ISSN 1009-3095 Journal of Zhejiang University SCIENCE V.3, No.5, P.579-583, Nov.-Dec., 2002
http://www.periodicals.com.cn; http://www.zju.edu.cn/English
http://www.zjupress.com; http://lib.zju.edu.cn/eindex.htm; jzu_s@mail.hz.zj.cn

Some limsup results for increments of stable processes in random scenery*

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Received Apr.28, 2001; revision accepted Dec.30, 2001

Abstract: In this paper, we prove some limsup results for increments and lag increments of $G(t)$, which is a stable processe in random scenery. The proofs rely on the tail probability estimation of $G(t)$.

Key words: Local time, Random walk in random scenery, Stable process in random scenery, Increments, Lag increments.
Document code: A **CLC number:** O211

INTRODUCTION

Let $\boldsymbol{Z}$, $\boldsymbol{N}_0$ and $\boldsymbol{N}$ denote the sets of integers, non-negative integers and positive integers, respectively. And let $\chi=\{X_i\}_{i\in \boldsymbol{Z}}$ denote a sequence of real-valued random variables such that

$$EX_i = 0 \text{ and } EX_i^2 = \sigma_X^2 \quad \forall\, i \in \boldsymbol{Z}.$$

Any realization of the sequence $\{X_i\}_{i\in \boldsymbol{Z}}$ is called a "random scenery". Let $y=\{Y_j\}_{j\in \boldsymbol{N}}$ be a sequence which satisfies

$$EX_j = 0 \text{ and } EX_j^2 = \sigma_Y^2 \quad \forall\, j \in \boldsymbol{N}.$$

We will assume that the collections χ and y are defined on a common probability space and that they generate independent σ-fields. Let $S_0=0$ and, for each $n\in\boldsymbol{N}$, let $S_n=\sum_{j=1}^{n} Y_j$. then $S=\{S_n\}_{n\in \boldsymbol{N}_0}$ is a random walk on $\boldsymbol{R}$ starting at $S_0=0$. If y is a sequence of integer-valued random variables, then S is a random walk on $\boldsymbol{Z}$. Now for each $n\in\boldsymbol{N}_0$, let

$$g_n = \sum_{k=0}^{n} X([S_k]). \tag{1}$$

The process $G=\{g_n : n\in\boldsymbol{N}_0\}$ is called random walk in random scenery, on which see Kesten and Spitzer (1979) and Révész (1990) for more details. For example, the model can be viewed as follows: if a random walker has to pay the amoumt of X_i dollars whenever he visits the site i, then g_n is the total amount he pays during the first n steps.

There is a continuous analogue for G introduced and analyzed by Kesten and Spitzer (1979). To describe this, Let $\boldsymbol{Y}_\pm=\{Y_\pm(t):t\geqslant 0\}$ denote two standard Brownian motions and let $\boldsymbol{X}=\{X(t):t\geqslant 0\}$ be a strictly stable Lévy process with index $\alpha(1<\alpha\leqslant 2)$. We will assume that $\boldsymbol{Y}_+$, $\boldsymbol{Y}_-$ and $\boldsymbol{X}$ are defined on a common probability space and that they generate independent σ-fields. And we will define a two-sided Brownian motion $\boldsymbol{Y}=\{Y(t):t\in\boldsymbol{R}\}$ accordint to the rule

$$Y(t) = \begin{cases} Y_+(t) & \text{if } t\geqslant 0,\\ Y_-(-t) & \text{if } t<0.\end{cases}$$

Give a function f: $\boldsymbol{R}\to\boldsymbol{R}$, we will let

$$\int_{\boldsymbol{R}} f(x)\,\mathrm{d}Y(x) := \int_0^\infty f(x)\,\mathrm{d}Y_+(x) + \int_0^\infty f(-x)\,\mathrm{d}Y_-(x)$$

provided that both of the Itô integrals on the right-hand side are defined.

Let $\boldsymbol{L}=\{L(t,x):t\geqslant 0, x\in\boldsymbol{R}\}$ denote the jointly continuous version of the local time pro-

* Project (No.10071072) supported by National Natural Science Foundation of China

cess of $\boldsymbol{X}$, in the sense that for any measurable function f: $\boldsymbol{R}\rightarrow\boldsymbol{R}$ and for each $t\geqslant 0$,

$$\int_0^t f(X(s))\mathrm{d}s = \int_{\boldsymbol{R}} f(a)L(t,a)\mathrm{d}a.$$

With this in mind, we can define the following process well. Let

$$G(t) := \int_{\boldsymbol{R}} L(t,x)\mathrm{d}Y(x) \quad \forall\, t \geqslant 0, \qquad (2)$$

then $\boldsymbol{G} = \{G(t): t\geqslant 0\}$ is called a stable process in random scenery.

Remark A Since $\boldsymbol{X}$ is a strictly stable Lévy process of index $\alpha(1<\alpha\leqslant 2)$, it is self-similar with index α^{-1}. The process of local time $\boldsymbol{L}$ inherits a scaling law from $\boldsymbol{X}$: for each $c>0$,

$$\{L(ct,x): t\geqslant 0, x\in\boldsymbol{R}\} \overset{d}{=} \{c^{1-1/\alpha}L(t, xc^{-1/\alpha}): t\geqslant 0, x\in\boldsymbol{R}\}.$$

Since a standard Brownian motion is self-similar with index 1/2, it follows that G is self-similar with index $\delta=1-1/(2\alpha)$.

RESULTS AND PROOFS

Before we state the results, we shall give a notion. Throughout this paper we denote $\log x = \ln\max\{e, x\}$. And let $\boldsymbol{G}$ be a stable process in random scenery, defined by Eq.(2). $\alpha(1<\alpha\leqslant 2)$ is the index of the Lévy process $\boldsymbol{X}$.

Csáki et al. (1999) and Zhang (2001a, b) proved that g_n can be approximated by $G(t)$ under some suitable conditions. This work is aimed at studying the increments of $G(t)$. Here are our main theorems.

Theorem 1 Let $h>0$ be a real number, we have

$$\limsup_{h\to 0}\sup_{0\leqslant t\leqslant 1-h}\sup_{0<s\leqslant h}\frac{|G(t+s)-G(t)|}{h^{\delta}(\log(1/h)/\gamma)^{1/\beta}}\leqslant 1 \text{ a.s.}, \qquad (3)$$

where $\beta=2\alpha/(1+\alpha)$, $\delta=1-1/(2\alpha)$ and $\gamma=\gamma(\alpha)$ is a positive real number whose value will be defined in Lemma 4.

Theorem 2 Let a_T be a continuous function of T, $0<a_T\leqslant T$, then

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0<s\leqslant a_T}\beta_T|G(t+s)-G(t)|\leqslant 1 \text{ a.s.}, \qquad (4)$$

where $\beta_T^{-1}=a_T^{\delta}\{(\log(T/a_T)+\log\log T)/\gamma\}^{1/\beta}$.

If we have also (I) $\lim\limits_{T\to\infty}\frac{a_T}{T}=\rho$, $0<\rho\leqslant 1$, then

$$\limsup_{T\to\infty}\beta_T|G(T+a_T)-G(T)|=1 \text{ a.s.}, \qquad (5)$$

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\beta_T|G(t+a_T)-G(t)|=1 \text{ a.s.}, \qquad (6)$$

$$\limsup_{T\to\infty}\sup_{0<s\leqslant a_T}\beta_T|G(T+s)-G(T)|=1 \text{ a.s.}, \qquad (7)$$

$$\limsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0<s\leqslant a_T}\beta_T|G(t+s)-G(t)|=1 \text{ a.s.} \qquad (8)$$

Theorem 3 We have

$$\limsup_{T\to\infty}\sup_{0<t\leqslant T}|G(T)-G(T-t)|/d(T,t)=1 \text{ a.s.}, \qquad (9)$$

$$\limsup_{T\to\infty}\sup_{0<t\leqslant T}\sup_{t\leqslant s\leqslant T}|G(s)-G(s-t)|/d(T,t)=1 \text{ a.s.}, \qquad (10)$$

where $d(T,t)=t^{\delta}\{(\log(T/t)+\log\log t)/\gamma\}^{1/\beta}$.

The proofs of the theorems depend on the follwing lemmas.

Lemma 4 (Khoshnevisan and Lewis, 1998) Let $\lambda\geqslant 0$, then there exists a positive real number $\gamma=\gamma(\alpha)$, such that

$$\lim_{\lambda\to\infty}\lambda^{-\beta}\log P\{G(1)\geqslant\lambda\}=-\gamma. \qquad (11)$$

Lemma 5 For any $\varepsilon>0$, there exists a constant $C=C(\alpha,\varepsilon)>0$ such that the inequality

$$P\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0<s\leqslant h}|G(t+s)-G(t)|\geqslant h^{\delta}\lambda\}\leqslant Ch^{-1}\exp\{-(1-\varepsilon)\gamma\lambda^{\beta}\} \qquad (12)$$

holds for every positive λ and $h<1$, where $\gamma=\gamma(\alpha)$ is a positive real number which is specified by Lemma 4.

Proof By Lemma 4 and Remark A, we obtain obviously, for any $\varepsilon>0$, there exists a positive real number $\lambda_0=\lambda_0(\alpha,\varepsilon)$, such that

$$P\{|G(t+h)-G(t)|\geqslant h^{\delta}\lambda\}\leqslant 2\exp\{-(1-\varepsilon/2)\gamma\lambda^{\beta}\} \qquad (13)$$

holds for every positive $h<1$ and $\lambda\geqslant\lambda_0$.

Using again the above notations, for a positive real number t and integer r, let $t_r=[2^r t]/2^r=\sum_{j=0}^{r}(\varepsilon_j(t)/2^j)$, where $t=$

$\sum_{j=0}^{\infty}(\varepsilon_j(t)/2^j)$, $\varepsilon_0(t)=0,1,2,\cdots$; $\varepsilon_k(t)=0$, 1, for $k=1,2,\cdots$, and $\varepsilon_k(t)$ should not be identically 1 from some k on. Also write $R=2^r$. Clearly, for each $\omega\in\Omega$, and s,t,r fixed, We have

$$|G(t+s)-G(t)|\leqslant|G((t+s)_r)-G(t_r)|+\sum_{j=0}^{\infty}|G((t+s)_{r+j+1})-G((t+s)_{r+j})|+\sum_{j=0}^{\infty}|G(t_{r+j+1})-G(t_{r+j})|.$$

Observe that

$$\sup_{0\leqslant t\leqslant 1-h}\sup_{0<s\leqslant h}|(t+s)_r-t_r|\leqslant h+R^{-1},$$

$$\sup_{0\leqslant t\leqslant 1-h}\sup_{0<s\leqslant h}|(t+s)_{r+j+1}-(t+s)_{r+j}|\leqslant 2^{-(r+j+1)},$$

$$\sup_{0\leqslant t\leqslant 1-h}\sup_{0<s\leqslant h}|t_{r+j+1}-t_{r+j}|\leqslant 2^{-(r+j+1)},$$

for any positives $h<1$, and non-negative integers r,j. So by Eq. (13) and the definition of the stochastic process $\boldsymbol{G}=\{G(t): t\geqslant 0\}$, we have

$$P\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0<s\leqslant h}|G(t+s)-G(t)|\geqslant(h+R^{-1})^{\delta}u+2\sum_{j=0}^{\infty}(2^{-(r+j+1)})^{\delta}x_j\}\leqslant 2R(Rh+1)\exp\{-(1-\varepsilon/2)\gamma u^{\beta}\}+8R\sum_{j=0}^{\infty}2^j\exp\{-(1-\varepsilon/2)\gamma x_j^{\beta}\}$$

for any positive $u\geqslant\lambda_0$, $x_j\geqslant\lambda_0$. Put $x_j=(\frac{j}{(1-\varepsilon/2)\gamma}+u^{\beta})^{1/\beta}$, then

$$P\Big\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0<s\leqslant h}|G(t+s)-G(t)|\geqslant u((h+R^{-1})^{\delta}+2(R^{-1})^{\delta}\sum_{j=0}^{\infty}(2^{-j-1})^{\delta})+2(R^{-1})^{\delta}\sum_{j=0}^{\infty}\Big(\frac{j}{(1-\varepsilon/2)\gamma}\cdot 2^{-\frac{2\alpha-1}{1+\alpha}(j+1)}\Big)^{1/\beta}\Big\}\leqslant(2R(Rh+1)+8R\sum_{j=0}^{\infty}(2/e)^j)\exp\{-(1-\varepsilon/2)\gamma u^{\beta}\}.$$

Put $R=2^r$ such that $R\leqslant k/h<2R$, where k is a positive constant which is large enough and will be specified later on. Then

$$2R(Rh+1)+8R\sum_{j=0}^{\infty}(2/e)^j\leqslant 2kh^{-1}(k+1)+8kh^{-1}\sum_{j=0}^{\infty}(2/e)^j=:Ah^{-1}k(k+1)=:Ch^{-1},$$

where A is a positive constant which does not depend on k. Here $C=C(k)$ is a positive constant only relying on k. In the sequel, C only indicates a constant, which can take different values in different places. Also,

$$u((h+R^{-1})^{\delta}+2(R^{-1})^{\delta}\sum_{j=0}^{\infty}(2^{-j-1})^{\delta})+2(R^{-1})^{\delta}\sum_{j=0}^{\infty}\Big(\frac{j}{(1-\varepsilon/2)\gamma}2^{-\frac{2\alpha-1}{1+\alpha}(j+1)}\Big)^{1/\beta}\leqslant uh^{\delta}((1+2/k)^{\delta}+2(2/k)^{\delta}\sum_{j=0}^{\infty}(2^{-j-1})^{\delta})+2h^{\delta}(2/k)^{\delta}\sum_{j=0}^{\infty}(\frac{j}{(1-\varepsilon/2)\gamma}2^{-\frac{2\alpha-1}{1+\alpha}(j+1)})^{1/\beta}=:h^{\delta}(u((1+2/k)^{\delta}+G/k^{\delta})+B/k^{\delta}),$$

where G,B are positive constants not relying on k. Now letting $\lambda=u((1+2/k)^{\delta}+G/k^{\delta})+B/k^{\delta}$, then

$$P\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0<s\leqslant h}|G(t+s)-G(t)|\geqslant h^{\delta}\lambda\}\leqslant Ch^{-1}\exp\{-(1-\varepsilon/2)\gamma u^{\beta}\}\leqslant Ch^{-1}\exp\{-(1-\varepsilon)\gamma\lambda^{\beta}\},$$

where the last inequality follows from the inequality

$$u=\frac{\lambda-B/k^{\delta}}{(1+2/k)^{\delta}+G/k^{\delta}}\geqslant\frac{\lambda}{((2-\varepsilon)/(2-2\varepsilon))^{1/\beta}}$$

for any given $\varepsilon>0$ upon taking k large enough. This proves our lemma with $\lambda\geqslant\lambda_0(\varepsilon)$, while it is trivially true for $\lambda\in(0,\lambda_0)$. Since, in the latter case, the right-hand side of Eq. (12) is larger than one for C big enough.

Lemma 6 For any $\varepsilon>0$, there exists a constant $C=C(\alpha,\varepsilon)>0$ such that the inequality

$$P\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0<s\leqslant h}|G(t+s)-G(t)|\geqslant h^{\delta}\lambda\}\leqslant CTh^{-1}\exp\{-(1-\varepsilon)\gamma\lambda^{\beta}\}$$

holds for every positive λ, T and $h<T$, where $\gamma=\gamma(\alpha)$ is a positive real number which is specified by Lemma 4.

Proof This lemma follows from Eq. (12) and the following observation: for any fixed $T>0$, we have

$$\{G(t):0\leqslant t\leqslant T\}\overset{d}{=}\{T^{\delta}G(t/T):0\leqslant t\leqslant T\}.$$

Remark The proofs of Lemma 5 and Lemma 6 come from the idea of Csörgö and Révész

(1981).

Proof of Theorem 1 Let $A_h = \sup_{0\leqslant t\leqslant 1-h}\sup_{0<s\leqslant h}|G(t+s)-G(t)|$. We apply Eq. (12) with $\lambda=(1+3\varepsilon)(\log(1/h)/\gamma)^{1/\beta}$, for any $0<\varepsilon<1/3$, then

$$P\left\{\frac{A_h}{h^{\delta}(\log(1/h)/\gamma)^{1/\beta}}\geqslant 1+3\varepsilon\right\}\leqslant Ch^{-1}\cdot \exp\{-(1-\varepsilon)(1+3\varepsilon)^{\beta}\log(1/h)\}\leqslant Ch^{\varepsilon}.$$

Take $T>1/\varepsilon$, and let $h_n=n^{-T}$, for $n=1,2,\cdots$, then

$$\sum_{n=1}^{\infty}P\left\{\frac{A_{h_n}}{h_n^{\delta}(\log(1/h_n)/\gamma)^{1/\beta}}\geqslant 1+3\varepsilon\right\}\leqslant C\sum_{n=1}^{\infty}n^{-T\varepsilon}<\infty$$

and the Borel-Cantelli lemma implies that

$$\limsup_{n\to\infty}\frac{A_{h_n}}{h_n^{\delta}(\log(1/h_n)/\gamma)^{1/\beta}}<1+3\varepsilon \text{ a.s.}$$

Let us take $h_{n+1}<h\leqslant h_n$, for $n=1,2,\cdots$, then for each $\omega\in\Omega$, and $0<\varepsilon<1/3$, we have

$$\limsup_{h\to 0}\frac{A_{h_n}}{h^{\delta}(\log(1/h)/\gamma)^{1/\beta}}\leqslant \limsup_{n\to\infty}\frac{A_{h_n}}{h_{n+1}^{\delta}(\log(1/h_n)/\gamma)^{1/\beta}}=\limsup_{n\to\infty}\frac{A_{h_n}}{h^{\delta}(\log(1/h_n)/\gamma)^{1/\beta}}\left(\frac{h_n}{h_{n+1}}\right)^{\delta}\leqslant 1+3\varepsilon \text{ a.s.}$$

Whence we obtain Eq. (3).

Proof of Theorem 2 This will be given in two steps.

Step1: Let $A(T)=\sup_{0\leqslant t\leqslant T-a_T}\sup_{0<s\leqslant a_T}\beta_T|G(t+s)-G(t)|$. We will prove

$$\limsup_{T\to\infty}A(T)\leqslant 1 \quad \text{a.s.} \tag{15}$$

Proof For any $0<\varepsilon<1/3$, take a positive real number θ such that $1<\theta<1+\varepsilon$, write

$$A_k:=\{T:\theta^k\leqslant T/a_T<\theta^{k+1}\},$$
$$A_{kj}:=\{T:\theta^j\leqslant a_T<\theta^{j+1},T\in A_k\},$$

for any integers $k\geqslant 0$ and j. We have

$$\limsup_{T\to\infty}A(T)=\limsup_{k+j\to\infty}\sup_{T\in A_{kj}}A(T)\leqslant\limsup_{k+j\to\infty}\sup_{0\leqslant t\leqslant\theta^{k+j+2}}\sup_{0<s\leqslant\theta^{j+1}}\frac{\theta^{\delta}|G(t+s)-G(t)|}{(\theta^{j+1})^{\delta}\{(\log(\theta^k\log\theta^{k+j}))/\gamma\}^{1/\beta}}. \tag{16}$$

On the other hand, by Lemma 6, we obtain

$$P\left\{\sup_{0\leqslant t\leqslant\theta^{k+j+2}}\sup_{0<s\leqslant\theta^{j+1}}\frac{|G(t+s)-G(t)|}{(\theta^{j+1})^{\delta}\{(\log(\theta^k\log\theta^{k+j}))/\gamma\}^{1/\beta}}\geqslant 1+3\varepsilon\right\}\leqslant C\theta^{k+1}\exp\{-(1-\varepsilon)(1+3\varepsilon)^{\beta}\}\log(\theta^k\log\theta^{k+j})\leqslant C\theta^{-k\varepsilon}j^{-(1+\varepsilon)}(\log\theta)^{-(1+\varepsilon)}.$$

Observing

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\theta^{-k\varepsilon}j^{-(1+\varepsilon)}<\infty,$$

together with Eq. (16), we proof Eq. (15), by the Borel-Cantelli lemma and letting $\theta\to 1$.

Step 2: Let $B(T)=\beta_T(G(T+a_T)-G(T))$. Suppose that Condition (I) of Theorem 2 is fulfilled, then

$$\limsup_{T\to\infty}B(T)\geqslant 1 \quad \text{a.s.} \tag{17}$$

Proof Let $T_k=\exp\{k^p\}$ for $k\geqslant 1$, where $p>1$ will be specified later on. And for any positive $\varepsilon<1$, define $\phi=((1-\varepsilon)/\gamma)^{1/\beta}$, this implies that $\gamma\phi^{\beta}=1-\varepsilon$.

It is easy to see that the proof of Eq. (17) can be reduced to the following result: for each $0<\varepsilon<1$,

$$\limsup_{k\to\infty}\xi_k\geqslant 0 \quad \text{a.s.}, \tag{18}$$

where $\xi_k=\dfrac{G(T_k+a_{T_k})-G(T_k)}{a_{T_k}^{\delta}}-\phi(\log(T_k/a_{T_k})+\log\log T_k)^{1/\beta}$, for $k\geqslant 1$.

By the definition of T_k, there exists an integer $N_1\geqslant 1$ large enough such that $T_k+a_{T_k}\leqslant 2T_k\leqslant T_{k+1}$, for each $k\geqslant N_1$. Thus, by Proposition 3.1 of Khoshnevisan and Lewis (1998), the collection of random variables $\{\xi_k, k\geqslant N_1\}$ is pairwise positively quardrant dependent. So it would suffice to establish items (a) and (b) of Proposition 4.1 of Khoshnevisan and Lewis (1998). Using Lemma 4, we have

$$\lim_{k\to\infty}\frac{\log P\{\xi_k\geqslant 0\}}{\log((T_k\log T_k)/a_{T_k})}=-\gamma\phi^{\beta}=-(1-\varepsilon).$$

Let $1-\varepsilon<q<1$, then there exists an integer $N_2\geqslant N_1$ such that, for each $k\geqslant N_2$, we get

$$P\{\xi_k\geqslant 0\}\geqslant(a_{T_k}/(T_k\log T_k))^q.$$

And let $0<h<\rho\leqslant 1$, by condition (I), we know we can take an integer $N_3\geqslant N_2$, which satisfies, for each $k\geqslant N_3$, $a_{T_k}/T_k\geqslant h$. Hence taking $1 < p \leqslant 1/q$, we obtain

$$\sum_{k=N_3}^{\infty}(a_{T_k}/(T_k \log T_k))^q \geqslant C\sum_{k=N_3}^{\infty} k^{-pq} \geqslant C\sum_{k=N_3}^{\infty} k^{-1} = \infty,$$

which verifies (a).

And clearly $Var((G(T_j + a_{T_j}) - G(T_j))/a_{T_j}^{\delta}) < \infty$, for $j = 1, 2, \cdots$. Put $0 < \lambda := 1/(1+h) < 1$ in Theorem 5.2 of Khoshnevisan and Lewis (1998), then, there exists a positive constant $C = C(\alpha, \lambda)$, such that

$$\mathrm{Cov}(\xi_j, \xi_k) = \mathrm{Cov}\left(\frac{G(T_j + a_{T_j}) - G(T_j)}{a_{T_j}^{\delta}}, \frac{G(T_k + a_{T_k}) - G(T_k)}{a_{T_k}^{\delta}}\right) \leqslant C\left[\frac{T_j + a_{T_j}}{T_k + a_{T_k}}\right]^{1/(2\alpha)},$$

for $N_3 \leqslant j < k$. While

$$\frac{T_j + a_{T_j}}{T_k + a_{T_k}} \leqslant \frac{2T_j}{T_k} = 2\exp\left\{-p\int_j^k x^{p-1}\mathrm{d}x\right\} \leqslant 2\exp\{-pj^{p-1}(k-j)\}.$$

For $j \geqslant N_3$, let $b_j = \exp\{-pj^{p-1}/(2\alpha)\}$, thus

$$\sum_{N_3 \leqslant j < k < \infty} Cov(\xi_j, \xi_k) \leqslant C\sum_{j=N_3}^{\infty}\sum_{k=j+1}^{\infty} b_j^{(k-j)} \leqslant C\frac{1}{1-b_1}\sum_{j=N_3}^{\infty} b_j < \infty,$$

which verifies (b). Thus Eq. (18) is proved.

Proof of Theorem 3 Similarly to the proof in Theorem 1.1.3 of Lin and Lu (1992), it is easy to see

$$\limsup_{T\to\infty}\sup_{0<t\leqslant T}\sup_{t\leqslant s\leqslant T}|G(s) - G(s-t)|/\mathrm{d}(T,t) \leqslant 1 \quad \text{a.s.} \tag{19}$$

On the other hand, using a law of iterated logarithm for stable processes in random scenery (see Khoshnevisan and Lewis, 1998), we get

$$\limsup_{T\to\infty}\sup_{0<t\leqslant T}|G(T) - G(T-t)|/\mathrm{d}(T,t) \geqslant \limsup_{T\to\infty} G(T)/(T^{\delta}(\gamma^{-1}\log\log T)^{1/\beta}) = 1 \text{ a.s.},$$

together with Eq. (19), we obtain our results immediately.

A NOTE

Let $\boldsymbol{B} = \{B(t): t \geqslant 0\}$ and $\boldsymbol{W} = \{W(x): x \in \boldsymbol{R}\}$ be independent real-valued standard Brownian motions with $B(0) = W(0) = 0$. Now define the process $\boldsymbol{G}' = \{G'(t): t \geqslant 0\}$, which will be called Brownian motion in Brownian scenery, by

$$G'(t) = \int_{\boldsymbol{R}} L'(t,x)\mathrm{d}W(x) \quad \forall t \geqslant 0,$$

where $\{L'(t,x): t \geqslant 0, x \in \boldsymbol{R}\}$ is the jointly continuous version of the local time process of $\boldsymbol{B}$. Then by the result of Csáki et al. (1999), we get the following corollary immediately.

Corollary 7 If we replace $\boldsymbol{G}$ by $\boldsymbol{G}'$ in Eqs. (3)-(10), then the above results hold with $\gamma = (81/32)^{1/3}$.

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