Contr*, 62(11):6012-6018.
https://doi.org/10.1109/TAC.2017.2702008

Liu Y, Zhao SW, 2011. Controllability for a class of linear time-varying impulsive systems with time delay in control input. *IEEE Trans Autom Contr*, 56(2):395-399.
https://doi.org/10.1109/TAC.2010.2088811

Lu JQ, Zhong J, Ho DWC, et al., 2016. On controllability of delayed Boolean control networks. *SIAM J Contr Optim*, 54(2):475-494.
https://doi.org/10.1137/140991820

Shen YW, Guo YQ, Gui WH, 2021. Stability of Boolean networks with state-dependent random impulses. *Front Inform Technol Electron Eng*, 22(2):222-231.
https://doi.org/10.1631/FITEE.1900454

Wang WQ, Zhong SM, 2012. Delay-dependent stability criteria for genetic regulatory networks with time-varying delays and nonlinear disturbance. *Commun Nonl Sci Numer Simul*, 17(9):3597-3611. https://doi.org/10.1016/j.cnsns.2011.12.024

Wang ZD, Gao HJ, Cao JD, et al., 2018. On delayed genetic regulatory networks with polytopic uncertainties: robust stability analysis. *IEEE Trans Nanobiosci*, 7(2):154-163. https://doi.org/10.1109/TNB.2008.2000746

Weiss E, Margaliot M, 2019. A polynomial-time algorithm for solving the minimal observability problem in conjunctive Boolean networks. *IEEE Trans Autom Contr*, 64(7):2727-2736. https://doi.org/10.1109/TAC.2018.2882154

Wu YH, Guo YQ, Toyoda M, 2021. Policy iteration approach to the infinite horizon average optimal control of probabilistic Boolean networks. *IEEE Trans Neur Netw Learn Syst*, 32(7):2910-2924. https://doi.org/10.1109/TNNLS.2020.3008960

Yang P, Xie GM, Wang L, 2009. Controllability of linear discrete-time systems with time-delay in state and control. *Int J Contr*, 82:1288-1296.

Zhang LJ, Zhang KZ, 2013. Controllability and observability of Boolean control networks with time-variant delays in states. *IEEE Trans Neur Netw Learn Syst*, 24(9):1478-1484. https://doi.org/10.1109/TNNLS.2013.2246187

Zhang X, Meng M, Wang YH, et al., 2021. Criteria for observability and reconstructibility of Boolean control networks via set controllability. *IEEE Trans Circ Syst II*, 68(4):1263-1267. https://doi.org/10.1109/TCSII.2020.3021190

Zhao Y, Qi HS, Cheng DZ, 2010. Input-state incidence matrix of Boolean control networks and its applications. *Syst Contr Lett*, 59(12):767-774. https://doi.org/10.1016/j.sysconle.2010.09.002

Zheng YT, Feng JE, 2020. Output tracking of delayed logical control networks with multi-constraint. *Front Inform Technol Electron Eng*, 21(2):316-323. https://doi.org/10.1631/FITEE.1900376

Zhong J, Lu JQ, Liu Y, et al., 2014. Synchronization in an array of output-coupled Boolean networks with time delay. *IEEE Trans Neur Netw Learn Syst*, 25(12):2288-2294.

Zhong J, Li BW, Liu Y, et al., 2020. Output feedback stabilizer design of Boolean networks based on network structure. *Front Inform Technol Electron Eng*, 21(2):247-259. https://doi.org/10.1631/FITEE.1900229

Zhou RP, Guo YQ, Gui WH, 2019. Set reachability and observability of probabilistic Boolean networks. *Automatica*, 106:230-241. https://doi.org/10.1016/j.automatica.2019.05.021

Zhu QX, Gao ZG, Liu Y, et al., 2021. Categorization problem on controllability of Boolean control networks. *IEEE Trans Autom Contr*, 66(5):2297-2303. https://doi.org/10.1109/TAC.2020.3002509

Zou YL, Zhu JD, 2015. Kalman decomposition for Boolean control networks. *Automatica*, 54:65-71. https://doi.org/10.1016/j.automatica.2015.01.023