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SOLUTION OF NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS BY GENERAL ORTHOGONAL POLYNOMIALS

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Abstract: A proposed method for finding an approximate solution of the nonlinear ordinary differential equations two-point boundary value problem is proposed. It simplifies the problem approximately to a problem of solving a set of nonlinear algebraic equations. The basic idea of the method is to utilize the properties of orthogonal polynomials and the approximate operational matrices of the nonlinear functional $f(x(t), u(t), t)$, and also the transformation matrix between the back vector and the current time vector for the general orthogonal polynomials. A method for solving the nonlinear two-point boundary value problems for descriptor systems is also given.

Key words: nonlinear systems, two-point boundary value problems, approximate solution, orthogonal polynomials.
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INTRODUCTION

In recent years, orthogonal polynomials and functions developed by Chang et al. (1986) have been successfully applied in the field of dynamic systems, for analysis and identification of linear systems and the optimal control (Tsay et al., 1987). The main advantage of this technique is the reduction of these problems to one of solving systems of algebraic equations. Thus it greatly simplifies the problem.

The ordinary differential equation of two-point boundary value problem is very important in control theory. It plays an important role in deriving the solution of the matrix Riccati equation which appears in most control system designs (Razzaghi, 1987). It is well known that the integration associated with the differential equation of these problems is difficult and needs long computation.

Recently (Chang et al., 1986), applied the operational matrix of integration to generalize orthogonal polynomials. Horng et al. (1987) used the transformation matrix relating the back vector to the current time vector for the shifted Chebyshev polynomials of the first kind to derive solutions of linear two-point boundary value problem with constant coefficients. Razzaghi et al. (1989) proposed a technique to derive solutions of a linear two-point boundary value problem with time-varying coefficients via Taylor series.

However, this technique cannot be as successfully applied to nonlinear systems as compared to linear systems. By using the approximation-operational vector and matrix of the nonlinear functionals

$$f(x(t),\ u(t),\ t),$$

together with the integration-operational matrix of general orthogonal polynomials, Wang et al. (1992) derived a method for solving nonlinear differential equations with initial value problems.

Here we solve the nonlinear differential equation of a two-point boundary value problem via general orthogonal polynomial series, by using the approximation operational vector and matrix of the nonlinear functionals, and the transformation matrix relating the back vector to the current time vector for the general orthogonal polynomials. The main characteristic of this technique is its ability to reduce the nonlinear differential equation of two-point boundary value problems to that of solving systems of nonlinear algebraic equations.

At the same time, a method for solving the nonlinear two-point boundary value problems for descriptor systems is also given.

PROPERTIES OF THE GENERAL ORTHOGONAL POLYNOMIALS

In general, the general orthogonal polynomials $\varphi_i(t)$ with respect to the weight function $w(t)$ over the interval $[a, b]$ are defined, by (if they satisfy) the condition

$$\int_a^b w(t)\varphi_i(t)\varphi_j(t)dt = \begin{cases} r_i, & i = j, \\ 0, & i \neq j. \end{cases} \quad (1)$$

The general orthogonal polynomials $\varphi_i(t)$ defined above have the following recurrence relations (Szego, 1975; Luke, 1969):

(a) the pure recurrence relation

$$\varphi_{i+1}(t) = (a_i t + b_i)\varphi_i(t) - c_i\varphi_{i-1}(t), \quad (2)$$

with

$$\varphi_{-1}(t) = 0, \quad \varphi_0(t) = 1;$$

(b) the differential recurrence relation

$$\varphi_i(t) = A_i\varphi'_{i+1}(t) + B_i\varphi'_i(t) + C_i\varphi'_{i-1}(t), \quad (3)$$

where a_i, b_i, c_i and A_i, B_i, C_i are the recurrence coefficients and the differential recurrence coefficients, respectively, whose values are specified by the particular orthogonal polynomials under consideration.

The polynomial $\varphi_m(t)$ has m distinct real zeros, all within the interval(a, b). Let $t_1, t_2, \cdots, t_m$ be the m zeros of $\varphi_m(t)$.

Note that an arbitrary function $h(t)$ can be approximated by the general orthogonal polynomial:

$$h(t) \approx h_{(m)}(t) = \sum_{i=0}^{m-1} h_i\varphi_i(t), \quad (4)$$

or, in vector form

$$h_{(m)}(t) = h^T\Phi(t), \quad (5)$$

where the superscript T denotes transpose, h is called the general orthogonal coefficient vector, and $\Phi(t)$ is the general orthogonal polynomial vector. These two vectors are defined as

$$h = (h_0, h_1, \cdots, h_{m-1})^T, \quad (6)$$

and

$$\Phi(t) = (\varphi_0(t), \varphi_1(t), \cdots, \varphi_{m-1}(t))^T. \quad (7)$$

The general orthogonal coefficients h_i can be determined by minimizing the integrated square error

$$G(h) = \int_a^b w(t)\left[h(t) - \sum_{j=0}^{m-1} h_j\varphi_j(t)\right]^2 dt, \quad (8)$$

to yield

$$h_i = \frac{1}{r_i}\int_a^b w(t)h(t)\varphi_i(t)dt, \quad i = 0, 1, \cdots, m-1, \quad (9)$$

Proposition 1

Let

$$D_i = -A_i\varphi_{i+1}(0) - B_i\varphi_i(0) - C_i\varphi_{i-1}(0), \quad i = 0, 1, \cdots, m-1,$$

$A_0 = \dfrac{1}{a_0}$, $B_0 = -\dfrac{b_0}{a_0}$, C_0 is an arbitrary constant,

$A_1 = \dfrac{1}{2a_1}$, $B_1 = \dfrac{1}{2}\left(\dfrac{b_0}{a_0} - \dfrac{b_1}{a_1}\right)$, C_1 is an arbitrary constant,

$$P = \begin{pmatrix} B_0 & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ C_1 + D_1 & B_1 & A_1 & 0 & \cdots & 0 & 0 & 0 \\ D_2 & C_2 & B_2 & A_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ D_{m-2} & & 0 & 0 & \cdots & C_{m-2} & B_{m-2} & A_{m-2} \\ D_{m-1} & 0 & 0 & 0 & \cdots & 0 & C_{m-1} & B_{m-1} \end{pmatrix},$$

then

$$\int_0^t \Phi(t)dt \approx P\Phi(t), \quad (10)$$

where P is called the operational matrix of integration of the general orthogonal polynomials.

In this paragraph, approximation of the nonlinear functional $f(x(t), u(t), t)$ is given by using the general orthogonal coefficients of $x(t)$ and $u(t)$, where $f(x(t), u(t), t)$ is assumed to be continuous with respect to $x(t)$, $u(t)$ and t.

Let

$$x(t) \approx x_{(m)}(t) = X^T\Phi(t) \quad (11)$$

and

$$u(t) \approx u_{(m)}(t) = U^T\Phi(t) \quad (12)$$

be the m order approximation of $x(t)$ and $u(t)$, respectively.

Proposition 2 (Wang et al., 1992)

Let

$$N_f = ((N_f)_{ij}), i = 0, 1, \cdots, m-1,$$

$$j = 0,1,\cdots,m-1, \tag{13}$$

$$(N_f)_{i\ j} = \frac{\varphi_j(t_{i+1})\left(\prod_{k=j+1}^{m-1}\frac{c_k}{a_k}\right)\left(\prod_{k=0}^{j-1}\frac{1}{a_k}\right)}{\varphi_{m-1}(t_{i+1})\prod_{\substack{k=1\\k\neq i+1}}^{m}(t_{i+1}-t_k)} \tag{14}$$

then

$$f(x(t),u(t),t) \approx (f_1(X,U),f_2(X,U),\cdots,f_m(X,U))N_f\Phi(t), \tag{15}$$

where

$$f_i(X,U) = f(X^T\Phi(t_i),U^T\Phi(t_i),t_i),$$

$t_i(i=1,2,\cdots,m)$ are m zeros of $\varphi_m(t)$, N_f is called the operational matrix of approximation of the nonlinear functional $f(x(t),u(t),t)$.

In the following, we can see that the transformation matrix relating the back vector to the current time vector for the general orthogonal polynomials will also be used.

Proposition 3

The back vector $\Phi(s-t)$ is related to the current time vector $\Phi(t)$ for the general orthogonal polynomials through the transformation

$$\Phi(s-t) = S\Phi(t), \tag{16}$$

where S is the $m\times m$ transformation matrix given by

$$S = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ S_{10} & -1 & 0 & \cdots & 0 & 0 \\ S_{20} & S_{21} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{m-2,0} & S_{m-2,1} & S_{m-2,2} & \cdots & (-1)^{m-2} & 0 \\ S_{m-1,0} & S_{m-1,1} & S_{m-1,2} & \cdots & S_{m-1,m-2} & (-1)^{m-1} \end{pmatrix}. \tag{17}$$

Here S is a lower triangular matrix whose (i,j)th element $S_{i\ j}$ can be determined from a_i, b_i, c_i and s, which are constants, where

$$S_{ij} = (-1)^i, \quad i = 0,1,\cdots,m-1,$$

$$S_{ij} = 0, \quad j > i.$$

When $j < i$, we have the following recurrence relations:

$$S_{10} = a_0 s + 2b_0,$$

$$S_{i+1,0} = \left(a_i s + b_i + \frac{a_i b_0}{a_0}\right)S_{i0} - \frac{a_i c_1}{a_1}S_{i1} - c_i S_{i-1,0},$$

$$S_{i+1,1} = \left(a_i s + b_i + \frac{a_i b_1}{a_1}\right)S_{i1} - \frac{a_i}{a_0}S_{i0} - c_i S_{i-1,1},$$

$$\vdots$$

$$S_{i+1,i-1} = \left(a_i s + b_i + \frac{a_i b_{i-1}}{a_{i-1}}\right)S_{i,i-1},$$

$$S_{i+1,i} = (-1)^i(a_i s + 2b_i) - \frac{a_i}{a_{i-1}}S_{i,i-1}.$$

Proof From(2), we have

$$\varphi_{i+1}(s-t) = (a_i s + 2b_i)\varphi_i(s-t) - (a_i t + b_i)\varphi_i(s-t) - c_i\varphi_{i-1}(s-t),$$

$$i = 0,1,\cdots,m-1,$$

with

$$\varphi_0(s-t) = \varphi_0(t) = 1,$$

$$\varphi_1(s-t) = (a_0 s + 2b_0)\varphi_0(t) - \varphi_1(t) = S_{10}\varphi_0(t) - \varphi_1(t).$$

Let

$$\varphi_i(s-t) = S_{i0}\varphi_0(t) + S_{i1}\varphi_1(t) + \cdots + S_{i,i-1}\varphi_{i-1}(t) + (-1)^i\varphi_i(t)$$

and notice that

$$(a_i t + b_i)\varphi_k(t) = \frac{a_i}{a_k}\left[(a_k t + b_k) - b_k + \frac{a_k b_i}{a_i}\right]\cdot \varphi_k(t) + \frac{a_i}{a_k}(c_k - c_k)\varphi_{k-1}(t)$$
$$= \frac{a_i}{a_k}\varphi_{k+1}(t) + \frac{a_i}{a_k}\left(\frac{a_k b_i}{a_i} - b_k\right)\cdot \varphi_k(t) + \frac{a_i c_k}{a_k}\varphi_{k-1}(t),$$

$$k = 0,1,\cdots,i,$$

then we obtain

$$\varphi_{i+1}(s-t) = [(a_i s + 2b_i)\varphi_i(s-t) - c_i\varphi_{i-1}\cdot(s-t)] - (a_i t + b_i)\varphi_i(s-t)$$
$$= [(a_i s + 2b_i)S_{i,0} - c_i S_{i-1,0}]\varphi_0(t) + [(a_i s + 2b_i)S_{i,1} - c_i S_{i-1,1}]\cdot \varphi_1(t) + \cdots + [(a_i s + 2b_i)S_{i,i-1}$$

$$-(-1)^{i-1}c_i]\varphi_{i-1}(t)+(-1)^i\cdot(a_is+2b_i)\varphi_i(t)-[\frac{a_i}{a_0}\varphi_1(t)+(b_i-\frac{a_ib_0}{a_0})\varphi_0(t)]S_{i0}-[\frac{a_i}{a_1}\varphi_2(t)+\frac{a_i}{a_1}(\frac{a_1b_i}{a_i}-b_1)\cdot\varphi_1(t)+\frac{a_ic_1}{a_1}\varphi_0(t)]S_{i,1}-\cdots-[\frac{a_i}{a_{i-1}}\varphi_i(t)+\frac{a_i}{a_{i-1}}(\frac{a_{i-1}b_i}{a_i}-b_{i-1})\varphi_{i-1}(t)+\frac{a_ic_{i-1}}{a_{i-1}}\cdot\varphi_{i-2}(t)]S_{i,j-1}-(-1)^i\varphi_{i+1}(t)-(-1)^ic_i\varphi_{i-1}(t)$$

$$=[(a_is+b_i+\frac{a_ib_0}{a_0})S_{i,0}-\frac{a_ic_1}{a_1}S_{i,1}-c_iS_{i-1,0}]\varphi_0(t)+[(a_is+b_i+\frac{a_ib_1}{a_1})S_{i,1}-\frac{a_i}{a_0}S_{i,0}-c_iS_{i-1,1}]\cdot\varphi_1(t)+\cdots+[(a_is+b_i+\frac{a_ib_{i-1}}{a_{i-1}})S_{i,i-1}]\varphi_{i-1}(t)+[(-1)^i(a_is+2b_i)-\frac{a_i}{a_{i-1}}S_{i,i-1}]\varphi_i(t)+(-1)^{i+1}\varphi_{i+1}(t)$$

$$=S_{i+1,0}\varphi_0(t)+S_{i+1,1}\varphi_1(t)+\cdots+S_{i+1,i-1}\varphi_{i-1}(t)+S_{i+1,i}\cdot\varphi_i(t)+(-1)^{i+1}\varphi_{i+1}(t),$$

$$i=0,1,\cdots,m-2.$$

Thus, the proposition is proven.

SOLUTION OF NONLINEAR TWO-POINT BOUNDARY-VALUE PROBLEMS

Consider the following nonlinear system with the two-point boundary conditions:

$$\begin{cases}x'(t)=f(x(t),u(t),t), & 0\le t\le s, & (18)\\ x_1(0)=x_{10}, \quad x_2(s)=x_{2s}, & & (19)\end{cases}$$

where $u(t)$ is the known control variable, $u\in R^k$, and $x(t)$ is the unknown state variable to be solved, $x\in R^n$, and

$$x(t)=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix},u(t)=\begin{pmatrix}u_1(t)\\u_2(t)\end{pmatrix},\quad f=\begin{pmatrix}f_1\\f_2\end{pmatrix},x_0=\begin{pmatrix}x_{10}\\x_{2s}\end{pmatrix}, \tag{20}$$

where $x_i, f_i\in R^{n_i}$, $u_i\in R^{k_i}$, $i=1,2$, and $n_1+n_2=n$, $k_1+k_2=k$.

Denote by $y(t)$ and $v(t)$ the $n\times 1$ and $k\times 1$ matrices defined in the following way:

$$y(t)=\begin{pmatrix}y_1(t)\\y_2(t)\end{pmatrix}=\begin{pmatrix}x_1(t)\\x_2(s-t)\end{pmatrix}, \tag{21}$$

$$v(t)=\begin{pmatrix}v_1(t)\\v_2(t)\end{pmatrix}=\begin{pmatrix}u_1(t)\\u_2(s-t)\end{pmatrix}, \tag{22}$$

then (18), (19) can be expressed as

$$\begin{cases}y'(t)=g(x(t),u(t),t), & (23)\\ y(0)=x_0, & (24)\end{cases}$$

where

$$g(x(t),u(t),t)=\begin{pmatrix}f_1(x(t),u(t),t)\\-f_2(x(s-t),u(s-t),s-t)\end{pmatrix}. \tag{25}$$

In order to solve (23), (24), we let

$$x_{(m)}(t)=X^T\Phi(t) \tag{26}$$

be the m order approximation of $x(t)$ and expand $u(t)$ in the following general orthogonal polynomials

$$u_{(m)}(t)=U^T\Phi(t). \tag{27}$$

Notice that x_0 can be expanded into general orthogonal polynomials as

$$x_0=\begin{pmatrix}x_{10} & 0\cdots 0\\x_{2s} & 0\cdots 0\end{pmatrix}\Phi(t). \tag{28}$$

Now, by using (15), the nonlinear functional (25) can be approximated by the following general orthogonal polynomials

$$g_{(m)}(t,x(t),u(t))=(g_1(X,U),g_2(X,U),\cdots,g_m(X,U))N_f\Phi(t), \tag{29}$$

where the operational matrix N_f is determined from the formulas (13), (14), and

$$g_i(X,U)=\begin{pmatrix}f_1(X^T\Phi(t_i),U^T\Phi(t_i),t_i)\\-f_2(X^T\Phi(s-t_i),U^T\Phi(s-t_i),s-t_i)\end{pmatrix}. \tag{30a}$$

By using the proposition 3, it reduces to the following

$$g_i(X,U) = \begin{pmatrix} f_1(X^T\Phi(t_i),U^T\Phi(t_i),t_i) \\ -f_2(X^T\Phi(S-t_i),U^T\Phi(S-t_i),s-t_i) \end{pmatrix}. \tag{30b}$$

Integrating (23), (24) from $t=0$ to t, we obtain

$$\begin{cases} x_1(t) - x_{10} = \int_0^t f_1(x(t),u(t),t)\,dt, \\ x_2(s-t) - x_{2s} = -\int_0^t f_2(x(s-t), \\ \qquad u(s-t),s-t)\,dt, \end{cases} \tag{31a}$$

i.e.

$$\begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 0 & I_{n_2} \end{pmatrix} x(s-t) - x_0 = \begin{pmatrix} \int_0^t f_1(x(t),u(t),t)\,dt \\ -\int_0^t f_2(x(s-t),u(s-t),s-t)\,dt \end{pmatrix}. \tag{31b}$$

Substituting (26) - (30) into (31), and using (10) and proposition 3, and equating the coefficient vectors of the general orthogonal polynomial vectors, we obtain

$$\begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix} X^T + \begin{pmatrix} 0 & 0 \\ 0 & I_{n_2} \end{pmatrix} X^T S - \begin{pmatrix} x_{10} & 0\cdots 0 \\ x_{2s} & 0\cdots 0 \end{pmatrix} = F(X^T), \tag{32a}$$

where

$$F(X^T) = \begin{pmatrix} f_1(X^T\Phi(t_1),U^T\Phi(t_1),t_1) & \cdots & f_1(X^T\Phi(t_m),U^T\Phi(t_m),t_m) \\ -f_2(X^TS\Phi(t_1),U^TS\Phi(t_1),s-t_1) & \cdots & -f_2(X^TS\Phi(t_m),U^TS\Phi(t_m).,s-t_m) \end{pmatrix} N_f P. \tag{32b}$$

Thus, the approximate solution (26) of the problem defined by (18) and (19) can be obtained by using the iterative Newton method or its modified version, or using the fixed-point algorithm for better convergence to solve the nonlinear algebraic equations (32a).

SOLUTION OF NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS FOR DESCRIPTOR SYSTEMS

In recent years, control problems for descriptor systems(or singular systems, or generalized state-space systems) have drawn considerable attention from many researchers due to extensive applications of descriptor systems in large-scale systems, singular perturbation theory, electrical networks, economic systems, macro-economic systems, and other areas. In this section, we consider the following nonlinear descriptor systems with the two-point boundary conditions:

$$\begin{cases} Ex'(t) = f(x(t),u(t),t), \quad 0 \le t \le s, & (33) \\ x_1(0) = x_{10}, \quad x_2(s) = x_{2s}, & (34) \end{cases}$$

where E is a constant square matrix and may be singular, and

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}, \tag{35}$$

here E_{11}, E_{12}, E_{21}, E_{22} are the $n_1 \times n_1$, $n_1 \times n_2$, $n_2 \times n_1$, $n_2 \times n_2$ submatrixes, respectively. Other terms have the same meaning as in Section 3.

Denote by $y(t)$ and $v(t)$ the $n \times 1$ and $k \times 1$ matrices defined in (21), (22). Then (33), (34) can be expressed as

$$\begin{cases} Ey'(t) = g(x(t),u(t),t), & (36) \\ y(0) = x_0, & (37) \end{cases}$$

where g and x_0 are defined as in (25) and (28), respectively. The mth order approximation $x_{(m)}(t)$ of $x(t)$ and $u_{(m)}(t)$ of $u(t)$ are expressed as in (26) and (27), respectively.

Integrating(36), (37) from $t=0$ to t, and notice (21), (22), we obtain

$$\begin{pmatrix} E_{11} & 0 \\ E_{21} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & E_{12} \\ 0 & E_{22} \end{pmatrix} x(s-t) - Ex_0 = \begin{pmatrix} \int_0^t f_1(x(t),u(t),t)\,dt, \\ -\int_0^t f_2(x(s-t),u(s-t),s-t)\,dt \end{pmatrix} \tag{38}$$

Substituting(26) - (30) into (38), and using (10) and the proposition 3, and equating the coefficient vectors of the general orthogonal polynomial vectors, we obtain

$$\begin{pmatrix} E_{11} & 0 \\ E_{21} & 0 \end{pmatrix} X^T + \begin{pmatrix} 0 & E_{12} \\ 0 & E_{22} \end{pmatrix} X^T S - E\begin{pmatrix} x_{10} & 0\cdots0 \\ x_{2s} & 0\cdots0 \end{pmatrix}$$

$$= F(X^T), \tag{39}$$

where $F(X^T)$ is given from (32b).

Thus, the approximate solution(26) of the problem (33), (34), can be obtained by using the iterative Newton method, or the modified version, or the fixed-point algorithm for better convergence to solve the nonlinear algebraic equations(39).

CONCLUSION

Deriving a solution for the ordinary differential equation of two-point boundary value problems is generally difficult. This is more so for nonlinear systems. In the present work, a technique is developed for the solution of nonlinear ordinary differential equation of a two-point boundary value problem using general orthogonal polynomials. The main idea of this method is to reduce the problem to one of solving sets of nonlinear algebraic equations. In terms of solving the nonlinear algebraic equation, this method gives better convergence for digital computation. Here the approximation operational vector and matrix of the nonlinear functionals, together with the transformation matrix are used to relate the back vector to the current time vector. At the same time, a method for solving the nonlinear two-point boundary value problems for descriptor systems is given. Furthermore, the method is simple and can be conveniently applied in digital computation.

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