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A CLASS OF MARCINKIEWICZ INTEGRAL OPERATORS ON PRODUCT DOMAINS*

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Abstract: In this paper，the author proves that the L^p-boundedness of the Marcinkiewicz integral μ_Ω on product domains $R^n \times R^m$；for $\Omega \in (1) \cap (5)$ improves the result of Chen et al.（2000）.

Key words: Marcinkiewicz integral operator，rough kernel，product domains，square functions
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INTRODUCTION

For $n \geqslant 2, m \geqslant 2$，let S^{n-1} and S^{m-1} be the unit sphere in R^n and R^m repectively，equipped with the normalized lebesgue measure. Let Ω be a function defined on $S^{n-1} \times S^{m-1}$ satisfying $\Omega \in L^1(S^{n-1} \times S^{m-1})$ and

$$\int_{S^{n-1}} \Omega(u,v)\mathrm{d}u = 0, (\forall_v \in S^{m-1})$$
$$\int_{S^{m-1}} \Omega(u,v)\mathrm{d}u = 0, (\forall_v \in S^{n-1}) \qquad (1)$$

The corresponding Marcinkiewicz integral operator on product domain $R^n \times R^m$ is given by

$$\mu(f)(x,y) = \left(\int_0^{+\infty}\int_0^{+\infty} |F_{t,s}(x,y)|^2 \frac{\mathrm{d}t\mathrm{d}s}{t^3 s^3}\right)^{\frac{1}{2}} \qquad (2)$$

$$F_{t,s}(x,y) = \iint_{\substack{|u|\leqslant t \\ |u|\leqslant s}} \Omega(u,v)|u|^{1-n}|v|^{1-m} \cdot f(x-u, y-v)\mathrm{d}u\mathrm{d}v \qquad (3)$$

Recently，Chen et al.（2000）proved that for $\Omega \in L^q(S^{n-1} \times S^{m-1})(q>1)$，$\mu$ is L^p-bounded for $1 < p < +\infty$. On the other hand，the related Calderon-Zygmund singular integral operator，which is defined by

$$T(f)(x,y) = p.v.\iint_{R^n \times R^m} \frac{\Omega(u',v')}{|u|^n |v|^m} \cdot f(x-u, y-v)\mathrm{d}u\mathrm{d}v \qquad (4)$$

where $x' = \frac{x}{|x|}$ for $x \neq 0$，is known to be bounded on L^p under weaker conditions on Ω. One of such conditions，for $\Omega \in L^q(S^{n-1} \times S^{m-1})(q>1)$ was shown by Duoandikoetxea（1986）to imply the L^p- boundedness of $T(f)$ for all $p \in (1, +\infty)$. We defined another condition of this kind similar to the condition introduced by Grafakos and Stefanove（1998）and got the following results（Ying，1999）.

Theorem A Let $\alpha > 0$ and $\Omega \in (1)$，if Ω also satisfies

$$\sup_{\zeta \in S^{n-1}\, \eta \in S^{m-1}} \int_{S^{n-1}\times S^{m-1}} |\Omega(u',v')| \cdot \left\{\log\frac{1}{|u'\zeta|} + \log\frac{1}{|v'\eta|} + \log\frac{1}{|u'\zeta|}\log\frac{1}{|v'\eta|}\right\}^{\alpha+1} \mathrm{d}u'\mathrm{d}v' < \infty \qquad (5)$$

then we have

（i）For $\alpha > 1$，the singular integral operator T is L^p-bounded for $p \in (\frac{2\alpha}{2\alpha-1}, 2\alpha)$.

（ii）For $\alpha > \frac{1}{2}$，the Marcinliewicz integral operator μ is L^2-bounded.

We also showed that there are functions Ω that satisfy condition Eq.（5）for $\alpha > 1$ but are

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notcontained in the space $L(\log^+ L)(S^{n-1}\times S^{m-1})$; and that every function that lies in $L^q(S^{n-1}\times S^{m-1})$ for any $q>1$ satisfies condition Eq.(5) for all $\alpha>1$.

Hence we ask naturally whether the corresponding Marcinkiewicz integral operator $\mu(f)$ is bounded in L^p spaces under the condition Eq.(5). Inspired by the work (Chen et al. 2001), instead of dealing with the operator $\mu(f)$, we deal with the equivalent form $\mu_\Omega(f)$ defined by

$$\mu_\Omega(f)=\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|F_{t,s}^{\Omega}(x,y)|^2\frac{dtds}{2^{2t}2^{2s}}\right)^{\frac{1}{2}} \quad (6)$$

$$F_{t,s}^{\Omega}(x,y)=\iint_{\substack{|u|\leqslant 2^t\\ |u|\leqslant 2^s}}\Omega(u,v)|u|^{1-n}|v|^{1-m}f(x-u,y-v)dudv \quad (7)$$

Now let us state the main result of the paper.

Theorem 1 Let $\alpha>1$ and $\Omega\in(1)\cap(5)$; the operator μ_Ω is bounded in $L^p(R^n\times R^m)$ for $p\in\left(\frac{2\alpha}{2\alpha-1},2\alpha\right)$.

For non-product case, a similar result was presented by (Chen et al. 2001).

Corollary 2 If $\Omega\in L^q(S^{n-1}\times S^{m-1})$ where $q>1$, the Marcinkiewicz operator μ_Ω is bounded in $L^p(R^n\times R^m)$ for all $p\in(1,+\infty)$.

SOME LEMMAS

For $t,s\in R$, we define measures $\{\sigma_{t,s}\}$, $\{\lambda_{t,s}\}$by

$$\hat{\sigma}_{t,s}(\zeta_1,\zeta_2)=\iint_{\substack{|u|\leqslant 2^t\\ |u|\leqslant 2^s}}\Omega(u,v)|u|^{1-n}\cdot|v|^{1-m}e^{-iu\cdot\zeta_1-iv\cdot\zeta_2}dudv/2^{t+s}$$

$$\hat{\lambda}_{t,s}(\zeta_1,\zeta_2)=\iint_{\substack{|u|\leqslant 2^t\\ |u|\leqslant 2^s}}\Omega(u,v)|u|^{1-n}\cdot|v|^{1-m}e^{-iu\cdot\zeta_1-iv\cdot\zeta_2}dudv$$

and denote the maximal operator $\sigma^*(f)$ by

$$\sigma^*(f)(x,y)=\sup_{t,s\in R}|2^{-(t+s)}\lambda_{t,s}*f(x,y)|$$

Then it is easy to see that

$$2^{-t-s}F_{t,s}^{\Omega}(x,y)=\sigma_{t,s}*f(x,y) \quad (8)$$

Lemma 1

$$\iint_{\substack{|u|\leqslant 2^t\\ |u|\leqslant 2^s}}|u|^{1-n}|v|^{1-m}|\Omega(u,v)dudv/2^{t+s}\leqslant c\|\Omega\|_1 \quad (9)$$

$$\|\sigma^*(f)\|_p\leqslant c\|\Omega\|_1\|f\|_p \quad (10)$$

where c is indepent of $t,s\in R$.

Proof Eq.(10) can be easily obtained by the well-known method of rotation while (9) is proved obviously.

Lemma 2 Let $t,s\in R$ and Ω satisfies Eqs.(1),(5). Then

(a) $|\hat{\sigma}_{t,s}(\zeta_1,\zeta_2)|\leqslant C_\alpha|2^t\zeta_1\|2^s\zeta_2|$, for $|2^t\zeta_1|\leqslant 2$, and $|2^s\zeta_2|\leqslant 2$

(b) $|\hat{\sigma}_{t,s}(\zeta_1,\zeta_2)|\leqslant C_\alpha[\log|2^t\zeta_1|]^{-\alpha}|2^s\zeta_2|$, for $|2^t\zeta_1|\geqslant 2$, and $|2^s\zeta_2|\leqslant 2$

(c) $|\hat{\sigma}_{t,s}(\zeta_1,\zeta_2)|\leqslant C_\alpha[\log|2^s\zeta_2|]^{-\alpha}|2^t\zeta_1|$, for $|2^t\zeta_1|\leqslant 2$, and $|2^s\zeta_2|\geqslant 2$

(d) $|\hat{\sigma}_{t,s}(\zeta_1,\zeta_2)|\leqslant C_\alpha[\log|2^s\zeta_2|]^{-\alpha}\cdot[\log|2^t\zeta_1|]^{-\alpha}$, for $|2^t\zeta_1|\geqslant 2$, and $|2^s\zeta_2|\geqslant 2$

Proof The proof is exactly similar to the estimate in the paper (Ying, 1999), we omit it here.

Take $\Phi_1\in C_0^\infty(R^n)$; $\Phi_2\in C_0^\infty(R^m)$, radial, $\text{supp}(\Phi_i)\subset\{\frac{1}{2}\leqslant|x_i|\leqslant 1\}$ and $\Phi_i\geqslant 0$, $\int_0^{+\infty}\Phi_i(t)\frac{dt}{t}=1$ where $i=1,2$. let $\widetilde{\Psi}_{t,s}(\zeta_1,\zeta_2)=\Phi_1(2^t\zeta_1)\Phi_2(2^s\zeta_2)$ and $\widetilde{\Psi}_{t,s}(\zeta_1,\zeta_2)=\Phi_1(t\zeta_1)\Phi_2(s\zeta_2)$. Now we have the following lemma known as Calderon reproducing formula:

$$f(x,y)=\int_0^{+\infty}\int_0^{+\infty}\widetilde{\Psi}_{t,s}*f(x,y)\frac{dtds}{ts} \quad (11)$$

As references, the reader can consult Chen et al.(2001) and Frazier(1991) for more detail.

For Ψ defined as above, we define the corresponding square g-function on the product domais $R^n\times R^m$,

$$g_\Psi(f)(x,y)=\left(\int_0^{+\infty}\int_0^{+\infty}|\Psi_{t,s}*f(x,y)|dtds\right)^{\frac{1}{2}}$$

Then we can immediately prove the following lemma, once we make a slight modification on the paper (Fefferman, 1982).

Lemma 3 For Ψ defined as above, Eq. (13) below holds

$$\| g_{\Psi}(f) \|_{p} \leqslant c \| f \|_{p} \tag{12}$$

where C is independent of the function $f \in L^{p}(R^{n} \times R^{m})$.

According to Eq. (8), we have, for $f \in S(R^{n} \times R^{m})$

$$\mu_{\Omega}(f)(x,y) = \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\sigma_{t,s} * f(x,y)|^{2} dt ds\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sigma_{t,s} * \Psi_{t+t_1,s+s_1} * f(x,y) dt_1 ds_1\right|^{2} dt ds\right)^{\frac{1}{2}} \leqslant \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\sigma_{t,s} * \Psi_{t+t_1,s+s_1} * f(x,y)|^{2} dt ds\right]^{\frac{1}{2}} dt_1 ds_1\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I_{t_1,s_1}(f)(x,y) dt_1 ds_1 \tag{13}$$

where the second equality follows from lemma 3, and the third inequality from Minkowski's inequality.

Lemma 4 For Ψ, $\{\sigma_{t,s}\}$ and $I_{t_1,s_1}(f)$ defined as above, then we have

$$\| I_{t_1,s_1}(f) \|_{p_0} \leqslant C_{n,p_0} \| f \|_{p_0} \ \forall p_0 \in (1, +\infty) \text{ and } f \in L^{p_0}(R^{n} \times R^{m}) \tag{14}$$

Proof Without loss of generality, suppose $p_0 > 2$ while the case $1 < p_0 < 2$ can be obtained by a way similar to that in the paper (Duoandikotxea, 1986).

It is easy to see that there exists a nonnegative function g satisfying $\| g \|_{(\frac{p_0}{2})'} = 1$ such that

$$\| I_{t_1,s_1}(f) \|_{p_0}^{2} = \iint_{R^n \times R^m}\iint_{(R)^2} |\sigma_{t,s} * \Psi_{t+t_1,s+s_1} * f(x,y)|^{2} \cdot g(x,y) dt ds dx dy \leqslant c\iint_{(R)^2}\iint_{R^n \times R^m} |\Psi_{t+t_1,s+s_1} * f|^{2} * |\sigma_{t,s}|(x,y) \cdot g(x,y) dt ds dx dy \leqslant c\left[\iint_{(R)^2}\left(\iint_{R^n \times R^m} |\Psi_{t+t_1,s+s_1} * f|^{2} dt ds\right)^{\frac{p_0}{2}} \cdot dx dy\right]^{\frac{2}{p_0}} \leqslant c \| f \|_{p_0}$$

where the second to the last inequality follows from Holder inequality and lemma 1, and the last one from lemma 3.

On the other hand, when $p_0 = 2$, Eq. (14) can be directly obtained from Plancherel's theorem and lemma 2.

PROOF OF THEOREM1

According to Eq.(13), i.e. $\mu_{\Omega}(f)(x,y) \leqslant \int_{0}^{+\infty}\int_{0}^{+\infty} \| I_{t_1,s_1}(f)(x,y) \|_{p} dt_1 ds_1$. By Minkowski's inequality, we have $\forall p \in (1+\infty)$,

$$\| \mu_{\Omega}(f) \|_{p} \leqslant \int_{0}^{+\infty}\int_{0}^{+\infty} \| I_{t_1,s_1}(f) \|_{p} dt_1 ds_1 \leqslant \left(\int_{-1}^{1} ds_1 \int_{-\infty}^{-1} + \int_{-1}^{1} ds_1 \int_{1}^{-\infty} + \int_{-1}^{1} ds_1 \int_{-1}^{1} + \int_{-\infty}^{-1} ds_1 \int_{-\infty}^{-1} + \int_{-\infty}^{-1} ds_1 \int_{1}^{+\infty} + \int_{-\infty}^{-1} ds_1 \int_{-1}^{1} + \int_{1}^{-\infty} ds_1 \int_{-\infty}^{-1} + \int_{1}^{+\infty} ds_1 \int_{1}^{+\infty} + \int_{1}^{-\infty} ds_1 \int_{-1}^{1}\right) \| I_{t_1,s_1}(f) \|_{p} dt_1$$

By lemma 4, we immediately get

$$\int_{-1}^{1}\int_{-1}^{1} \| I_{t_1,s_1}(f) \|_{p} dt_1 ds_1 \leqslant c \| f \|_{p}$$

For simplicity, we only discuss the follwing four cases and the others can be treated similarly.

(1) for $s_1 \geqslant 1$, $t_1 \geqslant 1$, then

$$\| I_{t_1,s_1}(f) \|_{2}^{2} \leqslant \iint_{(R)^2}\iint_{R^n \times R^m} |\hat{f}(\zeta_1,\zeta_2)|^{2} \cdot |\hat{\Phi}_1(2^{t+t_1}\zeta_1)\hat{\Phi}_2(2^{s+s_1}\zeta_2)|^{2} \cdot |\hat{\sigma}(\zeta_1,\zeta_2)|^{2} d\zeta_1 d\zeta_2 dt ds \leqslant c\iint_{(R)^2}\iint_{E_{t,s}} |\hat{f}(\zeta_1,\zeta_2)|^{2} \cdot |2^{t}\zeta_1|^{2} d\zeta_1 d\zeta_2 dt ds$$

where $E_{t,s} = \{(\zeta_1,\zeta_2) \in R^{n} \times R^{m} : 2^{-1-t_1} \leqslant |\zeta_1 2^{t}| \leqslant 2^{-t_1}, 2^{-1-s_1} \leqslant |\zeta_2 2^{s}| \leqslant |2^{-s_1}\}$. Hence

$$\| I_{t_1,s}(f) \|_2^2 \leqslant$$
$$c2^{-2t_1 2^{2s_1}} \iint_{(R)^2} \iint_{E_{t,s}} | \hat{f}(\zeta_1, \zeta_2) |^2 d\zeta_1 d\zeta_2 dt ds \leqslant$$
$$c2^{-2t_1} 2^{-2s_1} \| f \|_2^2 \quad (15)$$

Interpolating between Eq.(14) and Eq.(15), we can easily have

$$\int_1^{+\infty} \int_1^{+\infty} \| I_{t_1,s_1}(f) \|_p dt_1 ds_1 \leqslant c \| f \|_p \ \forall p \in (1, \infty)$$

(2) For $s_1 \geqslant 1$, $-1 \leqslant t_1 \leqslant 1$ then

$$\| I_{t_1,s_1}(f) \|_2^2 \leqslant c \iint_{(R)^2} \iint_{E_{t,s}} | \hat{f}(\zeta_1, \zeta_2) |^2 \cdot | 2^t \zeta_2 |^2 d\zeta_1 d\zeta_2 dt ds \leqslant c2^{-2s_1} \| f \|_2^2 \quad (16)$$

Interpolating between and Eqs (14) and (16), we get

$$\int_1^{+\infty} ds_1 \int_{-1}^{1} \| I_{t_1,s_1}(f) \|_p dt_1 ds_1 \leqslant c \| f \|_p \quad \forall p \in (1, \infty)$$

(3) For $s_1 \geqslant 1$, $t_1 < -1$, we have

$$\| I_{t_1,s_1}(f) \|_2^2 \leqslant c \iint_{(R)^2} \iint_{E_{t,s}} | \hat{f}(\zeta_1, \zeta_2) |^2 \cdot | 2^s \zeta_2 |^2 [\log | 2^t \zeta_1 |]^{-2\alpha} \cdot d\zeta_1 d\zeta_2 dt ds \leqslant c2^{-2s_1} \| t_1 \|^{2\alpha} \| f \|_2^2$$

Then $\forall p \in (\frac{2\alpha}{2\alpha - 1}, 2\alpha)$ exist $\theta_p > 1$ and $\varepsilon_p > 0$ such that

$$\| I_{t_1,s_1}(f) \|_2^2 \leqslant c \, | t_1 |^{-\theta p} 2^{-\varepsilon_p s_1} \| f \|_2^2$$

Hence for $p \in (\frac{2\alpha}{2\alpha - 1}, 2\alpha)$

$$\int_1^{+\infty} ds_1 \int_{-\infty}^{-1} \| I_{t_1,s_1}(f) \|_p dt_1 \leqslant C_{p,\alpha} \| f \|_p$$

(4) For $s_1 < -1$ and $t_1 < -1$; then we have

$$\| I_{t_1,s_1}(f) \|_2^2 \leqslant c \iint_{(R)^2} \iint_{E_{t,s}} | \hat{f}(\zeta_1, \zeta_2) |^2 \cdot [\log | 2^s \zeta_2 |]^{-2\alpha} [\log | 2^t \zeta_1 |]^{-2\alpha} \cdot d\zeta_1 d\zeta_2 dt ds \leqslant c \, | s_1 |^{-2\alpha} \cdot | t_1 |^{-2\alpha} \| f \|_2^2 \quad (17)$$

Interpolating again between Eq.(14) and Eq.(17), we see for $p \in (\frac{2\alpha}{2\alpha - 1}, 2\alpha)$, there exists $\theta_p > 1$ such that

$$\| I_{t_1,s_1}(f) \|_2^2 \leqslant c \, | t_1 |^{-\theta_p} | s_1 |^{-\theta_p} \| f \|_2^2$$

Therefore, for $p \in (\frac{2\alpha}{2\alpha - 1}, 2\alpha)$

$$\int_{-\infty}^{-1} \int_{-\infty}^{-1} \| I_{t_1,s_1}(f) \|_p dt_1 ds_1 \leqslant C_{p,\alpha} \| f \|_p$$

which completes the proof of Theorem1.

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