ESTIMATION METHOD FOR MIXED-EFFECT COEFFICIENT SEMIPARAMETRIC REGRESSION MODEL*

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Abstract: Consider the mixed-effect coefficient semiparametric regression model

$$Z = X'\alpha + Y'\beta + g(T) + e,$$

where X, Y and T are random vectors on $R^p \times R^q \times [0,1]$, α is a p-dimensional fixed-effect parameter, β is a q-dimensional random-effect parameter ($E\beta = b, \mathrm{Cov}(\beta) = \sum$), $g(.)$ is an unknown function on $[0,1]$, e is a random error with mean zero and variance σ^2, and (X,Y,T) and (β,e), β and e are mutually independent. We estimate α, b and $g(.)$ by the nearest neighbor and the least square method. In this paper, we prove that estimations of α, b have asymptotic normality and obtain the best convergence rate $n^{-1/3}$ for the estimation of $g(.)$.

Key words: mixed-effect coefficient, semiparametric regression model, the nearest neighbor estimation, asymptotic normality, the best convergence rate.
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INTRODUCTION

This paper considers the mixed-effect coefficient semiparametric regression model

$$Z = X'\alpha + Y'\beta + g(T) + e, \quad (1)$$

where X, Y and T are random vectors on R^p, R^q and $[0,1]$ respectively, α is a p-dimensional fixed-effect parameter, β is a q-dimensional random-effect parameter ($E\beta = b$, $\mathrm{Cov}(\beta) = \sum$), $g(.)$ is an unknown function on $[0,1]$, e is a random error with mean zero and variance σ^2, and (X, Y, T) and (β, e), β and e are mutually independent. We estimate α, b and $g(.)$ by the nearest neighbor and the least square method and obtain the asymptotic normality and the best convergence rate $n^{-1/3}$ for estimations of α, b and $g(.)$.

Up to now, for model (1) there are no results which yield the limit properties of the estimators. But for the fixed-effect coefficient semiparametric regression model (Chen, 1988; Speckman, 1988; Hong, 1991; Gao, et al., 1992; Hong, et al., 1994; Gao, et al., 1995), the mixed-effect (Gao, 1993; Zhuang, et al., 1996) and the random-effect coefficient linear regression model (Rao, 1965; Swamy, 1971; Johansen, 1982), a series of very useful results had been established. Our investigations in this respect yielded ideal results under more general conditions.

Assume that $\{X_i = (X_{i1}, \cdots, X_{ip})', Y_i = (Y_{i1}, \cdots, Y_{iq})', T_i, Z_i, 1 \le i \le n\}$ is a sequence of i.i.d. (independent identically distributed) random vectors from the model (1), i.e.,

$$Z_i = X'_i\alpha + Y'_i\beta + g(T_i) + e_i, i = 1, \cdots, n. \quad (2)$$

where $\{e_i, 1 \le i \le n\}$ is a sequence of i.i.d. random errors and $e_i \sim (0, \sigma^2)$, $\{\beta_i, 1 \le i \le n\}$ is a sequence of i.i.d. random parameter vectors, $\beta_i \sim (b, \sum)$, and $\{(X_i, Y_i, T_i), 1 \le i \le n\}$ and $\{(\beta_i, e_i), 1 \le i \le n\}$, $\{e_i, 1 \le i \le n\}$ and $\{\beta_i, 1 \le i \le n\}$ are mutually independent. $T_1, \cdots, T_n$ are rearranged as follows.

$$| T_{R(1,t)} - t | \le | T_{R(2,t)} - t | \le \cdots \le | T_{R(n,t)} - t |. \quad (3)$$

Let $\{v_{ni}, 1 \le i \le n\}$ and $\{k = k_n, n \ge 1\}$ be a set of nonnegative real numbers and positive in-

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tegers respectively, which satisfies

a) $k/\sqrt{n}\log n \to \infty$, $k/n^{3/4} \to 0$

b) $\sum_{i=1}^{n} v_{ni} = 1, k \max_{1\le i\le k} v_{ni} = O(1)$,

$\sum_{i>k} v_{ni} = o(n^{-1/2})$.

By (3), we define a nonnegative random variables sequence $\{W_{ni}(t) = W_{ni}(t, T_1, \cdots, T_n), 1 \le i \le n\}$, and have

$$W_{nR(i,t)}(t) = v_{ni} \quad 1 \le i \le n$$

Equation (2) can be rewritten as

$$Z_i - X'_i\alpha - Y'_i b = g(T_i) + e_i + Y'_i(\beta_i - b). \tag{4}$$

If α and b are known and $e_i + Y'_i(\beta_i - b)$ is regarded as an error term, by the nearest neighbor and the least square method, we structure the nearest neighbor estimation $\hat{g}_n(.)$ of $g(.)$ as follows.

$$\hat{g}_n(t) \triangleq \hat{g}_{1n}(t) - \hat{g}'_{2n}(t)\alpha - \hat{g}'_{3n}(t)b,$$

where

$$\hat{g}_{1n}(t) = \sum_{i=1}^{n} W_{ni}(t)Z_i,$$

$$\hat{g}_{2n}(t) = \sum_{i=1}^{n} W_{ni}(t)X_i,$$

$$\hat{g}_{3n}(t) = \sum_{i=1}^{n} W_{ni}(t)Y_i$$

are the nearest neighbor estimations of $g_1(t) = E(Z_1 \mid T_1 = t)$, $g_2(t) = E(X_1 \mid T_1 = t)$, and $g_3(t) = E(Y_1 \mid T_1 = t)$ respectively. In Equation (2), $\hat{g}(T_i)$ is used instead of $g(T_i)$. Let

$$d = (\alpha', b'), \tilde{X} = (\tilde{X}_1, \cdots, \tilde{X}_n)',$$

$$\tilde{Y} = (\tilde{Y}_1, \cdots, \tilde{Y}_n)', \tilde{Z} = (\tilde{Z}_1, \cdots, \tilde{Z}_n)',$$

where $\tilde{X}_i = X_i - \hat{g}_{2n}(T_i)$, $\tilde{Y}_i = Y_i - \hat{g}_{3n}(T_i)$, $\tilde{Z}_i = Z_i - \hat{g}_{1n}(T_i)$. Then by the least square method, we have

$$\hat{d}_n = \begin{pmatrix} \hat{a}_n \\ \hat{b}_n \end{pmatrix} = \left[\begin{pmatrix} \tilde{X}' \\ \tilde{Y}' \end{pmatrix} \tilde{X}\, \tilde{Y}\right]^{-1} \begin{pmatrix} \tilde{X}' \\ \tilde{Y}' \end{pmatrix} \tilde{Z}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} \tilde{X}_i\tilde{X}'_i & \sum_{i=1}^{n} \tilde{X}_i\tilde{Y}'_i \\ \sum_{i=1}^{n} \tilde{Y}_i\tilde{X}'_i & \sum_{i=1}^{n} \tilde{Y}_i\tilde{Y}'_i \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{n} \tilde{X}_i\tilde{Z}_i \\ \sum_{i=1}^{n} \tilde{Y}_i\tilde{Z}_i \end{pmatrix}$$

$$\triangleq \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^{-1} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}.$$

Thus the final estimation of $g(t)$ is obtained as follows

$$\hat{g}^*_n(t) \triangleq \hat{g}_{1n}(t) - \hat{g}'_{2n}(t)\hat{a}_n - \hat{g}'_{3n}(t)\hat{b}_n. \tag{5}$$

By Lemma 2 in Hong(1991), $\frac{1}{n}(\tilde{X}\,\tilde{Y})'(\tilde{X}\,\tilde{Y})$ is a positive definite matrix almost surely under the conditions listed below. So we may assume that for $\begin{pmatrix} \tilde{X}'\tilde{X} & \tilde{X}'\tilde{Y} \\ \tilde{Y}'\tilde{X} & \tilde{Y}'\tilde{Y} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ there exists an inverse matrix(when $n \to \infty$). To estimate α and b, we solve the group of equations

$$\begin{cases} C_{11}\hat{a}_n + C_{12}\hat{b}_n = \zeta \\ C_{21}\hat{a}_n + C_{22}\hat{b}_n = \eta \end{cases}$$

and get

$$\begin{cases} \hat{a}_n = A_n(\zeta - C_{12}C_{22}^{-1}\eta) \\ \quad = A_n \sum_{i=1}^{n} (\tilde{X}_i - C_{12}C_{22}^{-1}\tilde{Y}_i)\tilde{Z}_i & (6) \\ \hat{b}_n = B_n(\eta - C_{21}C_{11}^{-1}\zeta) \\ \quad = B_n \sum_{i=1}^{n} (\tilde{Y}_i - C_{21}C_{11}^{-1}\tilde{X}_i)\tilde{Z}_i, & (7) \end{cases}$$

where $A_n = (C_{11} - C_{12}C_{22}^{-1}C_{21})^{-1}$,

$B_n = (C_{22} - C_{21}C_{11}^{-1}C_{12})^{-1}$.

Remark: For A_n and B_n there exist inverse matrices, since

$$C_{11} - C_{12}C_{22}^{-1}C_{21} = \tilde{X}'[I_n - \tilde{Y}(\tilde{Y}'\tilde{Y})^{-1}\tilde{Y}']\tilde{X}$$

$$C_{22} - C_{21}C_{11}^{-1}C_{12} = \tilde{Y}'[I_n - \tilde{X}(\tilde{X}'\tilde{X})^{-1}\tilde{X}']\tilde{Y}$$

and $I_n - \tilde{Y}(\tilde{Y}'\tilde{Y})^{-1}\tilde{Y}'$ is a symmetric matrix with equal power. The rank of the matrix equals to $(n - q)$ and its characteristic roots are 0 or 1. Thus we may find an orthogonal square matrix Γ, which satisfies

$$C_{11} - C_{12}C_{22}^{-1}C_{21} = \tilde{X}'\Gamma'\begin{pmatrix} I_{n-q} & 0 \\ 0 & 0 \end{pmatrix}\Gamma\tilde{X}.$$

Because of the invertibility of $(\tilde{X}\,\tilde{Y})'(\tilde{X}\,\tilde{Y})$, we may get $n \ge p + q$, $\text{Rank}(\Gamma\tilde{X}) = p$ and $\text{Rank}(C_{11} - C_{12}C_{22}^{-1}C_{21}) = p$. Thus for A_n there exists an inverse matrix. Similarly, for B_n, there also exists an inverse matrix.

THE MAIN RESULTS AND PROOFS

In what follows, we assume that

1. T_1 has the density function $r(t)$ and $0 < \inf_{0\le t\le 1} r(t) \le \sup_{0\le t\le 1} r(t) < \infty$

2. The functions $g(t), g_1(t), g_{2j}(t)(1 \le j \le p)$ and $g_{3j}(t)(1 \le j \le q)$ satisfy the Lipschitz condition of order 1, where $g_{2j}(t)$ and $g_{3j}(t)$ are the j-th components of $g_2(t)$ and $g_3(t)$,

3. $E(Z_1^2 \mid T_1 = t)$ is a bounded function of t. X and Y are two bounded vectors.

4. $\sum_0 = \mathrm{Cov}\left(\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} - E\left(\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \mid T_1\right)\right)$

$$= \begin{pmatrix} \mathrm{Cov}(X_1 - E(X_1 \mid T_1)) & \mathrm{Cov}(X_1 - E(X_1 \mid T_1), Y_1 - E(Y_1 \mid T_1)) \\ \mathrm{Cov}(Y_1 - E(Y_1 \mid T_1), X_1 - E(X_1 \mid T_1)) & \mathrm{Cov}(Y_1 - E(Y_1 / T_1)) \end{pmatrix}$$

$$\triangleq \begin{pmatrix} \sum_{11} & \sum_{12} \\ \sum_{21} & \sum_{22} \end{pmatrix}$$

is a positive definite matrix,

5. $E(Y'_1 \sum Y_1) = \sigma_1^2$ exists.

The main results of the paper are as follows.

Theorem 1: In the conditions (1) ~ (5) and a), b), we have

$$\sqrt{n}(\hat{a}_n - \alpha) \xrightarrow{\mathscr{D}} N(0, (\sigma^2 + \sigma_1^2) \cdot (\textstyle\sum_{11} - \sum_{12}\sum_{22}^{-1}\sum_{21})^{-1}) \quad (8)$$

and

$$\sqrt{n}(\hat{b}_n - b) \xrightarrow{\mathscr{D}} N(0, (\sigma^2 + \sigma_1^2) \cdot (\textstyle\sum_{22} - \sum_{21}\sum_{11}^{-1}\sum_{12})^{-1}). \quad (9)$$

Theorem 2: In the conditions (1) ~ (5), a), b), and $E \parallel X_1 \parallel^3 < +\infty$, $E \parallel Y_1 \parallel^3 < +\infty$, $E \mid e_1 \mid^3 < +\infty$, we have for $k = [Cn^{2/3}](0 < C < +\infty)$,

$$\hat{g}_n^*(t) - g(t) = O_p(n^{-1/3}) \quad 0 \le t \le 1 \quad (10)$$

Let

$$\tilde{G}(T) = (g(T_1) - \hat{g}_n(T_1), \cdots, g(T_n) - \hat{g}_n(T_n))', e = (e_1, \cdots, e_n)',$$

$h = (Y'_1(\beta_1 - b), \cdots, Y'_n(\beta_n - b))'$, then

$$\tilde{Z} = \tilde{X}\alpha + \tilde{Y}b + \tilde{G}(T) + e + h. \quad (11)$$

The proof of Theorem 1: From (6), (11) and the definition of A_n, we have

$$\begin{aligned} \sqrt{n}(\hat{a}_n - \alpha) &= \sqrt{n}\Big(A_n \sum_{i=1}^n (\tilde{X}_i - C_{12}C_{22}^{-1}\tilde{Y}_i)\tilde{Z}_i - \alpha\Big) \\ &= \sqrt{n}A_n \sum_{i=1}^n (\tilde{X}_i - C_{12}C_{22}^{-1}\tilde{Y}_i)(\tilde{Z}_i - \tilde{X}'_i\alpha) \\ &= \sqrt{n}A_n \sum_{i=1}^n (\tilde{X}_i - C_{12}C_{22}^{-1}\tilde{Y}_i)(\tilde{Y}'_i b + g(T_i) - \hat{g}_n(T_i) + e_i + Y'_i(\beta_i - b)) \\ &= \sqrt{n}A_n \sum_{i=1}^n (\tilde{X}_i - C_{12}C_{22}^{-1}\tilde{Y}_i)(g(T_i) - \hat{g}_n(T_i)) + \sqrt{n}A_n \sum_{i=1}^n (\tilde{X}_i - C_{12}C_{22}^{-1}\tilde{Y}_i)e_i \\ &\quad + \sqrt{n}A_n \sum_{i=1}^n (\tilde{X}_i - C_{12}C_{22}^{-1}\tilde{Y}_i)Y'_i(\beta_i - b) \\ &\triangleq I_{n1} + I_{n2} + I_{n3}. \end{aligned} \quad (12)$$

By Lemma 2 of Hong(1991),

$$\frac{1}{n}(\tilde{X}\,\tilde{Y}')(\tilde{X}\,\tilde{Y}) \xrightarrow{a.s.} \textstyle\sum_0, \text{ then}$$

$$\frac{1}{n}\tilde{X}'\tilde{X} \longrightarrow \textstyle\sum_{11}, \frac{1}{n}\tilde{X}'\tilde{Y} \longrightarrow \sum_{12},$$

$$\frac{1}{n}\tilde{Y}'\tilde{Y} \longrightarrow \textstyle\sum_{22} \quad a.s.$$

Thereby

$$\begin{aligned} \frac{1}{n}A_n^{-1} &= \frac{1}{n}\tilde{X}'(I_n - \tilde{Y}(\tilde{Y}'\,\tilde{Y})^{-1}\tilde{Y}')\tilde{X} \\ &= \frac{1}{n}\tilde{X}'\tilde{X} - \Big(\frac{1}{n}\tilde{X}'\tilde{Y}\Big)\Big(\frac{1}{n}\tilde{Y}'\tilde{Y}\Big)^{-1}\Big(\frac{1}{n}\tilde{Y}'\tilde{X}\Big) \\ &\to \textstyle\sum_{11} - \sum_{12}\sum_{22}^{-1}\sum_{12} \end{aligned}$$

i.e.,

$$nA_n \to \left(\textstyle\sum_{11} - \sum_{12}\sum_{22}^{-1}\sum_{21}\right)^{-1} \quad (13)$$

then

$$\begin{aligned} I_{n1} &= \sqrt{n}A_n(\tilde{X}'\tilde{G}(T) - C_{12}C_{22}^{-1}\tilde{Y}'\tilde{G}(T)) \\ &= nA_n\Big(\frac{1}{\sqrt{n}}\tilde{X}'\tilde{G}(T) - \Big(\frac{C_{12}}{n}\Big)\Big(\frac{C_{22}}{n}\Big)^{-1} \frac{1}{\sqrt{n}}\tilde{Y}'\tilde{G}(T)\Big). \end{aligned} \quad (14)$$

Similar to Hong(1991), we may get

$$\frac{1}{\sqrt{n}}\tilde{X}'\tilde{G}(T) \xrightarrow{\mathscr{P}} 0 \quad (15)$$

and

$$\frac{1}{\sqrt{n}}\tilde{Y}'\tilde{G}(T) \xrightarrow{\mathscr{P}} 0. \quad (16)$$

Since $\frac{C_{12}}{n}=\frac{\tilde{X}'\tilde{Y}}{n}\to\sum_{12},\frac{C_{22}}{n}=\frac{\tilde{Y}'\tilde{Y}}{n}\to\sum_{22}$, and substituting them in (14), we have

$$I_{n1}\xrightarrow{\mathscr{P}}0. \quad (17)$$

Now we discuss

$$I_{n2}=\sqrt{n}A_n\sum_{i=1}^{n}(\tilde{X}_i-C_{12}C_{22}^{-1}\tilde{Y}_i)e_i$$
$$=\sqrt{n}(\tilde{X}'_0\tilde{X}_0)^{-1}\tilde{X}'_0e, \quad (18)$$

where $\tilde{X}_0=[I_n-\tilde{Y}(\tilde{Y}'\tilde{Y})^{-1}\tilde{Y}']\tilde{X}$.

The following result was obtained by a method similar to that for getting Equation(54)of Hong(1991),

$$(\tilde{X}'_0\tilde{X}_0)^{-1/2}\tilde{X}'_0e\xrightarrow{\mathscr{D}}N(0,\sigma^2I_p) \quad (19)$$

By (13),

$$\sqrt{n}(\tilde{X}'_0\tilde{X}_0)^{-1/2}=\sqrt{n}A_n^{1/2}\to$$
$$\left(\sum_{11}-\sum_{12}\sum_{22}^{-1}\sum_{21}\right)^{-\frac{1}{2}} \quad (20)$$

then

$$I_{n2}\xrightarrow{\mathscr{D}}N(0,\sigma^2(\sum_{11}-\sum_{12}\sum_{22}^{-1}\sum_{21})^{-1}). \quad (21)$$

Therefore Equation (8) is equivalent to

$$I_{n3}\xrightarrow{\mathscr{D}}\mathrm{N}(0,\sigma_1^2)(\sum_{11}-\sum_{12}\sum_{22}^{-1}\sum_{21})^{-1}), \quad (22)$$

since

$$I_{n3}=\sqrt{n}A_n\sum_{i=1}^{n}(\tilde{X}_i-C_{12}C_{22}^{-1}\tilde{Y}_i)Y'_i(\beta_i-b)$$
$$=\sqrt{n}(\tilde{X}'_0\tilde{X}_0)^{-1}\tilde{X}'_0h. \quad (23)$$

From (20), we need only to prove that

$$(\tilde{X}'_0\tilde{X}_0)^{-1/2}\tilde{X}'_0h\xrightarrow{\mathscr{D}}N(0,\sigma_1^2I_p). \quad (24)$$

Obviously, it is sufficient for every p-dimension vector $\alpha=(\alpha_1,\cdots,\alpha_p)$ that we have

$$S_n=\alpha'(\tilde{X}'_0\tilde{X}_0)^{-1/2}\tilde{X}'_0h\xrightarrow{\mathscr{D}}N(0,\sigma_1^2\alpha'\alpha). \quad (25)$$

Putting

$$A_{n0}=(\tilde{X}'_0\tilde{X}_0)^{-\frac{1}{2}}\tilde{X}'_0=(a_{nij})_{p\times n},$$
$$a_j=A_n^{1/2}(\tilde{X}_j-C_{12}C_{22}^{-1}\tilde{Y}_j),$$

$\mathscr{F}_n=\{(X_1,Y_1,T_1),\cdots,(X_n,Y_n,T_n)\}$, then

$$S_n=\alpha'A_{n0}h=\sum_{j=1}^{n}\left(\sum_{i=1}^{p}a_{nij}\alpha_i\right)(Y'_j(\beta_j-b))$$
$$\triangleq\sum_{j=1}^{n}b_{nj}(Y'_j(\beta_j-b)) \quad (26)$$
$$b_{nj}=a'_j\alpha,$$

It holds by independence of β and (X,Y,T) and the following Lemma 2 that,

$$E(b_{nj}Y'_j(\beta_j-b)|\mathscr{F}_n)=b_{nj}Y'_jE(\beta_j-b)=0$$

$$\sum_{j=1}^{n}E[(b_{nj}Y'_j(\beta_j-b))^2]|\mathscr{F}_n]$$
$$=\sum_{j=1}^{n}\{b_{nj}^2Y'_jE(\beta_j-b)(\beta_j-b)'Y_j\}$$
$$=\alpha'\{A_n^{1/2}\sum_{j=1}^{n}(Y'_j\sum Y_j)(\tilde{X}_j-C_{12}C_{22}^{-1}\tilde{Y}_j)\cdot(\tilde{X}_j-C_{12}C_{22}^{-1}\tilde{Y}_j)'A_n^{1/2}\}\alpha$$
$$=\sum_{j=1}^{n}(b_{nj}^2Y'_j\sum Y_j)\to\sigma_1^2\alpha'\alpha. \quad (27)$$

For every $\varepsilon>0$ and $M>0$, it follows from (27) and $\sum_{j=1}^{n}b_{nj}^2=\alpha'\alpha$, that

$$\sum_{j=1}^{n}E[(b_{nj}Y'_j(\beta_j-b))^2I(|b_{nj}Y'_j(\beta_j-b)|\geq\varepsilon)|\mathscr{F}_n]$$
$$\leq\sum_{j=1}^{n}E\left\{(b_{nj}Y'_j(\beta_j-b))^2\left[I\left(|b_{nj}|\geq\frac{\varepsilon}{M}\right)+I(|Y'_j(\beta_j-b)|\geq M)\right]|\mathscr{F}_n\right\}.$$

By condition 3, we may assume that every component of Y is dominated by C, then

$$\sum_{j=1}^{n}E[(b_{nj}Y'_j(\beta_j-b))^2I(|b_{nj}Y'_j(\beta_j-b)|\geq\varepsilon)|\mathscr{F}_n]$$
$$\leq\sum_{j=1}^{n}b_{nj}^2(Y'_j\sum Y_j)I\left(|b_{nj}|\geq\frac{\varepsilon}{M}\right)+C^2\alpha'\alpha E\left\{\left(\sum_{j=1}^{q}|\beta_{1j}-b_j|\right)^2\cdot I\left(\sum_{j=1}^{q}|\beta_{1j}-b_j|\geq\frac{M}{C}\right)\right\}$$
$$\leq C\sigma_1^2\alpha'\alpha I\left(\max_{1\leq j\leq n}|b_{nj}|\geq\frac{\varepsilon}{M}\right)+C\alpha'\alpha E\left(\left(\sum_{j=1}^{q}|\beta_{1j}-b_j|\right)^2\cdot I\left(\sum_{j=1}^{q}|\beta_{1j}-b_j|\geq\frac{M}{C}\right)\right), \quad (28)$$

where β_{1j} is the j-th component of β_1. Letting

$M \to \infty$, then the second term of (28) can be made arbitrarily small. To get (25), we need only to prove the following result from Dvoretzky (1972),

$$\max_{1 \le j \le n} | b_{nj} | \xrightarrow{P} 0. \quad (29)$$

Equation (29) can be obtained by the same method for getting Equation (57) of Hong (1991). So that (22) is obtained.

$$I_{n3} \xrightarrow{\mathscr{D}} N\left(0, \sigma_1^2\left(\sum_{11} - \sum_{12}\sum_{22}^{-1}\sum_{21}\right)^{-1}\right).$$

Hence from (17)、(21)、(22), the first part of the theorem is proved.

Next we consider the second part of the theorem, i.e.,

$$\sqrt{n}(\hat{b}_n - b) \xrightarrow{\mathscr{D}} N\Big(0, (\sigma^2 + \sigma_1^2) \cdot \left(\sum_{22} - \sum_{21}\sum_{11}^{-1}\sum_{12}\right)^{-1}\Big)$$

Noting

$$\begin{aligned}\sqrt{n}(\hat{b}_n - b_n) &= \sqrt{n}B_n \sum_{i=1}^{n} (\tilde{Y}_i - C_{21}C_{11}^{-1}\tilde{X}_i)(g(T_i) - \hat{g}_n(T_i)) + \\ &\quad \sqrt{n}B_n \sum_{i=1}^{n} (\tilde{Y}_i - C_{21}C_{11}^{-1}\tilde{X}_i)e_i \\ &\quad + \sqrt{n}B_n \sum_{i=1}^{n} (\tilde{Y}_i - C_{21}C_{11}^{-1}\tilde{X}_i) Y'_i(\beta_i - b) \\ &\triangleq J_{n1} + J_{n2} + J_{n3}. \quad (30)\end{aligned}$$

Similarly, the following results are obtained,

$$J_{n1} \xrightarrow{\mathscr{P}} 0, \quad (31)$$

$$J_{n2} \xrightarrow{\mathscr{D}} N\left(0, \sigma^2\left(\sum_{22} - \sum_{21}\sum_{11}^{-1}\sum_{12}\right)^{-1}\right), \quad (32)$$

$$J_{n3} \xrightarrow{\mathscr{D}} N\left(0, \sigma_1^2\left(\sum_{22} - \sum_{21}\sum_{11}^{-1}\sum_{12}\right)^{-1}\right), \quad (33)$$

Consequently the second assertion holds.

The proof of Theorem 2: From Wei and Su (1986), it follows under the conditions of the theorem,

$$E(\hat{g}_{1n}(t) - g_1(t))^2 = O(n^{-2/3}),$$

$$E(\hat{g}_{2nj}(t) - g_{2j}(t))^2 = O(n^{-2/3}), \quad 1 \le j \le p$$

$$E(\hat{g}_{3nj}(t) - g_{3j}(t))^2 = O(n^{-2/3}). \quad 1 \le j \le q$$

where $\hat{g}_{2nj}(t)$ and $\hat{g}_{3nj}(t)$ are the j-th components of $\hat{g}_{2n}(t)$ and $\hat{g}_{3n}(t)$ respectively. Then

$$\hat{g}_{1n}(t) - g_1(t) = O_p(n^{-\frac{1}{3}}), \quad (34)$$

$$\hat{g}_{2nj}(t) - g_{2j}(t)) = O_p(n^{-\frac{1}{3}}), \quad (35)$$
$$1 \le j \le p$$

$$\hat{g}_{3nj}(t) - g_{3j}(t) = O_p(n^{-\frac{1}{3}}), \quad (36)$$
$$1 \le j \le q$$

To calculate the conditional expectation about $T = t$ in the model(1), we have

$$g(t) = g_1(t) - g'_2(t)\alpha - g'_3(t)b. \quad 0 \le t \le 1 \quad (37)$$

therefore

$$\begin{aligned}\hat{g}_n^*(t) - g(t) &= (\hat{g}_{1n}(t) - g_1(t)) \\ &- \sum_{j=1}^{p}(\hat{g}_{2nj}(t) - g_{2j}(t))(\hat{\alpha}_{nj} - \alpha_j) \\ &- \sum_{j=1}^{q}(\hat{g}_{3nj}(t) - g_{3j}(t))(\hat{b}_{nj} - b_j) \\ &- \sum_{j=1}^{p}(\hat{g}_{2nj}(t) - g_{2j}(t))\alpha_j \\ &- \sum_{j=1}^{q}(\hat{g}_{3nj}(t) - g_{3j}(t))b_j \\ &- \sum_{j=1}^{p} g_{2j}(t)(\hat{\alpha}_{nj} - \alpha_j) - \sum_{j=1}^{q} g_{3j}(t)(\hat{b}_{nj} - b_j).\end{aligned} \quad (38)$$

Theorem 1 and (34)、(35)、(36) are used to complete the proof of the theorem.

THE TWO LEMMAS

Lemma 1 (Tao et al., 1981): If $\{V_i, 1 \le i \le n\}$ is a sequence of random variables with mean zero, and $\{a_i, 1 \le i \le n\}$ is a sequence of constants satisfying $\sum_{i=1}^{n} a_i^2 = 1$, then

$$E\left(\sum_{i=1}^{n} a_i V_i\right)^{2s} \le 3^s(2s-1)!! \max_{1 \le i \le n} EV_i^{2s}$$

where s is a positive integer.

Lemma 2: If X, Y are two bounded random vectors, then

$$\frac{1}{n}\sum_{i=1}^{n}(Y'_i\sum Y_i-\sigma_1^2)(\tilde{X}_i-C_{12}C_{22}^{-1}\tilde{Y}_i)$$
$$(\tilde{X}_i-C_{12}C_{22}^{-1}\tilde{Y}_i)'\to 0\quad \text{a.s}$$
$$\frac{1}{n}\sum_{i=1}^{n}(Y'_i\sum Y_i-\sigma_1^2)(\tilde{Y}_i-C_{21}C_{11}^{-1}\tilde{X}_i)$$
$$(\tilde{Y}_i-C_{21}C_{11}^{-1}\tilde{X}_i)'\to 0\quad \text{a.s}$$

i.e.,

$$\frac{1}{n}\Big\{A_n^{1/2}\sum_{i=1}^{n}(Y'_i\sum Y_i)(\tilde{X}_i-C_{12}C_{22}^{-1}\tilde{Y}_i)$$
$$(\tilde{X}_i-C_{12}C_{22}^{-1}\tilde{Y}_i)'A_n^{\frac{1}{2}}-\sigma_1^2\Big\}\to 0\quad \text{a.s.}$$
$$\frac{1}{n}\Big\{B_n^{1/2}\sum_{i=1}^{n}(Y'_i\sum Y_i)(\tilde{Y}_i-C_{21}C_{11}^{-1}\tilde{X}_i)$$
$$(\tilde{Y}_i-C_{21}C_{11}^{-1}\tilde{X}_i)'B_n^{\frac{1}{2}}-\sigma_1^2\Big\}\to 0\quad \text{a.s.}$$

Proof: It follows under the conditions that

$$H_{i1}=(\tilde{X}_i-C_{12}C_{22}^{-1}\tilde{Y}_i)(\tilde{X}_i-C_{12}C_{22}^{-1}\tilde{Y}_i)',$$
$$H_{i2}=(\tilde{Y}_i-C_{21}C_{11}^{-1}\tilde{X}_i)(\tilde{Y}_i-C_{21}C_{11}^{-1}\tilde{X}_i)'$$

are two bounded matrices and $\left\{\left(Y'_i\sum Y_i-\sigma_1^2\right),1\le i\le n\right\}$ is a sequence of i.i.d. bounded random vectors with mean zero. Then we only need to prove that

$$\frac{1}{n}\sum_{i=1}^{n}\zeta_i\eta_i\to 0,\qquad \text{a.s.}\tag{39}$$

where $\{\zeta_i,1\le i\le n\}$ is a sequence of i.i.d. bounded random variables with mean zero and $\{\eta_i,1\le i\le n\}$ is a sequence of bounded random variables. Assume that

$$|\zeta_i|\le C_1,\quad |\eta_i|\le C_2.$$

Let

$$\zeta_{i1}=\sqrt{2/5}\,\zeta_i/C_1,\ \eta_{i1}=\sqrt{2/5}\,\eta_i/C_2,$$

and

$$T_n=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\zeta_{i1}\eta_{i1}.$$

Then, it follows by Lemma 1 that if, for every $\varepsilon>0$, $\eta_1,\cdots,\eta_n$ are known

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\zeta_i\eta_i\right|>\varepsilon\right)=P\left(|T_n|>\frac{2\varepsilon}{5C_1C_2}\sqrt{n}\right)$$
$$\le \exp(-Cn)Ee^{T_n^2}=\exp(-Cn)\sum_{s=0}^{\infty}\frac{1}{s!}ET_n^{2s}$$
$$\le \exp(-Cn)\Big[1+\sum_{s=1}^{\infty}\frac{1}{s!}3^s(2s-1)!!\max_{1\le i\le n}E(\zeta_{i1}\eta_{i1})^{2s}\Big]$$
$$\le \exp(-Cn)\Big[1+\sum_{s=1}^{\infty}\frac{1}{s!}3^s(2s-1)!!\left(\frac{2}{5}\right)^{2s}\Big]$$
$$\le \exp(-Cn)\Big[1+\sum_{s=1}^{\infty}\left(\frac{24}{25}\right)^{s}\Big]$$
$$=25\exp(-Cn).\tag{40}$$

Such that

$$\sum_{n=1}^{\infty}P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\zeta_i\eta_i\right|>\varepsilon\right)<+\infty.$$

By Borel-Cantelli lemma and the arbitrariness of ε, it holds that

$$\frac{1}{n}\sum_{i=1}^{n}\zeta_i\eta_i\to 0\quad \text{a.s.}$$

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