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Equilibrium bed-concentration of nonuniform sediment*

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Abstract: Knowledge of the equilibrium bed-concentration is vital to mathematical modeling of the river-bed deformation associated with suspended load but previous investigations only dealt with the reference concentration of uniform sediment because of difficulties in observation of the bed-concentration. This work is a first attempt to develop a theoretical formula for the equilibrium bed-concentration of any fraction of nonuniform sediment defined at the bed-surface. The formula is based on a stochastic-mechanistic model for the exchange of nonuniform sediment near the bed, and described as a function of incipient motion probability, non-ceasing probability, pick-up probability, and the ratio of the average single-step continuous motion time to static time. Comparison of bed-concentration calculated from the proposed formula with the measured data showed satisfactory agreement, indicating the present formula can be used for solving the differential equation governing the motion of suspended load.

Key words: Equilibrium bed concentration, Suspended load, Stochastic-mechanistic model, Nonuniform sediment.

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INTRODUCTION

The equilibrium bed-concentration, as the bed boundary condition for solving the differential equation of suspended load is vital to mathematical modeling of sediment transport. Because of difficulties in measurements of the bed concentration in the field and in the experiment flume, no attempt has been made so far to identify equilibrium bed-concentration. Previous investigators only dealt with suspended load concentration C_a at a reference level $\boldsymbol{a}$ (Einstein, 1950; Engelund et al., 1976; Smith et al., 1977; Van Rijn, 1984; Akiyama et al., 1986; Garcia et al., 1991; Zyserman et al., 1994; Cao, 1997 and others, see Table 1). They proposed several empirical or semi-empirical formulas for the reference concentration of suspended load but the choice of $\boldsymbol{a}$ was somewhat arbitrary and lacked theoretical foundation. Strictly speaking, such reference concentration C_a cannot be used as a bed boundary condition for solving the differential equation for the motion of suspended load. Like the boundary of flow velocity, the boundary of sediment concentration in the vertical direction should be the bed surface. Therefore, the bed-concentration, instead of a reference concentration, is key to understanding the motion of suspended load in turbulent flow. Not only that, most of previous relations were applicable to uniform sediment rather than to nonuniform sediment. However, in fact, sediments transported in natural flow and those comprising flow boundaries are usually nonuniform particles.

Based on the stochastic-mechanistic model for the exchange of nonuniform sediment near the bed, this work is a first attempt to develop a formula for equilibrium bed-concentration of nonunifrom sediment defined at the bed-surface.

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Table 1 Previous formulas for the near-bed concentration

Investigators	Formula for Ca	reference level **a** and parameters
Einstein (1950)	$C_a = 0.0431 \frac{q_b}{r_s D U'_*}$	$2D$, q_b = transport rate of bedload U'_* = friction velocity due to grain
Mercer (1964)	$C_a = 0.025\tau_*^4$	0.05h
Engelund-Fredsoe (1976)	$C_a = 0.65 \frac{0.027\tau_* \rho_s/\rho}{\tau_* - 0.06 - 0.0851\pi P}$	$2D$, $P = \left[1 + \left(\frac{0.267}{\tau_* - 0.06}\right)^4\right]^{-1/4}$
Smith-Mclean (1977)	$C_a = 0.65 \frac{2.4\times10^{-3}(\tau_*/\tau_{*c} - 1)}{1 + 2.4\times10^{-3}(\tau_*/\tau_{*c} - 1)}$	$26.3D(\tau_* - \tau_{*c}) + K_s$ K_s = roughness height of grain
Van Rijn (1984)	$C_a = 0.015 \frac{D}{a} \frac{(\tau_*/\tau_{*c} - 1)^{1.5}}{D_*^{0.3}}$	0.5Δor K_s, $a \geqslant 0.01$h $D_* = D_k[(\rho_s/\rho - 1)g/v^2]^{1/3}$
Garde (1985)	$C_a = 1.12\times10^{-7}\left(\frac{U_*}{\omega}\frac{U_* D}{\upsilon}\right)^{5.25}$	$2D$, ν = viscosity of water ω = fall velocity of sediment
Akiyama-Fukushimu (1986)	$C_a = \begin{cases} 0 & \text{for } Z < 5 \\ 3\cdot10^{-12} Z^{10}(1 - 5/Z) & \text{for } 5 \leqslant Z \leqslant 13.2 \\ 0.3 & \text{for } Z > 13.2 \end{cases}$	0.05h $Z = \frac{U'_*}{\omega} D_*^{0.75}$
Garcia-Parker (1991)	$C_a = \frac{0.39\times10^{-7} Z_{eff}^5}{0.3 + 1.3\times10^{-7} Z_{eff}^5}$	0.05h, $\lambda = 1 - 0.288\log_2\sigma_g$ $Z_{eff} = \lambda U'_* / \omega_k D_{*k}^{0.9} (D_k/D_{50})^{0.2}$
Zyserman-Fredsoe (1994)	$C_a = \frac{0.46\times0.331(\tau_* - 0.045)^{1.75}}{0.46 + 0.33(\tau_* - 0.045)^{1.75}}$	$2D$
Cao (1997)	$C_a = 1.2\times10^{-4}(\frac{U_*^2}{U_{*c}^2} - 1)\frac{U_*}{\omega}\frac{U_*}{\upsilon D}$	The same as Van Rijn's a

DERIVATION OF FORMULA

Sediments transported in water can be divided into two types referred to as bedload and suspended load, depending on their respective form of motion: discontinuous motion near the bed and continuous motion over the entire depth of flow. Imagine that volume contents of sediment in arbitrarily small neighborhoods of the two-side of bed surface (Fig.1) are density coefficient of sediment in the bed-surface active-layer $M(z)$ $[z \in (-\delta, 0^-)]$ and suspended load concentration near-bed water layer $C(z)[z \in (0^+, \delta)]$ respectively, where δ = an arbitrarily small positive real number, z = distance from the bed. Then the left and right limits at the bed-surface are taken respectively as $\delta \to 0$:

$$\lim_{z\to0^-} M(z) = M_0 \quad \text{(Sediment in active layer)} \tag{1a}$$

$$\lim_{z\to0^+} C(z) = C_0 \quad \text{(Suspended load)} \tag{1b}$$

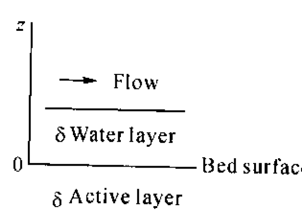

Fig.1 Arbitrarily small neighborhoods near the bed surface

The left limit M_0 is called the density coefficient of the active-layer surface material, which has a relatively stable value of 0.4. The right limit C_0, called the bed-concentration of suspended load, has a value less than M_0 and depends on flow conditions and sediment composition. Just as the left limit of sediment content at the water surface (i.e. water-surface concentration), with a value larger than zero, is unequal to its right limit (sediment concentration in air) with a zero value, sediment concentration at the interface, the bed surface, is discontinuous. For nonuniform sediment, let P_{0k} and P_k be the kth fraction percentages of bed material and suspended load at the bed-surface, then the corresponding kth fraction concentrations should equal $M_0 P_{0k}$ and $C_0 P_k$ (alternatively C_{0k}) respectively.

The time-averaged flux for the kth fraction of suspended load in turbulent flow is the sum of convection, settling and diffusion fluxes:

$$\Theta_i = (\bar{u}_i - \omega_k \delta_{iz}) C_k + \overline{u_i' C_k'} \quad (i = x, y, z) \tag{2}$$

where $\bar{u}_i$ and u_i' = the time-averaged and fluctuating velocities, C_k and C_k' = the time-averaged and fluctuating concentration for the kth fraction of nonuniform sediment, ω_k = settling velocity for the kth fraction of nonuniform sediment, $\delta_{iz} = \begin{cases} 1 & \text{for } i = z \\ 0 & \text{else} \end{cases}$. In the vertical direction, the time-averaged value of vertical bottom velocity approaches to zero when $z \to 0$. Thus, the vertical flux of suspended load at the bed-surface can be expressed as:

$$\Theta_{z0} = -\omega_k C_{0k} + \overline{V' C_k'} . \tag{3}$$

When the sediment exchange reaches an equilibrium state, the vertical flux of suspended load must be equal to zero, that is, $\Theta_{z0} = 0$, thus the upward diffusion flux of suspended load at the bed surface can be obtained as:

$$\overline{V' C_k'} = \omega_k C_{0k}^* . \tag{4}$$

Therefore,

$$\Theta_{z0} = \omega_k (C_{0k}^* - C_{0k}), \tag{5}$$

where C_{0k}^* = the equilibrium bed-concentration for the kth fraction of suspended load.

The bed-concentration for the kth fraction of suspended load C_{0k} is closely related to the concentration of active-layer material $M_0 P_{0k}$ because sediment exchange near the bed frequently occurs between suspended load and bed material. In steady turbulent flow, the sediment exchange near the bed will reach an equilibrium state after sufficient time adjustment so that the transition probability of sediment motion tends towards the stationary state probability. According to the stochastic-mechanistic model proposed by Sun (2000)* for the exchange of nonuniform sediment, the probability of suspension state P_{sk} for the kth fraction of nonuniform sediment denotes the percentage of particles entrained into suspension among the kth fraction of exchangeable particles within the arbitrarily small neighborhood of the bed-surface active-layer. The concentration of the kth fraction of exchangeable particles within this bed-surface active-layer is undoubtedly $M_0 P_{0k}$. From the viewpoint of sediment exchange, therefore, the equilibrium bed-concentration C_{0k}^* (or sediment entrainment capacity) for the kth fraction sediment should equal the product of the concentration for exchangeable material in the active-layer $M_0 P_{0k}$ multiplied by the suspension state probability P_{sk}, i.e.

$$C_{0k}^* = M_0 P_{0k} P_{sk} . \tag{6}$$

Let $C_0^*(k)$, the possible sediment entrainment capacity for the kth fraction of nonuniform sediment, represent the sediment concentration which can be entrained under given flow conditions and sediment composition. The value of $C_0^*(k)$ is equivalent to the sediment entrainment capacity when bed material is completely composed of the kth fraction particles:

$$C_0^*(k) = C_{0k}^* / P_{0k} = M_0 P_{sk} . \tag{7}$$

When the bed-surface active-layer consists of uniform sediment, the possible sediment entrainment capacity will equal the sediment entrainment capacity and the subscript k can be omitted, that is,

$$C_0^*(k) = C_0^* = M_0 P_s . \tag{8}$$

Based on Sun's (2000) model for sediment exchange, the probability of suspension state for the kth fraction of nonuniform sediment can be rewritten as:

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* Sun, Z., 2000. A model for the exchange of nonuniform sediment near the bed. submitted for publication.

$$P_{sk} = \frac{\dfrac{B\alpha_k\beta_k}{(1-\alpha_{0k})(1-\beta_k)(1+\alpha_{0k}\beta_k)}}{1+\dfrac{B_0\alpha_k(1-\alpha_{0k}\beta_k)}{(1-\alpha_{0k})(1+\alpha_{0k}\beta_k)}+\dfrac{B\alpha_k\beta_k}{(1-\alpha_{0k})(1-\beta_k)(1+\alpha_{0k}\beta_k)}} = \frac{BF(\cdot)}{1+G(\cdot)+BF(\cdot)} \quad (9)$$

where the second and third terms in the denominator represent the ratios of the state probability of discontinuous motion to static probability and the state probability of continuous motion to static probability, respectively. $B = q_1/q_3$ indicates the ratios of the average time intervals of single-step continuous motion to the average time interval of static state. These should be determined by measured data. For the case in which the quantity of bed load is much less than that of suspended load, the term $G(\cdot)$ can be neglected. As a consequence the combination of Eq. (6) or Eq. (7) with Eq. (9) reduces to

$$C_{0k}^{*} = M_0 P_{0k}\frac{BF(\cdot)}{1+BF(\cdot)}, \quad (10)$$

or

$$C_0^{*}(k) = M_0\frac{BF(\cdot)}{1+BF(\cdot)}, \quad (11)$$

where

$$F(\cdot) = \frac{\alpha_k\beta_k}{(1-\alpha_{0k})(1-\beta_k)(1+\alpha_{0k}\beta_k)}. \quad (12)$$

Eq. (10) and Eq. (11) present the theoretical relations for the equilibrium bed-concentration $C_0^{*}(k)$ and the possible sediment entrainment capacity $C_0^{*}(k)$ for the kth fraction of nonuniform sediment, which relate to three basic probabilities and hence to dimensionless shear stress θ_k. The formulas show theoretically that the value of $C_0^{*}(k)$ defined at the bed-surface is generally less than the density coefficient of bed material M_0 and approaches to M_0 as θ_k increases.

THREE BASIC PROBABILITIES

Three basic probabilities involved in Eq. 12, that is, incipient motion probability α_k, non-ceasing probability α_{0k}, and pick-up probability β_k for any kth fraction of nonuniform sediment can be determined as follows.

Incipient motion probability, α_k.

Incipient motion probability is a key parameter in the mechanics of sediment transport, but previous effort was mainly limited to investigating uniform sediment. Sun et al. (1997) detailedly analyzed the stochastic properties of instantaneous drag force, F_{Dk}; lift force, F_{Lk}; and the moment arms of drag force, lift force and gravity: L_{Dk}、L_{Dk} and L_{Gk}. Based on this analysis, they derived an expression for the incipient motion probability of nonuniform sediment:

$$\alpha_k = P_r(L_{Dk}F_{Dk} + L_{Lk}F_{Lk} > L_{Gk}G_k) = 1 - \frac{1}{\sqrt{2\pi}}\int_{-2.7(\sqrt{0.0822/\theta_k}+1)}^{2.7(\sqrt{0.0822/\theta_k}-1)} e^{-0.5x^2}dx; \quad (13a)$$

where G_k = the submerged weight of a particle, a determinant variable. Dimensionless shear stress θ_k can be expressed as:

$$\theta_k = \frac{\tau'_0\sigma_g^{0.25}}{(\rho_s-\rho)g(D_kD_m)^{0.5}}, \quad (13b)$$

in which a shelter-exposure coefficient of nonuniform sediment is considered. Here τ'_0 = the bed shear stress with respect to the grains; σ_g and D_m = the geometric standard deviation and mean diameter of bed material respectively; D_k = the mean diameter for the kth fraction of the moving particles; ρ_s and ρ = the mass densities of sediment and fluid respectively.

Non-ceasing probability, α_{0k}

For cohesive sediment, the moment of cohesive force exists in α_k but not in α_{0k}, so that $\alpha_{0k} > \alpha_k$ under the same flow conditions and sediment properties. For cohesionless sediment, however, there is disagreement on whether the critical hydraulic condition for ceasing motion differs from that for causing incipient motion. Some authors believe that no difference exists with the exception of cohesive sediment. This implies that Eq. (13) can be adopted for estimation of α_{0k}. Other authors contend that even for cohesionless sediment, incipient velocity is approximately 1.2 times ceasing velocity (Zhang et al., 1997). According to these authors, non-ceasing probability may be expressed as:

$$\alpha_{0k} = P_r\{L_{1k}F'_{Dk} + L_{2k}F'_{Lk} > L_{3k}G_k\} =$$

$$1-\frac{1}{\sqrt{2\pi}}\int_{-2.7(\sqrt{0.0571/\theta_k}+1)}^{2.7(\sqrt{0.0571/\theta_k}-1)} e^{-0.5x^2}dx \quad (14)$$

where F'_{Dk} and F'_{Lk} = instantaneous drag force and lift force respectively acting on a moving kth fraction particle of nonuniform sediment.

Pick-up probability β_k

On assumption that the instantaneous vertical velocity v_b near the bed follows normal probability distribution, $f(v_b)=\frac{1}{\sqrt{2\pi}\sigma_v}\exp\left[-0.5\left(\frac{v_b}{\sigma_v}\right)^2\right]$, the maximum pick-up probability equals 0.5 according to $\beta_k = P_r\{v_b \geqslant \omega_k\}$ since settling velocity $\omega_k > 0$. Considering that only upward-directed bottom eddies contribute to the suspension of sediment, pick-up probability is defined here as conditional (Sun, 1996) to make $\beta_k \in [0, 1]$ in Eq. (15):

$$\beta_k = P_r\{v_b \geqslant \omega_k \mid v_b \geqslant 0\} = \frac{2}{\sqrt{2\pi}}\int_{\omega/\sigma_v}^{\infty} e^{-0.5x^2}dx \quad (15)$$

where the standard deviation of vertical bottom velocity $\sigma_v \approx u_*$ (= shear velocity).

DETERMINATION OF THE ASSOCIATED PARAMETER

Experiment data on the distribution of suspended load concentration in flumes were selected from Vanoni (1946) and Einstein-Chien (1954). Three grain-sizes (0.1mm, 0.16mm, 0.133mm) and two flume slopes ($J=0.0025$ and 0.00125) were adopted in Vanoni's experiment. Concentrations at the reference point (0.05H) ranged from 0.995 g/l to 22.9 g/l, and suspended indices ranged from 0.423 – 1.42. Einstein-Chien used sediment with three size distributions in flume experiments and observed near-bed concentrations under the following conditions: discharge ranging from 73.62 l/s to 84.95 l/s, slopes from 0.0185 to 0.025, and sediment concentrations from 3.05 g/l to 695 g/l.

Because the lowest survey point of the concentration distribution is not on the bed, the development of a suitable forecasting model for C_{0k}^* is necessary. For the reason that some information about the concentration distribution is evident and the other may be obscure, application of a gray system model to predict C_{0k}^* is reasonable.

Concentration distribution curves derived from Vanoni's data (1946) are characterized basically by monotonous change. A corrected gray model GM(1, 1) can be used to predict the bed-concentration by taking the following steps:

1. Let a, the distance between the lowest survey point and bottom, be the interval of interpolation. Make a spline interpolation between water surface and the lowest survey point, based on the points where concentrations were measured, to obtain a sequence of sediment concentrations: $\{x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n)\}$, where $x^{(0)}(n)$ is the value of sediment concentration at the lowest survey point.

2. Generate a summation: $x^{(1)}(j) = \sum_{m=1}^{j} x^{(0)}(m)$, for $j = 1, 2\cdots n$ and take a data matrix

$$A = \begin{bmatrix} -0.5[x^{(1)}(1)+x^{(1)}(2)] & 1 \\ \cdots & \cdots \\ -0.5[x^{(1)}(j)+x^{(1)}(j+1)] & 1 \\ \cdots & \cdots \\ -0.5[x^{(1)}(n-1)+x^{(1)}(n)] & 1 \end{bmatrix}$$

$$Y_N = [x^{(0)}(2), x^{(0)}(3), \cdots, x^{(0)}(n)]^T$$

3. Calculate parameter sequence,

$$\hat{a} = \begin{bmatrix} a \\ u \end{bmatrix} = (A^TA)^{-1} - A^TY_N.$$

A gray model, GM(1, 1) is determined

$$\frac{dx^{(1)}}{dt} + ax^{(1)} = u, \quad (16)$$

or its discrete form:

$$x^{(1)}(j+1) = [x^{(0)}(j) - \frac{u}{a}]e^{-aj} + \frac{u}{a}. \quad (17)$$

Prediction of the concentration series below can be made according to the above formula, $\{\hat{x}^{(1)}(1), \hat{x}^{(1)}(2), \cdots, \hat{x}^{(1)}(n), \hat{x}^{(1)}(n+1)\}$.

4. If relative errors between predicted value $\hat{x}^{(1)}(j)$ and measured value $x^{(1)}(j)$ exceed a given threshold value, correction of errors is required:

$$\varepsilon^{(0)}(j) = x^{(1)}(j) - \hat{x}^{(1)}(j) \quad (j = 1, 2\cdots n).$$

The gray model below for errors is derived in the same manner:

$$\varepsilon^{(1)}(j+1)=[\varepsilon^{(0)}(1)-\frac{u_\varepsilon}{a_\varepsilon}]e^{-a_\varepsilon j}+\frac{u_\varepsilon}{a_\varepsilon}. \quad (18)$$

Derivative of Eq.(18) with respect to j is obtained:

$$\hat{\varepsilon}^{(0)}(j+1)=[u_\varepsilon - a_\varepsilon\varepsilon^{(0)}(1)]e^{-a_\varepsilon j}. \quad (19)$$

Eq.(20) is a prediction model GM(1,1), in which residual errors are corrected. $\hat{x}^{(0)}(j+1) =[x^{(0)}(1)-\frac{u}{a}]e^{aj}+\frac{u}{a}+[u_\varepsilon - a_\varepsilon\varepsilon^{(0)}(1)] e^{-a_\varepsilon j}$. (20)

5. According to the value $\tilde{x}^{(1)}(n+1)$ predicted from the corrected GM(1,1) model, we can obtain

$$C_{0k}^{*}=\tilde{x}^{(1)}(n+1)-x^{(1)}(n). \quad (21)$$

The bed-concentration calculated from the gray system model and the corresponding flow parameters for Vanoni's experimental data are listed in Table 2.

Table 2 Bed-concentration based on Vanoni's data

C_0(g/l)	θ_k	D(mm)	D^*	ω/U_*
9.487	1.391	0.160	4.019	0.260
8.343	1.391	0.160	4.019	0.260
5.066	0.905	0.160	4.019	0.323
21.604	1.432	0.160	4.019	0.257
15.199	1.419	0.160	4.019	0.258
15.162	1.419	0.160	4.019	0.258
17.652	1.432	0.160	4.019	0.257
22.502	1.495	0.160	4.019	0.251
34.054	1.437	0.160	4.019	0.256
23.934	1.394	0.160	4.019	0.260
45.994	1.368	0.160	4.019	0.263
16.304	0.863	0.160	4.019	0.331
19.093	1.380	0.160	4.019	0.262
34.695	1.555	0.160	4.019	0.246
10.446	0.792	0.160	4.019	0.345
7.916	0.775	0.160	4.019	0.349
2.182	0.578	0.160	4.019	0.404
8.952	1.067	0.100	2.512	0.157
3.803	0.546	0.100	2.512	0.219
9.139	1.220	0.100	2.512	0.147
7.742	1.081	0.100	2.512	0.156
11.289	1.048	0.130	3.266	0.228

In Einstein-Chien's experimental data (1954) on near-bed concentration, the highest survey point is below 0.25 H so that the number of equal-distance nodes in the interval from the bed to the lowest measured point is too small. Therefore, the above gray system model is unsuitable. We have to select another prediction approach. When Einstein-Chien's data (1954) are plotted as a semi-logarithmic graph (Fig.2), the near-bed concentrations basically show exponential distribution. The bed-concentrations calculated from the best-fitting line of relationship and the corresponding flow parameters are listed in Table 3.

Table 3 Bed-concentration based on Einstein-Chien's data

No	C_0(g/l)	θ_k	D(mm)	D_*	D_m(mm)	ω/U_*	P_0
1	4.865	1.174	0.920	24.501	1.30	0.839	0.036
2	47.253	1.041	1.170	31.159	1.30	0.976	0.234
3	167.086	0.885	1.620	43.143	1.30	1.181	0.721
4	2.054	0.817	1.900	45.671	1.30	1.284	0.009
5	55.910	1.205	1.170	28.124	1.35	0.888	0.175
6	295.580	1.024	1.620	38.940	1.35	1.080	0.719
7	39.926	0.945	1.900	45.671	1.35	1.182	0.080
8	91.997	1.411	1.170	29.067	1.38	0.819	0.138
9	528.885	1.199	1.620	40.247	1.38	0.995	0.713
10	149.792	1.107	1.900	47.203	1.38	1.089	0.165
11	82.036	1.406	1.170	30.777	1.40	0.823	0.092
14	3.454	1.416	0.920	24.614	0.99	0.818	0.044
15	115.950	1.255	1.170	31.302	0.99	0.952	0.796
16	23.024	1.101	1.520	40.666	0.99	1.111	0.155
17	18.318	1.416	0.920	23.154	0.95	0.818	0.055
18	338.606	1.256	1.170	29.446	0.95	0.955	0.804
19	60.277	1.102	1.520	38.255	0.95	1.117	0.141
20	16.105	1.399	0.920	24.388	0.95	0.830	0.075
21	200.331	1.240	1.170	31.016	0.95	0.967	0.773
22	58.500	1.088	1.520	40.294	0.95	1.128	0.143
23	25.131	1.441	0.920	23.976	0.95	0.816	0.056
24	398.416	1.278	1.170	30.491	0.95	0.951	0.809
25	65.071	1.121	1.520	39.612	0.95	1.110	0.132
26	67.227	1.607	0.920	23.976	0.95	0.773	0.091
27	514.767	1.425	1.170	30.491	0.95	0.900	0.755
28	96.205	1.250	1.520	39.612	0.95	1.051	0.144
32	5.600	4.022	0.215	5.463	0.33	0.205	0.024
33	80.734	3.382	0.304	7.725	0.33	0.335	0.325
34	136.847	2.964	0.396	10.063	0.33	0.454	0.492
35	44.188	2.492	0.560	14.230	0.33	0.629	0.152
36	5.274	4.182	0.215	5.341	0.33	0.197	0.013
37	107.035	3.517	0.304	7.553	0.33	0.323	0.243
38	284.495	3.081	0.396	9.838	0.33	0.440	0.541
39	108.975	2.591	0.560	13.913	0.33	0.612	0.188
40	3.061	5.301	0.215	5.263	0.33	0.172	0.008
41	108.650	4.458	0.304	7.442	0.33	0.283	0.209
42	352.318	3.906	0.396	9.694	0.33	0.387	0.558
43	157.842	3.285	0.560	13.709	0.33	0.541	0.210
44	513.071	4.033	0.396	9.344	0.33	0.373	0.540
45	199.973	4.376	0.304	7.418	0.33	0.285	0.237
46	5.317	5.474	0.215	5.073	0.33	0.162	0.008
47	222.611	3.392	0.560	13.213	0.33	0.525	0.230
48	180.058	4.603	0.304	7.173	0.33	0.271	0.207

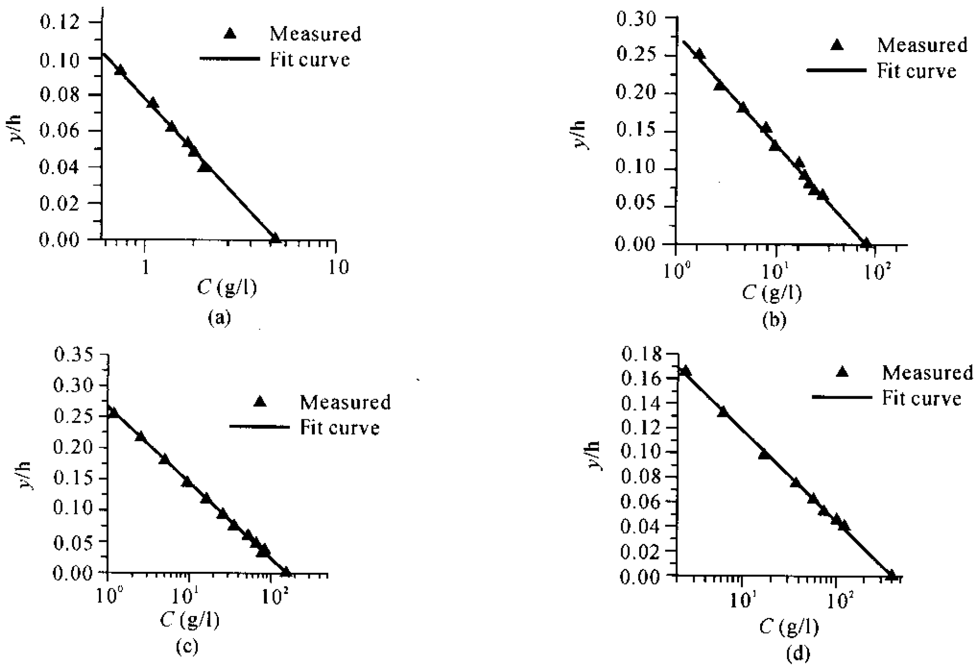

Fig.2 Some of Einstein-Chien's near-bed concentration profiles
(a) Run 1; (b) Run 11; (c) Run 44; (d) Run 24

The above data were used to determine the coefficient B in Eq.(11) for the equilibrium bed concentration as

$$B = 10^{-5}\theta_k^2 D_*^{1.84}. \qquad (22)$$

Therefore,

$$C_{0k}^* = M_0 P_{0k}\frac{10^{-5}\theta_k^2 D_*^{1.84}F(\cdot)}{1+10^{-5}\theta_k^2 D_*^{1.84}F(\cdot)} \qquad (23)$$

or

$$C_0^*(k) = M_0\frac{10^{-5}\theta_k^2 D_*^{1.84}F(\cdot)}{1+10^{-5}\theta_k^2 D_*^{1.84}F(\cdot)} \qquad (24)$$

where D_* is dimensionless grain-size of sediment.

Comparison of predicted equilibrium bed-concentration from Eq.(23) with measured values shown in Fig.3, indicates that the result using the theoretical formula agrees satisfactorily with experimental observation and thus can be used as the bottom boundary condition for solving the differential equation governing the motion of suspended load. Fig.4 displays the relationship between the possible sediment entrainment capacity and dimensionless shear stress when $D_* = 4.019$. It seems that C_{0k}^*/P_{0k} is approximately related to the fourth power of θ_k and increases with θ_k. When θ_k become very large, C_{0k}^*/P_{0k} gradually approaches to the density coefficient of bed surface material M_0.

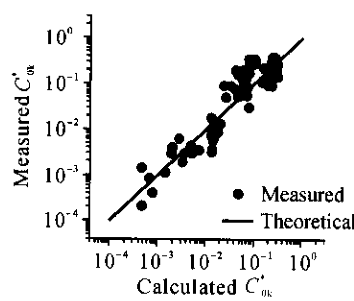

Fig.3 Comparison of calculated C_{0k}^* with measured

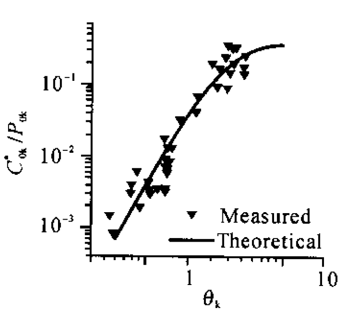

Fig.4 Relationship between C_{0k}^*/P_{0k} and θ_k

The equilibrium bed concentration may be used to extend the measurement data for suspended load from the lowest observing point to the bed surface (about 0.5m thickness of water layer near the bed) in natural streams. This provides insight into the transport of suspended load near the bed, which plays a very important role in the riverbed evolution. Use of the equilibrium bed-concentration can yield the vertical distribution of absolute concentration for the equilibrium case, while previous formulas, such as Rouse's formula, can only give relative concentrations. Especially, equilibrium bed-concentration can be applied to the first type of bed boundary condition for the convection-diffusion equation for sediment concentration. In order to use the equilibrium bed-concentration for the boundary condition, the turbulent diffusion coefficient ε_s in the convection-diffusion equation for sediment concentration should be modified to avoid zero value of ε_s at the bed.

CONCLUSIONS

Based on the stochastic-mechanistic model proposed by the author for the exchange of nonuniform sediment near the bed, a theoretical formula was derived for the equilibrium bed-concentration of nonuniform sediment C_{0k}^{*} defined at the bed-surface. The formula shows that the equilibrium bed-concentration is a function of three basic probabilities and the ratio of the average single-step continuous motion time to static time and, hence, of dimensionless shear stress θ_k and dimensionless grain-size D_*. Satisfactory agreement between the equilibrium bed-concentration calculated from Eq.23 and measured data means that the proposed formula can be applied to the bottom boundary condition for solving the differential equation of suspended load. The formula also shows that the equilibrium bed-concentration C_{0k}^{*} is approximately related to the fourth power of θ_k and increases with θ_k for a given D_*, and that the possible sediment entrainment capacity C_{0k}^{*}/P_{0k} gradually approaches to the density coefficient of bed-material M_0 as θ_k become very large.

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