Study on one-dimensional consolidation of saturated soil with semi-pervious boundaries and under cyclic loading*

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Abstract: The variation of effective stress ratio of stratfied soil with semi-pervious boundaries and under cyclic loading was analyzed on the basis of Terzaghi's one-dimensional consolidation assumptions. A solution by Laplace Transform was obtained for the case when the soil was under time-varied loading. With numerical inversion of Laplace Transform，some useful results were obtained for several kinds of commonly encountered loadings. The results can be meaningful in engineering practice.

Key words: Semi-pervious boundaries，Saturated soil，Cyclic loading，One-dimensional consolidation，Laplace transform，Effective stress ratio
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INTRODUCTION

Sand mat can be effective for soil improvement. In practice，time-varying loading is sometimes applied on soils under some structures such as roads，oil-tanks，coal storages and vibrating machines. Traditional consolidation theory regards sand mats as pervious. However，when the permeability of sand mats is not very satisfactory，results from traditional consolidation theory will not be perfectly reliable. So semi-pervious boundaries conditions should be considered. The one-dimensional consolidation of saturated soil with semi-pervious boundaries and under cyclic loading was analyzed on the basis of above engineering background.

Terzaghi (1943) proposed a one-dimensional consolidation theory of saturated soils under suddenly imposed constant loading for calculating the settlements at any time. After his pioneering work，Schiffman (1970) obtained a general solution to consolidation considering loadings increase linearly with time. Wilson (1974) studied the consolidation of soils under cyclic loadings and Alonso (1974) considered random loading of a linear (elastic) clay layer. Baligh (1978) developed a simple prediction method for an inelastic (nonlinear) clay layer that is initially in a normally consolidated state and is subjected to cyclic loading. Pyrah (1996) studied problems considering the soil is not homogeneous. Rahal (1998) developed a solution to the consolidation equation with boundary consolidations that are cyclic with time. Zhu (1999) presented a mathematical solution for consolidation analysis of a double-layered soil profile under depth-dependent ramp loading. Liang (2001) analyzed the settlement of granular fill on soft soil and circular loading. As to consolidation problems with semi-pervious boundaries，Gray (1945)，Schiffman (1970)，and Xie (1996) obtained fruitful results for cases of constant loading. But to the author's knowledge，there are few papers considering consolidation problems of soils under cyclic loading and with semi-pervious boundaries. This work on such problems yielded general solutions. Examples are discussed with numerical inversion of Laplace Transform and the significantly useful conclusions are presented for possible application in engineering practice.

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EQUATIONS AND SOLUTIONS

Fig. 1 shows the one-dimensional foundation model of saturated soil with semi-pervious boundaries and under the time-varying loading $q(t)$. $2H$, k_v, C_v and E_s are the thickness, coefficient of permeability, coefficient of consolidation, and the modulus of compressibility of the stratified soil. L_1, L_2, k_1 *and* k_2 are the thickness and the permeability coefficients of the semi-pervious layers on the top and at the bottom, respectively.

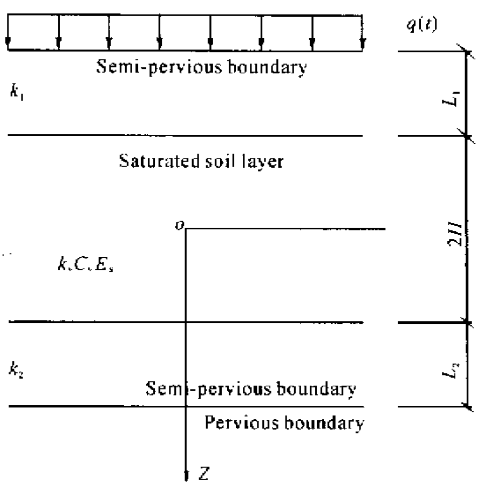

Fig. 1 1-D foundation model of soil layer with semi-pervious boundaries

With the use of all the assumptions from Terzaghi's consolidation theory except for the time-varying loadings, the consolidation equation can be expressed as:

$$\frac{\partial \bar{\sigma}(z,t)}{\partial t} = C_v \frac{\partial^2 \bar{\sigma}(z,t)}{\partial z^2} \quad (1)$$

where $\bar{\sigma}(z,t)$ is the increment of effective stress at the depth z and instant t.

After Laplace Transform, Eq. (1) is:

$$s \cdot \bar{\sigma}_1(z,s) - \bar{\sigma}_1(z,0) = C_v \frac{\partial^2 \bar{\sigma}_1(z,s)}{\partial z^2} \quad (2)$$

where $\bar{\sigma}_1(z,s)$ is the Laplace Transform of $\bar{\sigma}(z,t)$.

With the initial conditions $\bar{\sigma}(z,0)=0(|z| \leqslant H)$, the solution to Eq. (2) is:

$$\bar{\sigma}_1(z,s) = c_1 e^{\beta z} + c_2 e^{-\beta z} \quad (3)$$

where β is $\sqrt{\frac{s}{C_v}}$.

As far as the soil layer with double semi-pervious boundaries is concerned, the following initial and boundary conditions exist:

$$\frac{\partial u(z,t)}{\partial z} - \frac{R}{2H} \times u(z,t) = 0 \quad z=-H \quad t>0 \quad (4)$$

$$\frac{\partial u(z,t)}{\partial z} + \frac{R}{2H} \times u(z,t) = 0 \quad z=H \quad t>0 \quad (5)$$

where R is $\frac{k_1 \cdot 2H}{k_v \cdot L_1}$ and R' is $\frac{k_2 \cdot 2H}{k_v \cdot L_2}$.

Since $q(t)=u(z,t)+\bar{\sigma}(z,t)$, at any instant t, Eqs. (4) and (5) can also be written as:

$$\frac{\partial \bar{\sigma}(z,t)}{\partial z} + \frac{R}{2H}[q(t) - \bar{\sigma}(z,t)] = 0 \quad z=-H \quad t>0 \quad (6)$$

$$\frac{\partial \bar{\sigma}(z,t)}{\partial z} + \frac{R'}{2H}[q(t) - \bar{\sigma}(z,t)] = 0 \quad z=H \quad t>0 \quad (7)$$

The following expressions can be obtained after Laplace Transforms of Eqs. (6) and (7):

$$c_1 = \frac{\frac{Q(s)}{2H}[\alpha_2 R' + \alpha_4 R]}{\alpha_2\alpha_3 - \alpha_1\alpha_4},$$

$$c_2 = \frac{\frac{Q(s)}{2H}[\alpha_1 R' + \alpha_3 R]}{\alpha_2\alpha_3 - \alpha_1\alpha_4},$$

in which $Q(s)$ is the Laplace Transform of $q(t)$ and $\alpha_1 = e^{-\beta H}\left[\beta - \frac{R}{2H}\right]$, $\alpha_2 = e^{\beta H}\left[\beta + \frac{R}{2H}\right]$, $\alpha_3 = e^{\beta H}\left[\beta + \frac{R'}{2H}\right]$, $\alpha_4 = e^{-\beta H}\left[\beta - \frac{R'}{2H}\right]$.

Then Eq. (3) will yield:

$$\bar{\sigma}_1(z,s) = \frac{Q(s)\{[\alpha_2 R' + \alpha_4 R]e^{\beta z} + [\alpha_1 R' + \alpha_3 R]e^{-\beta z}\}}{[\alpha_2\alpha_3 - \alpha_1\alpha_4] \times 2H} \quad (8)$$

$\bar{\sigma}(z, t)$ can be obtained after inversion of Laplace Transform of Eq. (8):

$$\bar{\sigma}(z,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\bar{\sigma}_1(z,s)e^{st}ds \tag{9}$$

When the expression $\bar{\sigma}(z,t)$ is complicated, the analytical solution is very difficult to obtain. Durbin (1974) did a lot of research on the numerical inversion of the Laplace Transform (NILT). In the following analysis, due to the difficulty in achieving the analytic solution of Eq. (9), NILT was adopted by the authors who develop a program in this paper specially. The effect of soil parameters on the change of the effective stress ratio $\sigma_{esr}(=\bar{\sigma}(z,t)/\sigma_0)$ for some typical kinds of cyclic loading is discussed below.

TYPICAL CYCLIC LOADINGS AND THE LAPLACE TRANSFORM

1. Cases under suddenly-imposed constant loading

The imposed loading is shown in Fig. 2 (a),

$$q(t) = \sigma_0,\ t > 0. \tag{10}$$

The Laplace Transform is

$$Q(s) = \frac{\sigma_0}{s} \tag{11}$$

When R and R' approach infinity (which means the pervious boundary condition), Eq. (8) reduces as:

$$\bar{\sigma}(z,s) = \frac{Q(s)(e^{\beta z}+e^{\beta z})}{e^{\beta H}+e^{-\beta H}} = \frac{\mathrm{sh}[\beta(z+H)]-\mathrm{sh}[\beta(z-H)]}{\mathrm{sh}(2\beta H)}Q(s) \tag{12}$$

Eq. (12) yields the following analytical solution to the one-dimensional consolidation equation after NILT:

$$\bar{\sigma}(z,t) = \sigma_0\times\left[1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi}{2}\cdot\cos\frac{n\pi z}{2H}\exp\left(-\frac{C_v}{4H^2}n^2\pi^2 t\right)\right] \tag{13}$$

$$U(z,t) = \left[1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi}{2}\cdot\cos\frac{n\pi z}{2H}\exp\left(-\frac{C_v}{4H^2}n^2\pi^2 t\right)\right] \tag{14}$$

where $U(z,t)$ is the degree of consolidation at depth z and instant t, M is $\frac{\pi}{2}(2m+1)$, T_v is $\frac{C_v t}{4H^2}$.

The average degree of consolidation $\bar{U}(t)$ is:

$$\bar{U}(t) = 1-\sum_{m=1}^{\infty}\frac{1}{M^2}\exp(-M^2T_v),$$

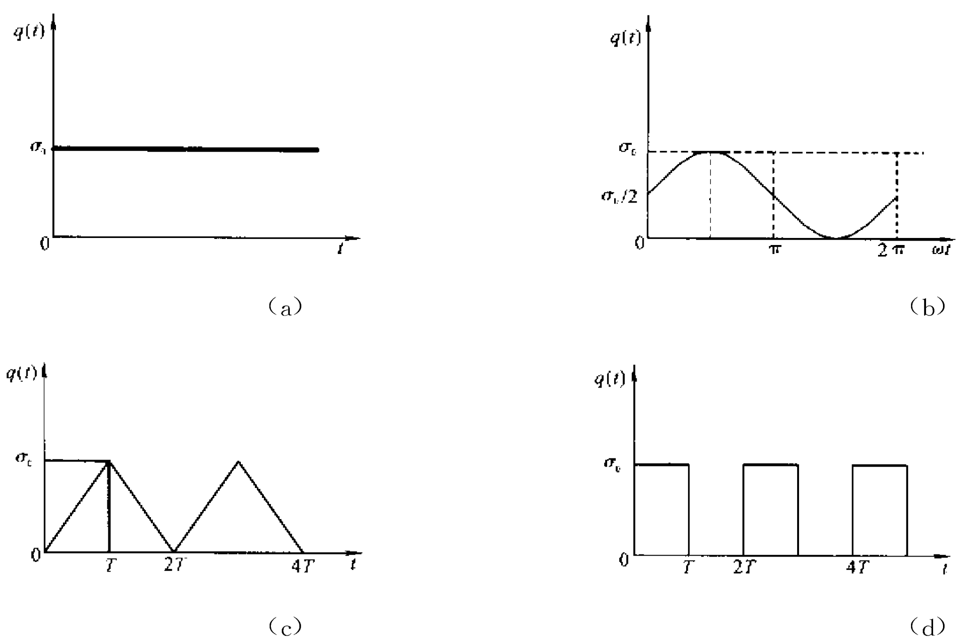

Fig. 2 Waveforms of different loadings

(a) constant; (b) sinusoidal; (c) triangular; (d) rectangular

$$M = \frac{\pi}{2}(2m+1),\ T_v = \frac{C_v t}{4H^2} \tag{15}$$

It is obvious that the solution with the method presented in this paper is just the solution proposed by Terzaghi.

2. Cases under sinusoidal loadings

The imposed loading is shown in Fig. 2 (b),

$$q(t) = \sigma_0(1+\sin\omega t),\ t>0 \tag{16}$$

The Laplace Transform is:

$$Q(s) = \sigma_0\left(\frac{1}{s} + \frac{\omega}{s^2+\omega^2}\right) \tag{17}$$

3. Cases under triangular loadings

The imposed loading is shown in Fig. 2 (c),

$$q(t) = \begin{cases} \frac{\sigma_0}{T}t,\ 0<t<T \\ \frac{\sigma_0}{T}(2T-t),\ T<t<2T \end{cases}$$
$$q(t+2T) = q(t) \tag{18}$$

The Laplace Transform is:

$$Q(s) = \frac{\sigma_0}{Ts^2}\text{th}\frac{Ts}{2} \tag{19}$$

4. Cases under rectangular loadings

The imposed loading is shown in Fig. 2 (d),

$$q(t) = \begin{cases} \sigma_0,\ 0<t<T \\ 0,\ T<t<2T \end{cases}$$
$$q(t+2T) = q(t) \tag{20}$$

The Laplace Transform is:

$$Q(s) = \frac{\sigma_0}{2s}\left(1+\text{th}\frac{Ts}{2}\right) \tag{21}$$

EXAMPLE AND DISSCUSSION

In Fig. 1, parameters of soils are: $H = 2.5$ m, $L_1 = L_2 = 0.5$ m, $k_1 = k_2 = 2\times10^{-8}$ m/s, $k_v = 5\times10^{-10}$ m/s, $E_s = 6$ MPa, $T = 1728000$ s (20 days). The change of the effective stress ratio $\sigma_{esr}(=\bar{\sigma}(z,t)/\sigma_0)$ with different kinds of imposed loadings is discussed.

Fig. 3 shows curves of effective stress ratio under different kinds of cyclic loadings at the depth $z = 2$ m. When the parameters mentioned above are all the same except for the type of loadings, the effective stress under constant loading ($\sigma_0/2$) increases gradually and finally becomes a constant. For other cyclic loadings, the average value between the maximum effective stress and the minimum effective stress in each cycle will be equal to the loadings ($\sigma_0/2$) eventually if the duration is long enough; Because saturated soil is composed of two phases of mediums, the change of σ_{esr} lags cyclic loadings. σ_{esr} under cyclic loadings oscillate around that of $\sigma_0/2$ but becomes almost constant as time goes on. The amplitude in the case of rectangular and sinusoidal loading is the largest and smallest respectively.

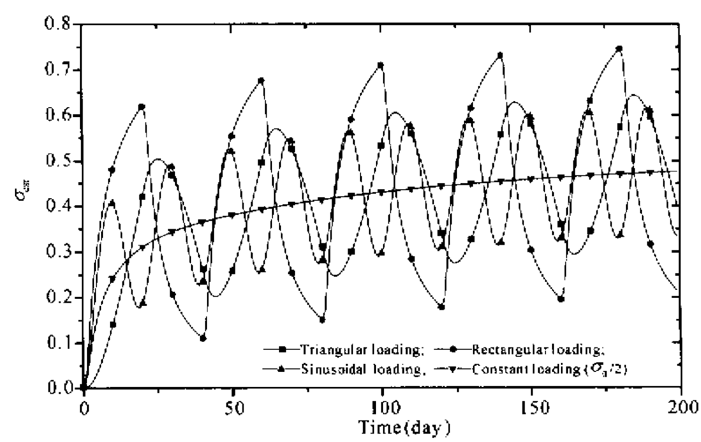

Fig. 3 Curves of effective stress ratio under different kinds of cyclic loadings

Fig. 4 shows curves of effective stress ratio for different modulus of compressibility under constant loadings ($\sigma/2$) at the depth $z=2$ m. This indicates that the larger Es is, the faster is the consolidation. However, as Es increases, the change rate of the effective stress ratio becomes slower. Table. 1 shows the value of σ_{esr} when t is 200 days.

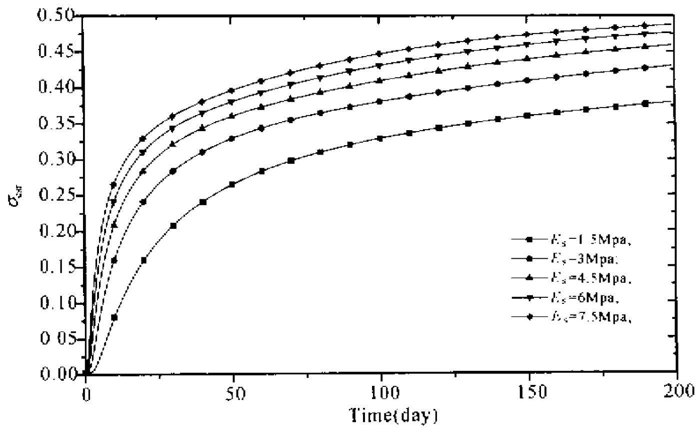

Fig. 4 Curves of effective stress ratio for different modulus of compressibility under constant loadings ($\sigma_0/2$)

Table 1 The value of σ_{esr} ($t = 200$ days)

	$E_s=1.5$MPa	$E_s=3$MPa	$E_s=4.5$MPa	$E_s=6$MPa	$E_s=7.5$MPa
σ_{esr}	0.38	0.43	0.46	0.477	0.487

When E_s increases from 1.5 MPa to 4.5 MPa, the corresponding σ_{esr} increases from 0.38 to 0.46 and when E_s increases from 4.5 MPa to 7.5 MPa, the corresponding σ_{esr} only increases from 0.46 to 0.487.

Rahal (1998) considered loading induced by a silo or a tank filling and discharging. Such cycles will cause loading change as shown in Fig. 5. The imposed loading is:

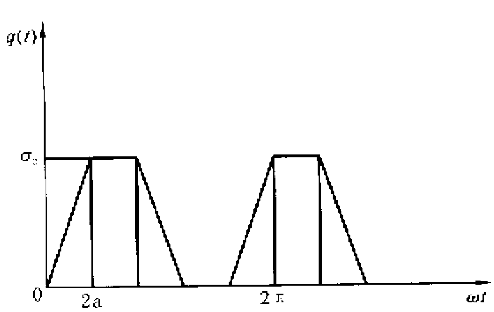

Fig. 5 Loading induced by filling and discharging of a silo or a tank

$$q(t) = \sigma_0 + \frac{4\sigma_0}{a\pi}\sum_{n=1,3,\cdots}^{\infty}\frac{1}{n}\sin na \cdot \sin n(\omega t + a),$$

The Laplace Transform is: $Q(s) = \frac{\sigma_0}{2s} + \frac{\sigma_0}{a\pi}\sum_{n=1,3,\cdots}^{\infty}\left(\frac{\sin na}{n^2}\cdot\frac{n\omega}{s^2+n^2\omega^2}\right)$.

The above parameters used to plot curves of effective stress ratio for different modulus of compressibility under such trapezoidal loading ($z=2$ m and 0 m) are presented in Fig. 6 and Fig. 7, showing that soils with smaller modulus of compressibility take longer lag time to respond to the change of the imposed loading. Soil with larger modulus of compressibility has greater amplitude of σ_{esr} and exhibits stronger reaction to the change of $q(t)$. The results also indicated that the longer the distance from a calculated point to the semi-pervious boundary is, the smaller the amplitude will be. In the center of the soil, σ_{esr} increases linearly when E_s is relatively small (<3MPa).

Fig. 8 presents curves of the effective stress ratio for different coefficient of permeability of the semi-pervious boundaries under trapezoidal loadings for depth $z = 2$ m. Results indicatd that the larger the coefficient of permeability of the semi-pervious boundaries is, the greater is the amplitude of σ_{esr}, and the shorter is the lag time for the soil to respond to the change of the imposed loading. It also indicates that when the values of R and R' are small enough (the boundaries are almost impervious), the amplitude of σ_{esr} almost approaches zero and σ_{esr} increases slowly and linearly. Many calculations indicated that when R and R' are larger than 40, the boundaries can be regarded as pervious; that when R and R' are smaller than 0.04, the boundaries can be regarded as impervious; and that when R and R' range between 40 and 0.04, the boundaries should be treated as semi-pervious. Problems about consolidation are often encountered in many engineering projects, wherein boundary conditions are generally treated simply as pervious or impervious; but will bring about inaccuracy in the result. It is much more proper to adopt the method presented above to solve such problems.

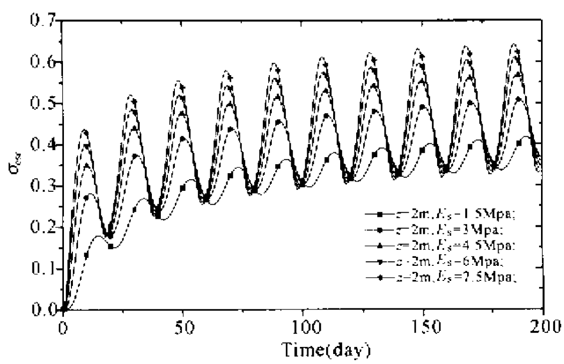

Fig. 6 Curves of effective stress ratio for different modulus of compressibility under trapezoridal loadings ($z = 2m$)

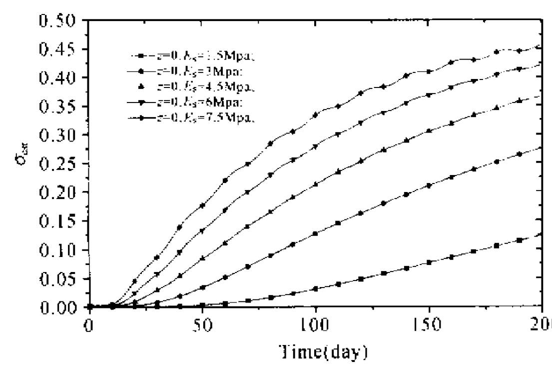

Fig. 7 Curves of effective stress ratio for different modulus of compressibility under trapezoridal loadings ($z = 0$)

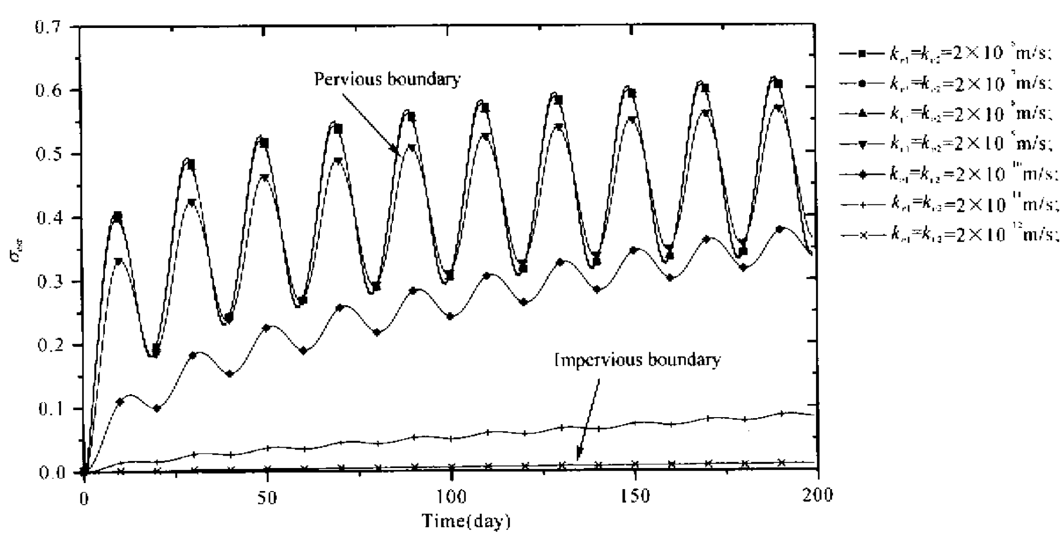

Fig. 8 Curves of effective stress ratio for different coefficient of permeability of the semi-pervious boundaries under trapezoridal loadings ($z = 2$m)

CONCLUSIONS

This study led to the following conclusions:

1. The method presented in this paper can be used for calculating the degree of consolidation and σ_{esr} of saturated soils at any instant t with different boundary conditions (pervious, semi-pervious, and impervious). After Laplace Transform of $\bar{\sigma}(z,t)$ and with NILT, such problems can be easily solved.

2. In practical engineering projects, semi-pervious boundary conditions should be determined for more accurate results.

3. σ_{esr} in soils under all kinds of cyclic loadings will reach to a constant value eventually. The change of σ_{esr} lags that of the imposed loadings and the amplitude does not reduce as time goes on.

4. σ_{esr} at the point with the longest drainage path changes slowest. Larger *Es* makes soils consolidate faster. However, as *Es* increases, it shows decreased effect on the change rate of the effective stress ratio.

5. For larger values of R and R', the amplitude of σ_{esr} is greater and the lag time the soil takes to respond to the change of the imposed loadings is shorter.

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