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Improvement of the termination criterion for subdivision of the rational Bézier curves*

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Abstract: By using some elementary inequalities, authors in this paper makes further improvement for estimating the heights of Bézier curve and rational Bézier curve. And the termination criterion for subdivision of the rational Bézier curve is also improved. The conclusion of the extreme value problem is thus further confirmed.

Key Words: Rational Bézier curves, Subdivision, Termination criterion.
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INTRODUCTION

It is a fundamental problem in computer aided geometric design (CAGD) and manufacture to find the intersection (Lane et al., 1980; Wang, 1984; Sederberg et al., 1986) of the parametric curves. Since most of the curves in CAGD are denoted by polynomials or rational polynomials, to find their intersection is to find the set of roots of their equation system. So finding their intersection is a difficult non-linear problem in mathematics (Wang et al., 1991). One of the methods used to solve it is subdivision algorithm (Filip et al., 1986; Sheng et al., 1992; Anglada et al., 1999). The basic idea of subdivision algorithm is "subdivision" and "straightening". The so-called "subdivision" is the process of splitting the curve into the true curve segment by central point subdivision algorithm and the so-called "straightening" is the process of each curve segment being approximated (Nairn et al, 1999) by the chord between successive points. Thus, it is a key for a termination criterion to determine the more or less of calculation. We ask, under what condition for a curve may the "subdivision" stop and then the "straightening" can begin? In other words, what is a termination criterion for subdivision of the rational Bézier curves is better? For the following rational Bézier curve of degree n,

$$p_n(t) = \sum_{i=0}^{n} B_n^i(t) w_i P_i \Big/ \sum_{i=0}^{n} B_n^i(t) w_i, \quad (P_i \in R^3, \ w_i > 0, \ 0 \leqslant t \leqslant 1), \tag{1}$$

(where $B_n^i(t) = C_n^i t^i (1-t)^{n-i}$) when w_i is a constant, the curve Eq.(1) degenerates into the following Bézier curve:

$$p_n(t) = \sum_{i=0}^{n} B_n^i(t) P_i, \ (p_i \in R^3, \ 0 \leqslant t \leqslant 1). \tag{2}$$

In 1980, Lane and Riesenfeld (Lane et al., 1980) proposed a convergence test for piecewise linear approximation of the curve Eq.(2),

$$d_i := d(P_i, l(P_0, P_n)) \leqslant \varepsilon, \quad (i = 1, 2, \cdots, n-1). \tag{3}$$

Where $l(P_0, P_n)$ is the line segment from P_0 and P_n, and $d(P_i, l(P_0, P_n))$ are the Euclidean distances from P_i to the straight line $l(P_0, P_n)$, and ε is not greater than the tolerance desired. This criterion is simple and rough. Some improvements were made on it in Wang et al. (1991). Let $h_n(t)$ be the height of the rational Bézier curve corresponding to parameter t, that is

$$h_n(t) = d(p_n(t), l(P_0, P_n)), \ (0 \leqslant t \leqslant 1) \tag{4}$$

Wang et al. (1991) gave the following two theorems.

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Theorem A For the Bézier curve Eq. (2) of degree 3, we have

$$h_3(t) \leqslant \frac{3}{4} \max_{i=1,2} d_i . \qquad (5)$$

therefore the termination criterion Eq. (3) for subdivision of the curve Eq. (2) may be improved into the following expression: $\frac{3}{4}\max_{i=1,2} d_i \leqslant \varepsilon$, $(i = 1,2)$.

Theorem B For the following rational Bézier curve of degree n, we have

$$h_n(t) \leqslant \left\{1 - \left[1 + \frac{(\max_{i=0,n} w_i)(\max_{1\leqslant i\leqslant n-1} w_i)}{w_0\ w_n}(2^{n-1})\right]^{-1}\right\} \max_{1\leqslant i\leqslant n-1} d_i . \qquad (6)$$

By the theorem B, we have

Corollary A For the Bézier curve of degree n, we have

$$h_n(t) \leqslant \left(\frac{2^n - 2}{2^n}\right) \max_{1\leqslant i\leqslant n-1} d_i . \qquad (7)$$

Therefore the termination criterion for subdivision of the Bézier curve of degree n may be improved into the following expression:

$$\left(\frac{2^n - 2}{2^n}\right) \max_{1\leqslant i\leqslant n-1} d_i \leqslant \varepsilon,\ (i = 1,2,\cdots,n-1). \qquad (8)$$

Eq. (7) becomes Eq. (5) when $n = 3$. We point out the equality sign in Eq. (5) and Eq. (7) can be taken, so in some sense the constants in Eq. (5) and Eq. (7) cannot be improved.

Let l be a straight line and let ε ($\varepsilon > 0$) be the tolerance desired. The closed cylinder with radius ε and central axis l can be expressed as

$$N(\varepsilon, l) = \{P \mid d(P, l) \leqslant \varepsilon, P \in R^3\}, \qquad (9)$$

and for the rational Bézier curve (1), the expression of the relations between the maximum normal distance d_n and the tolerance ε can be written as

$$\delta_n = \sup_{\substack{0\leqslant t\leqslant 1 \\ P_i \in N(\varepsilon, l(P_0, P_n)) \\ (i=1,2,\cdots,n-1)}} d(P_n(t), l(P_0, P_n)). \qquad (10)$$

Thus, Wang et al. (1991) obtained the following theorem on an extreme value problem:

Theorem C For the rational curve Eq. (1), if we write the maximum value points of the extreme value problem Eq. (10) as $P_i^* \in N(\varepsilon, l(P_0, P_n))$ $(i = 1,2,\cdots,n-1)$, $t^* \in [0,1]$, then $P_0, P_1^*, \cdots, P_{n-1}^*, P_n$ must be coplanar.

This paper makes further improvement for the results of Wang et al. (1991).

RESULT AND PROOF

Theorem 1 For the degree n rational Bézier curve (1), d_i $(i = 1, 2, \cdots, n-1)$ defined by (3), $P_0, P_1, \cdots, P_n$ must be coplanar and $P_1, P_2, \cdots, P_{n-1}$ must lie on the same side of the line P_0P_n when δ_n attains its maximum.

Proof By the geometric invariance of Bézier curves, without loss of generality, choosing the x-axis and the origin of coordinates to be the straight line P_0P_n and the point P_0, respectively, and writing $P_i = (x_i, y_i, z_i)$, $(i = 0, 1, \cdots, n)$, then the rational Bézier curve can be expressed as

$$p_n(t) = \sum_{i=1}^{n} B_n^i(t) w_i / P_i \sum_{i=0}^{n} B_n^i(t) w_i .$$

The condition of the theorem and choice of coordinate system imply that

$$d_i = \sqrt{y_i^2 + z_i^2},\ P_0 = (0,0,0),$$
$$P_n = (x_n, 0, 0),$$

then the square of the height of the Bézier curve corresponding to parameter t is

$$h_n^{\ 2}(t) = \Big(\sum_{i=1}^{n-1} B_n^i(t) w_i y_i / \sum_{i=0}^{n} B_n^i(t) w_i\Big)^2 + \Big(\sum_{i=1}^{n-1} B_n^i(t) w_i z_i / \sum_{i=0}^{n} B_n^i(t) w_i\Big)^2 .$$

On the other hand, when $P_0, P_1, \cdots, P_n$ are coplanar and $P_1, P_2, \cdots, P_{n-1}$ lie on the same side of the line P_0P_n, choose P_0P_n and P_0 as x-axis and the origin of planar coordinates system respectively, write $P_i = (x_i, d_i)$, then the square of the height of the Bézier curve corresponding to parameter t is

$$\hat{h}_n^2(t) = \Big(\sum_{i=1}^{n-1} B_n^i(t) w_i d_i / \sum_{i=0}^{n} B_n^i(t) w_i\Big)^2$$

To finish the proof of the theorem, first we need to prove

$$h_n^{\ 2}(t) \leqslant \hat{h}_n^2(t) \qquad (11)$$

holds for every parameter $t \in [0,1]$. Write

$$C_i = B_n^i(t) w_i / \sum_{i=0}^{n} B_n^i(t) w_i, \quad (i = 1, 2, \cdots, n-1),$$

then we only need to prove the following inequality

$$\left(\sum_{i=1}^{n-1} C_i y_i\right)^2 + \left(\sum_{i=1}^{n-1} C_i z_i\right)^2 \leqslant \left(\sum_{i=1}^{n-1} C_i \sqrt{y_i^2 + z_i^2}\right)^2$$

or

$$\left(\sum_{i=1}^{n-1} C_i y_i\right)\left(\sum_{j=1}^{n-1} C_j y_j\right) + \left(\sum_{i=1}^{n-1} C_i z_i\right)\left(\sum_{j=1}^{n-1} C_j z_j\right) \leqslant \left(\sum_{i=1}^{n-1} C_i \sqrt{y_i^2 + z_i^2}\right)\left(\sum_{j=1}^{n-1} C_j \sqrt{y_j^2 + z_j^2}\right).$$

The above inequality can be obtained by the following obvious inequality

$$C_i C_j y_i y_j + C_i C_j z_i z_j \leqslant C_i C_j \sqrt{y_i^2 + z_i^2}\sqrt{y_j^2 + z_j^2} \quad (12)$$

summing up for i, j from 1 to $n-1$. Finally, we point out the equality sign in Eq.(12) is taken if and only if

$$z_i y_j = z_j y_i;\ z_i z_j \geqslant 0;\ y_i y_j \geqslant 0, \quad (i, j = 1, \cdots, n-1).$$

Thus, the equality sign in Eq.(11) is taken if and only if

$$\frac{z_i}{y_i} = k; z_i \geqslant 0 \text{ or } (\leqslant 0); y_i \geqslant 0 \text{ or} (\leqslant 0). \quad (13)$$

Where k is constant and setting $\frac{0}{0} = k$. The expression Eq.(13) means $P_1, P_2, \cdots, P_{n-1}$ lie in the plane $z = ky$ and on the same side of the line P_0P_n. This completes the proof.

By the above theorem, the conclusion of theorem C can be strengthened. We have

Corollary 1 For the rational curve Eq.(1), if we write the maximum value points of the extreme value problem (10) as $P_i^* \in N(\varepsilon, l(P_0, P_n))(i = 1, 2, \cdots, n-1)$, $t^* \in [0,1]$, then $P_0, P_1^*, \cdots, P_{n-1}^*, P_n$ lie on the line which is parallel P_0P_n, and the distance from the line to P_0P_n is ε.

Proof By theorem 1, $P_0, P_1, \cdots, P_n$ must be coplanar when δ_n attained its maximum. By the geometric invariance of Bézier curves, we can choose the plane determined by $P_0, P_1, \cdots, P_n$ and P_0P_n as the xy plane and x axis, respectively. Let $P_i = (x_i, y_i)$ $(i = 0, 1, \cdots, n)$, let the rational Bézier curve be as expressed in Eq.(1), thus its height corresponding to parameter t is

$$h_n(t) = \sum_{i=1}^{n-1} B_n^i(t) w_i y_i / \sum_{i=0}^{n} B_n^i(t) w_i.$$

Noting that the condition $P_i^* \in N(\varepsilon, l(P_0, P_n))(i = 1, 2, \cdots, n-1)$ has become $|y_i| \leqslant \varepsilon$ in substance, it is easy to see that when $y_i = \varepsilon$, $(i = 1, 2, \cdots, n-1)$ or $y_i = -\varepsilon$, $(i = 1, 2, \cdots, n-1)$, namely, $h_n(t)$ gets its maximum when $P_1, P_2, \cdots, P_{n-1}$ are on the line. Thus completes the proof.

Theorem 2 For Bézier curve of degree 3, we have

$$h_3(t) \leqslant \frac{1}{3}\frac{(\sqrt{\Delta} + d_1 + d_2)^2}{2\sqrt{\Delta} + d_1 + d_2} \quad (n = 3), \quad (14)$$

where $\Delta = d_1^2 + d_2^2 - d_1 d_2$. Eq.(14) holds accurately for the plane curve with two central control points lying on the same side of the line connecting two endpoints.

Proof By theorem 1, we only need to consider the plane curve with two central control points lying on the same side of the two endpoints. It is easy to see that for this case, we have

$$h_3(t) = 3t(1-t)^2 d_1 + 3t^2(1-t) d_2, \quad (d_1 \geqslant 0,\ d_2 \geqslant 0)\ (0 \leqslant t \leqslant 1).$$

In order to find the maximum of $h_3(t)$ in $[0, 1]$, differentiation of $h_3(t)$ yields

$$h'_3(t) = 3[(3t^2 - 4t + 1)d_1 + (2t - 3t^2)d_2] = 3[(3d_1 - 3d_2)t^2 - (4d_1 - 2d_2)t + d_1].$$

It is easy to calculate the extreme points of $h_3(t)$ as follows:

$$t_1 = \frac{2d_1 - d_2 - \sqrt{\Delta}}{3(d_1 - d_2)}, \quad t_2 = \frac{2d_1 - d_2 + \sqrt{\Delta}}{3(d_1 - d_2)}.$$

Without loss of generality, suppose $d_1 > d_2$, by elementary calculation, we may only find one extreme point of $h_3(t)$ in $[0, 1]$, that is

$$t_1 = \frac{2d_1 - d_2 - \sqrt{\Delta}}{3(d_1 - d_2)}.$$

Noticing that $h_3(0) = h_3(1) = 0$ and $h_3(t) \geqslant 0$, it is easy to see that t_1 is the maximum point of $h_3(t)$ in $[0,1]$. We have

$$h_3(t) \leqslant h_3(t_1) = 3\left(\frac{d_1 - 2d_2 + \sqrt{\Delta}}{3(d_1 - d_2)}\right)^2 \frac{2d_1 - d_2 - \sqrt{\Delta}}{3(d_1 - d_2)} d_1 + 3\, \frac{d_1 - 2d_2 + \sqrt{\Delta}}{3(d_1 - d_2)}\left(\frac{2d_1 - d_2 - \sqrt{\Delta}}{3(d_1 - d_2)}\right)^2 d_2 = \frac{1}{9(d_1 - d_2)^3}(d_1 - 2d_2 + \sqrt{\Delta})(2d_1 - d_2 - \sqrt{\Delta})(d_1 - d_2)(d_1 + d_2 + \sqrt{\Delta}) = \frac{d_1 + d_2 + \sqrt{\Delta}}{9(d_1 - d_2)^2}[d_1^2 + d_2^2 - 4d_1 d_2 + (d_1 + d_2)\cdot\sqrt{\Delta}] = \frac{d_1 + d_2 + \sqrt{\Delta}}{9(d_1 - d_2)^2}[2\sqrt{\Delta} - (d_1 + d_2)]\cdot[\sqrt{\Delta} + d_1 + d_2] = \frac{(d_1 + d_2 + \sqrt{\Delta})^2}{9(d_1 - d_2)^2}\cdot\frac{4\Delta - (d_1 + d_2)^2}{d_1 + d_2 + 2\sqrt{\Delta}} = \frac{1}{3}\frac{(\sqrt{\Delta} + d_1 + d_2)^2}{2\sqrt{\Delta} + d_1 + d_2}.$$

Because of symmetry, it is easy to see the above result still holds when $d_1 < d_2$. Note that Eq.(14) still holds when $d_1 = d_2$. This completes the proof.

Corollary 2 For plane Bézier curve of degree 3 with two central control points lying on the two sides of the line connecting two endpoints, we have

$$\max_{0\leqslant t\leqslant 1} h_3(t) = \frac{1}{3}\frac{(\sqrt{d_1^2 + d_2^2 + d_1 d_2} + |d_1 - d_2|)^2}{2\sqrt{d_1^2 + d_2^2 + d_1 d_2} + |d_1 - d_2|}.$$

Proof The proof of Corollary 2 is similar to that of theorem 2, so we omit it.

According to Theorem 2, the termination criterion for subdivision of the degree 3 rational Bézier curve may be improved into the following expression:

$$\frac{1}{3}\frac{(\sqrt{\Delta} + d_1 + d_2)^2}{2\sqrt{\Delta} + d_1 + d_2} \leqslant \varepsilon.$$

Theorem 3 For the Bézier curves of degree 3 or 4, we have respectively

$$h_3(t) \leqslant \frac{3}{4}\sqrt{\frac{d_1^2 + d_2^2}{2}} = \frac{\sqrt{6}}{8}\sqrt{3d_1^2 + 3d_2^2} \quad (n = 3), \tag{15}$$

$$h_4(t) \leqslant \frac{\sqrt{14}}{16}\sqrt{4d_1^2 + 6d_2^2 + 4d_3^2} \quad (n = 4). \tag{16}$$

Proof The proofs of Eq.(15) and Eq.(16) are similar, so we just prove Eq.(16). From the proof procedure of Theorem 1 and Theorem 2, it is easy to see that

$$h_4(t) \leqslant 4t(1 - t)^3 d_1 + 6t^2(1 - t)^2 d_2 + 4t^3(1 - t)d_3.$$

Using Cauchy's inequality, we have

$$h_4(t) \leqslant \sqrt{4d_1^2 + 6d_2^2 + 4d_3^2}\cdot\sqrt{4t^2(1 - t)^6 + 6t^4(1 - t)^4 + 4t^6(1 - t)^2},$$

write

$$f(t) := 4t^2(1 - t)^6 + 6t^4(1 - t)^4 + 4t^6(1 - t)^2.$$

Differentiation of $f(t)$ yields

$$f'(t) = 8t(1 - t)(1 - 2t)\{2(1 - t)^2[(1 - t) - \frac{1}{4}t]^2 + \frac{7}{4}(1 - t)^2 t^2 + 2t^2[t - \frac{1}{4}(1 - t)]^2\}.$$

From the above equality, we may see that $f(t)$ possesses the unique extreme point $t = \frac{1}{2}$ and that $f(t)$ gets its maximum $\frac{14}{256}$ at $t = \frac{1}{2}$. Thus Eq.(16) holds.

Remark 1 The estimate Eq.(15) is better than Eq.(5) and worse than Eq.(14), however Eq.(15) is more simple and elegant than Eq.(14).

By Eq.(16) in Theorem 3, the termination criterion for subdivision of the degree 4 Bézier curve may be improved into the following expression:

$$\max_{0\leqslant t\leqslant 1} h_4(t) \leqslant \frac{\sqrt{14}}{16}\sqrt{4d_1^2 + 6d_2^2 + 4^2 d_3} < \varepsilon.$$

Note that from considering Eq.(15) and Eq.(16), it is natural to think that the following inequality

$$h_n(t) \leqslant \frac{\sqrt{2^n - 2}}{2^n}\sqrt{\sum_{i=1}^{n-1} C_n^i d_i^2}, \quad (n \geqslant 5) \tag{17}$$

holds for the Bézier curve of degree n. But this is not true. To show it, we just put up an example. Let the plane Bézier curve of degree n such that $d_2 = d_3 = \cdots = d_{n-1} = 0$ and $d_1 \neq 0$, then

$$p_n(t) = nt(1-t)^{n-1} d_1.$$

By straightforward calculation, we have $\lim\limits_{n\to\infty}\max\limits_{0\leqslant t\leqslant 1} h_n(t) = \frac{1}{e} d_1$. However, according to the estimate Eq. (17) we have $\lim\limits_{n\to\infty}\max\limits_{0\leqslant t\leqslant 1} h_n(t) = 0$. This criterion shows that Eq. (17) is not true when n is large enough. By more careful calculation, we can prove Eq. (17) is not true when $n \geqslant 5$. To establish better estimation when $n \geqslant 5$, we will obtain it by different method than that above.

Theorem 4 For the rational Bézier curve of degree $n (\geqslant 4)$, $d_i (i = 1, 2, \cdots, n-1)$ is defined by (3), write $D = \max\limits_{1\leqslant i\leqslant n-1} d_i$, $w = \max\limits_{1\leqslant i\leqslant n-1} w_i$; then

$$h_n(t) \leqslant \left(1 + \frac{\min(w_0, w_n)}{(2^{n-1}-1)w - \frac{1}{2}\min(2(n-1)w, B)}\right)^{-1} D. \tag{18}$$

Where $B =$

$$\begin{cases} 2\sum\limits_{i=1}^{[\frac{n}{2}-1]} C_n^i \sqrt{(w-w_i)(w-w_{n-i})}, & n \text{ is odd}, \\ 2\sum\limits_{i=1}^{[\frac{n}{2}-1]} C_n^i \sqrt{(w-w_i)(w-w_{n-i})} + C_n^{\frac{n}{2}}(w - w_{\frac{n}{2}}), & n \text{ is even}. \end{cases} \tag{19}$$

Proof From the proof procedure of Theorem 1 and Theorem 2, it is easy to see that

$$h_n(t) \leqslant \frac{\sum\limits_{i=1}^{n-1} B_n^i(t) w_i d_i}{\sum\limits_{i=0}^{n} B_n^i(t) w_i},$$

thus,

$$h_n(t) \leqslant \frac{\sum\limits_{i=1}^{n-1} B_n^i(t) w_i}{\sum\limits_{i=0}^{n} B_n^i(t) w_i} D = \left(1 + \frac{w_0(1-t)^n + w_n t^n}{\sum\limits_{i=1}^{n-1} B_n^i(t) w_i}\right)^{-1} D. \tag{20}$$

Therefore

$$\sum_{i=1}^{n-1} B_n^i(t) w_i = \sum_{i=1}^{n-1} B_n^i(t) w - \sum_{i=1}^{n-1} B_n^i(t)(w - w_i) = w - \left[t^n w + (1-t)^n w + \sum_{i=1}^{n-1} B_n^i(t)(w - w_i)\right]. \tag{21}$$

Using the following inequality

$$B_n^i(t)(w - w_i) + B_n^{n-i}(t)(w - w_{n-i}) \geqslant 2C_n^i t^{\frac{n}{2}}(1-t)^{\frac{n}{2}} \sqrt{(w-w_i)(w-w_{n-i})}, \quad \left(i < \frac{n}{2}\right). \tag{22}$$

We have

$$\sum_{i=1}^{n-1} B_n^i(t) w_i \leqslant w - \left[t^n w + t^{\frac{n}{2}}(1-t)^{\frac{n}{2}} B + (1-t)^n w\right]. \tag{23}$$

Write

$$g(t) := t^n w + t^{\frac{n}{2}}(1-t)^{\frac{n}{2}} B + (1-t)^n w, \tag{24}$$

then differentiation of $g(t)$ yields

$$g'(t) = nt^{n-1} w + \frac{n}{2} t^{\frac{n}{2}-1}(1-t)^{\frac{n}{2}-1}(1-2t) \cdot B - n(1-t)^{n-1} w = n(2t-1)[t^{n-2} + t^{n-3}(1-t) + \cdots + (1-t)^{n-2}]w + \frac{n}{2} t^{\frac{n}{2}-1} \cdot (1-t)^{\frac{n}{2}-1}(1-2t) B = n(2t-1)\left\{[t^{n-2} + t^{n-3}(1-t) + \cdots + (1-t)^{n-2}]w - \frac{1}{2} t^{\frac{n}{2}-1} \cdot (1-t)^{\frac{n}{2}-1} B\right\}.$$

Therefore

$$[t^{n-2} + t^{n-3}(1-t) + \cdots + (1-t)^{n-2}]w - \frac{1}{2} t^{\frac{n}{2}-1}(1-t)^{\frac{n}{2}-1} B \geqslant \frac{1}{2} t^{\frac{n}{2}-1}(1-t)^{\frac{n}{2}-1} \cdot (2(n-1)w - B). \tag{25}$$

Now we distinguish two cases:

1. When $B \leqslant 2(n-1)w$, from Eq. (25), we know that $g'(t)$ has only extreme point $\frac{1}{2}$. It is easy to see $g(t) \geqslant \frac{2}{2^n} w + \frac{n}{2^n} B$.

2. When $B \geqslant 2(n-1)w$, noting Eq. (25), we know that

$$g(t) \geqslant t^n w + 2t^{\frac{n}{2}}(1-t)^{\frac{n}{2}}(n-1)w +$$

$$(1-t)^n w \geqslant \frac{n}{2^n}w + \frac{n}{2^n}2(n-1)w = \frac{2n}{2^n}w.$$

Combining 1 with 2, we have

$$g(t) \geqslant \frac{n}{2^n}w + \frac{n}{2^n}\min(2(n-1)w, B). \quad (26)$$

Using Eqs.(20),(21),(24),(26) and the following obvious inequality

$$w_0(1-t)^n + w_n t^n \geqslant \frac{\min(w_0, w_n)}{2^{n-1}},$$

we get $h_n(t) \leqslant$

$$\left(1 + \frac{w_0(1-t)^n + w_n t^n}{\frac{2^n-2}{2^n}w - \frac{1}{2^n}\min(2(n-1)w, B)}\right)^{-1} D \leqslant$$

$$\left(1 + \frac{\min(w_0, w_n)}{(2^{n-1}-1)w - \frac{1}{2}\min(2(n-1)w, B)}\right)^{-1} D.$$

This completes the proof.

Corollary 2 For the Bézier curve of degree n ($\geqslant 4$), d_i ($i = 1, 2, \cdots, n-1$) is defined by Eq.(3), writing $D = \max\limits_{1\leqslant i\leqslant n-1} d_i$, then

$$h_n(t) \leqslant \left(\frac{2^n-2}{2^n}\right)D - \frac{1}{2^n}\min(2(n-1)D, T). \quad (27)$$

Where $T =$

$$\begin{cases} 2\sum\limits_{i=1}^{[\frac{n}{2}-1]} C_n^i \sqrt{(D-d_i)(D-d_{n-i})}, & n \text{ is odd}, \\ 2\sum\limits_{i=1}^{[\frac{n}{2}-1]} C_n^i \sqrt{(D-d_i)(D-d_{n-i})} + C_n^{\frac{n}{2}}(D-d_{\frac{n}{2}}), & n \text{ is even}. \end{cases} \quad (28)$$

By Corollary 3, the termination criterion for subdivision of the degree n Bézier curve may be improved into the following expression:

$$\left(\frac{2^n-2}{2^n}\right)D - \frac{1}{2^n}\min(2(n-1)D, T) \leqslant \varepsilon.$$

Proof The proof of Corollary 3 is similar to that of Theorem 4, so we omit it.

Remarks 2:

1. When $T \geqslant 2(n-1)D$, Eq.(27) becomes $h_n(t) \leqslant \left(\frac{2^n-2n}{2^n}\right)D$, this is better than Eq.(7).

2. When $n = 4$ estimate Eq.(27) is sometimes better, sometimes worse, than estimate Eq.(16). We suggest computing both estimates and using their minimum.

3. When n is odd, degree elevation for the Bézier curve before using Corollary 2, but this requires increased calculations.

Finally, we point out that the estimate Eq.(27) is accurate for some kinds of Bézier curve of degree n. For example, for the plane Bézier curve of degree 4 such that $d_1 = d_3$, $d_1 \leqslant d_2 \leqslant 4d_1$, the estimate Eq.(27) holds accurately. We show it below. In this case we have

$$h_4(t) = 4t(1-t)^3 d_1 + 6t^2(1-t)^2 d_2 + 4t^3(1-t)d_1 = 2(3d_2 - 4d_1)[t(1-t)]^2 + 4d_1 t(1-t).$$

Noting that $0 \leqslant t(1-t) \leqslant \frac{1}{4}$, it is easy to get

$$\max_{0\leqslant t\leqslant 1} h_n(t) = h\left(\frac{1}{2}\right) = \frac{1}{2}d_1 + \frac{3}{8}d_2.$$

On the other hand, by the definition of T, we get $T = 8(d_2 - d_1) \leqslant 6d_2$. From the estimate Eq.(27), we immediately get

$$h_n(t) \leqslant \frac{1}{2}d_1 + \frac{3}{8}d_2.$$

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