http://www.zju.edu.cn/jzus; E-mail: jzus@zju.edu.cn
ISSN 1009-3095 *Journal of Zhejiang University SCIENCE* V.4, No.5, P.503-510, Sep.-Oct., 2003

Soliton dynamics in planar ferromagnets and anti-ferromagnets*

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Received Aug.3, 2003; revision accepted Aug.16, 2003

Abstract: The aim of this paper is to present a rigorous mathematical proof of the dynamical laws for the topological solitons (magnetic vortices) in ferromagnets and anti-ferromagnets. It is achieved through the conservation laws for the topological vorticity and the weak convergence methods.

Key words: Magnetic vortices, Topological vorticity, Conservation law, Soliton dynamics
Document code: A **CLC number:** O189

INTRODUCTION

Topological solitons have been the subject of much research for the past several decades. They are static solutions to geometric evolution equations with certain topological properties. Topological solitons arise in a variety of physical problems such as domain walls and magnetic bubbles in ferromagnetic continuum (O'Dell, 1981; Malozemoff and Slonzewski, 1979; Komineas and Papanicolaou, 1996; Papanicolaou and Tomaras, 1991), vortices in superfluids and superconductors (Huebener, 1979), and defects in liquid crystals (De Gennes and Prost, 1993). They also arise in the study of Bose-Einstein condensates, skyrmions, monopoles and instantions which are solutions of models of high-energy physics (Fetter and Svidzinsky, 2002; Jaffe and Taubes, 1980; Makhankov *et al.*, 1993; Rajaraman, 1982).

One important area of research is study of the dynamical behavior when many solitons are present in the system. Lin and Xin (1999) successfully employed various conservation laws, in particular the conservation of topological vorticity, to establish rigorously the Kirchoff law for vortex dynamics described by Gross-Pitaevskii equations. The idea of employing conservation laws, such as the conservation of the topological vorticity, to study dynamical behavior of vortices may not be new to many physicists working on the subject. However the purpose of Lin and Xin (1999) and this work was to present rigorous mathematical proofs. These rigorous proofs have led to several subtle mathematical issues which pose a challenge to explain at the level of physics. In particular, it is not known to the authors if there are any numerical simulations or physical experiments to study the effects of "sound waves" or "radiations" from the infinity on the dynamics of these topological solitons. In the present article, we shall concentrate on two particular examples: Topological solitons in planar ferromagnet described by the Landau-Lifschitz equations, and soliton in two dimensional anti-ferromagnets. One of the reasons for choosing these two specific examples (although our method applies to several other models as well) is that there is sparse experimental study of topological solitons in ferromagnetic films with an easy-plane anisotropy for which the relevant topological solitons are theoretically predicted to be half-bubbles or vortices. In fact, direct experimental evidence for the existence of pure anti-ferromagnetic solitons is absent although theoretical arguments suggests that such solitons should exist for the essentially the same reason as in the ordinary ferromagnets, although their dynamics should be significantly different as we will show below. Note that the rigorous static theory for the

* Project supported partially by the NSF grant, USA and Chang-Jiang Professor Fund at Zhejiang University, China

planar ferromagnetic vortices was recently established in Hang and Lin (2001). The paper is organized as follows. In Section 2, we describe the main results, in particular, the dynamical law for solitons in planar ferromagnets and anti-ferromagnets. Various conservation laws, as well as their physical relevance, are also stated. In Section 3, we present the proofs of these results. The research of the second author was partially supported by an NSF grant (USA). The research of the first author was supported by a NSF grant (USA) and a Chang-Jiang Professor Fund at Zhejiang University.

MAIN RESULTS

Ferromagnets The Landau-Lifschitz equations describing planar ferromagnets with easy plane anisotropy are given by

$$\frac{\partial}{\partial t}m = m \times f, \quad f = \Delta m - m_3 \boldsymbol{e}_3, \tag{1}$$

where $m: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{S}^2$ is the spin density vector with $|m|^2 = m_1^2 + m_2^2 + m_3^2 = 1$, and $\boldsymbol{e}_3 = (0, 0, 1)$ is the unit vector along the symmetry axis. Physical units have been normalized in Eq. (1) and, as a consequence, the magnon speed in the long-wavelength limit is normalized to unity ($c \equiv 1$). Eq. (1) can also be rewritten as

$$m \times \frac{\partial m}{\partial t} = -\Delta m + m_3 \boldsymbol{e}_3 - (|\nabla m|^2 + m_3^2) m. \tag{2}$$

These equations comprise an infinite dimensional Hamiltonian system with the Hamiltonian E given by

$$e(m) = \frac{1}{2}(|\nabla m^2| + m_3^2)$$
$$E(m) = \int_{\mathbb{R}^2} e(m) \mathrm{d}x. \tag{3}$$

Besides the energy the flow keeps the symplectic form invariant. From the Lagrangian formulation of the equations one can easily compute the symplectic form to be $\omega(x, t) \in \mathbb{R}$,

$$\omega(x, t) = m \cdot \frac{\partial m}{\partial x_1} \wedge \frac{\partial m}{\partial x_2}. \tag{4}$$

which is also the topological vorticity. Introducing the stress tensor

$$\sigma_{ij} = \frac{|\nabla m|^2 + m_3^2}{2}\delta_{ij} - \frac{\partial m}{\partial x_i} \cdot \frac{\partial m}{\partial x_j}, \tag{5}$$

$1 \leqslant i, j \leqslant 2$, then the conservation law for the topological vorticity is

$$\frac{\partial}{\partial t}\omega = -\epsilon_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}\sigma_{jk}. \tag{6a}$$

Here (ϵ_{ij}) is the 2-D anti-symmetric tensor, and here the usual summation convention is employed in Eq. (6a). Note that Eq. (6a) can be rewritten as

$$\frac{\partial \omega}{\partial t} = -\operatorname{curl}(\operatorname{div} \sigma). \tag{6b}$$

Eqs. (6a) or (6b) are derived formally under the assumption that the solution $m(x, t)$ of Eq. (1) is classical (say C^2). From these equations one can formally deduce that the total topological vorticity $\int_{\mathbb{R}^2} \omega(x, t) \mathrm{d}x$ is a constant of time (assuming the vector $m(x, t)$ is essentially a constant vector at $|x| = \infty$). In addition to this conserved quantity we can derive from Eq. (6a) the conservation of the linear momentum defined by $p_i = -\epsilon_{ij} I_j$,

$$I_j = \int_{\mathbb{R}^2} x_j \, \omega(x, t) \mathrm{d}x.$$

and the angular momentum a

$$a = \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 \omega(x, t) \mathrm{d}x.$$

In addition to these conserved quantities, the easy plane anisotropy conserves the total (azimuthal) spin:

$$M = \int_{\mathbb{R}^2} m_3(x, t) \mathrm{d}x.$$

In the study of the dynamics of topological solitons (magnetic vortices) of Eq. (1) formal arguments indicate that the behavior of these vortices are very similar to the ordinary vortices observed in classical fluids and superfluids (Papanicolaou and Tomaras, 1991; Komineas and Papanicolaou, 1996). That is when these magnetic half-bubbles are far apart from each other, and they move slowly in time, in other words, it is in a regime of "particles + fields" (Neu, 1990), these magnetic bubbles dynamics are very much like the dynamics of vortices in classical fluids whose essential features we now recall. A single vortex or anti-vortex is always spontaneously pinned and hence can move only together with the background fluid. On the other hand, in the presence of other vortices, it is possible that there is a nontrivial relative motion of vortices in the fluid; and, in fact, they display characteristics of the 2-D Hall motion of interacting electric charges in a uniform magnetic field. In particular, two similar vortices orbit around each other while a vortex antivortex pair goes through Kelvin motion along parallel trajectories perpendicular to the line connecting the vortex and the antivortex. In Hang and Lin (2002), we shall prove rigorously this latter fact is indeed true globally in time for the planar ferromagnetism. To study the dynamics of topological vortices we consider the Landau-Lifschitz Eq.(1) with given initial data $m(x,0) = m_0(x)$. Typical data $m_0(x)$ that we consider would have n degree 1 vortices and n degree -1 vortices (so that the total topological vorticity is zero). These vortices are moving at the speed $\simeq \epsilon^2 \ll 1$ (note the speed of light is normalized to be 1 in Eq.(1)), and mutual distance between any two is of order $1/\epsilon$ (we shall see this is the right scale to have nontrivial vortex dynamics). After a proper scaling in both time and space, Eq.(1) reduces to

$$\begin{cases} m^\epsilon \times \dfrac{\partial m^\epsilon}{\partial t} = -\Delta m^\epsilon + \dfrac{m_3}{\epsilon^2}e_3 - \\ \left(|\nabla m^\epsilon|^2 + \dfrac{m_e^2}{\epsilon^2}\right) m^\epsilon \\ m^\epsilon|_{t=0} = m_0^\epsilon(x). \end{cases} \tag{7}$$

Note that in the Eq.(7), vortices move at speed one, the distance between vortices is also of size one, and the "core" size of the magnetic bubbles are $\simeq \epsilon$. Our main result follows:

Theorem 1 Assume that the initial data have total $2n$-vortices such that it has n degree $+1$ vortices and n degree -1 vortices; the total initial energy $E(m_0^\epsilon) = 2\pi n \log\dfrac{1}{\epsilon} + O(1)$, and $m_0^\epsilon(x) \simeq$ constant for $|x| \simeq +\infty$. Then there is a solution of Eq.(7) with the following properties:

(i) $$\frac{|\nabla m^\epsilon|^2(x,t) + |m_3^\epsilon(x,t)|^2/\epsilon^2}{2\pi \log \dfrac{1}{\epsilon}} \to \sum_{j=1}^{2n} \delta_{a_j(t)}$$

as Radon measures when $\epsilon \to 0^+$, for every $t \in [0, T]$. Moreover, $a_j(t)$'s are Lifschitz continuous in $t \in [0, T]$, for some $T > 0$.

(ii) If, in addition, the initial energy is given by

$$E(m_0^\epsilon) = 2\pi n \log\frac{1}{\epsilon} + W(a(0)) + o_\epsilon(1),$$

then

$$(*)\quad \frac{d}{dt}a_j(t) = n_j \boldsymbol{J} \nabla_{a_j} W(a(t)),$$
for $t \in [0, T]$.

Here n_j = degree of vortices at $a_j = \pm 1$, $a(t) = (a_1(t), a_2(t), \ldots a_{2n}(t))$ and $\boldsymbol{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Where W is the so-called renormalized energy (Bethuel *et al.*, 1994) which is a function of $a_1, a_2, \ldots a_{2n} \in \mathbb{R}^2$.

We remark that the Kirchoff dynamic law for vortices (*) is established under the restrictive assumptions on the energy of the initial data m_0^ϵ. The reason for these assumption is to avoid the difficulty of dealing with the so-called "sound wave" effects due to the excess of energy in the data and the difficulty with the possible radiation effect from the infinity due to nonzero total topological vorticity. For more precise explanation see remarks in Section 3.

Antiferromagnets From the hydrodynamics approach of V.G. Baryakhar, M.V. Chetkin, B.A. Ivanov and S.N. Gadeskii (Dynamics of Topological Magnetic Solitons --- Experiment and Theory, Berlin, Springer), and the preliminary analysis in Komineas and Papanicolaou (1996), the equations for planar antiferromagnets are given by

$$\begin{cases} n_{tt} - \Delta n + n_3 e_3 + e_3 \wedge n_t = f, \ (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+, \\ n \times f = 0, \quad n \in \mathbb{S}^2. \end{cases} \tag{8}$$

Here $f = (|\nabla n|^2 - |n_t|^2 + {n_3}^2 + n_2 \dfrac{\partial}{\partial t} n_1 -$

$n_1 \frac{\partial}{\partial t} n_2) n$. We have to point out the term $e_3 \wedge n_t$ plays a key role in the vortex dynamics of Eq.(8) even though it is a first order term in a second order system. This term is due to a constant applied magnetic field whose strength is normalized to be 1. Indeed, without the applied magnetic field, Eq.(8) would be a relativistic σ-model which is Lorentz invariant, and thus one would not expect any particular motion law for vortices.

Since we are interested in the dynamics of the topological solitons we introduce an appropriate space and time scaling and consider the system

$$\begin{cases} \epsilon^2 n_{tt} - \Delta n + \frac{n_3 e_3}{\epsilon^2} + e_3 \wedge n_t = f, \\ f \times n = 0, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+, \end{cases} \tag{9}$$

As for the ferromagnets problem, these equations comprise an infinite dimensional Hamiltonian system whose energy density and energy are given by

$$e_\epsilon(n) \overset{\text{def}}{=} \frac{1}{2}\left(\epsilon^2 |n_t|^2 + |\nabla n|^2 + \frac{n_3^2}{\epsilon^2}\right)$$

$$E(n) = \int_{\mathbb{R}^2} e_\epsilon(n) \mathrm{d}x.$$

The symplectic form ω is easily derived from the Lagrangian of Eq.(9)

$$L = \frac{1}{2}\left[\epsilon^2 |n_t|^2 - |\nabla n|^2 - \frac{n_3^2}{\epsilon^2} + e_3 \cdot (n \wedge n_t)\right]$$

to obtain

$$\omega = \gamma + \epsilon_{ij} \epsilon^2 \partial_{x_i}\left(n_t \cdot \frac{\partial}{\partial x_j} n\right)$$

$$\gamma \overset{\text{def}}{=} e_3 \cdot n_{x_1} \wedge n_{x_2} = \frac{1}{2}(\partial_{x_2} e_3 \cdot n_{x_1} \wedge n + \partial_{x_1} e_3 \cdot n \wedge n_{x_2}).$$

The invariance of the symplectic form under the flow of Eq.(9) implies

$$\frac{\partial}{\partial t}\omega = -\epsilon_{ij} \partial_{x_i} \partial_{x_j} (\sigma_{ij}), \tag{10}$$

where $\sigma_{ij} = e_\epsilon(n) \delta_{ij} - n_{x_i} \cdot n_{x_j}$ is the stress tensor. From the above equation one can derive the following quantities for a classical solution $n(x,t)$:

(i) Total topological vorticity $\int_{\mathbb{R}^2} \gamma \mathrm{d}x$;

(ii) Total linear momentum $p_i \overset{\text{def}}{=} -\epsilon_{ij} I_j$ where

$$I_j = \int_{\mathbb{R}^2} x_j \omega(x,t) \mathrm{d}x;$$

(iii) angular momentum

$$a = \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 \omega(x,t) \mathrm{d}x.$$

(iv) $M = \int_{\mathbb{R}^2}\left(e_3 \cdot (n \wedge n_t) - \frac{n_3^2}{\epsilon^2}\right)\mathrm{d}x$

is also a constant in time though it has no physical meaning that we are aware of.

The linear and angular momentum can be written explicitly

$$p_i = -\int_{\mathbb{R}^2} (\epsilon^2 n_t \cdot \partial_{x_i} n + \epsilon_{ij} x_j \gamma) \mathrm{d}x = -\int_{\mathbb{R}^2} (\epsilon^2 n_t \cdot \partial_{x_i} n + \frac{1}{2} n_{x_i} \cdot (e_3 \wedge n)) \mathrm{d}x$$

$$a = \int_{\mathbb{R}^2} (-\epsilon_{ij} x_i \epsilon^2 n_t \cdot \partial_{x_j} n + \frac{1}{2} |x|^2 \gamma) \mathrm{d}x$$

Using these conservation laws, which at least hold under the assumption that we have a classical solution which tends to a constant unit vector at $|x| \simeq \infty$, we can state the following.

Theorem 2 Under the same assumptions as for Theorem 1, one has

$$\frac{\epsilon^2 |n_t|^2 + |\nabla n|^2 + \frac{n_3^2}{\epsilon^2}}{2\pi \log \frac{1}{\epsilon}} \rightharpoonup \sum_{j=1}^{2n} \delta_{a_j(t)}$$

as Random measures when ϵ tends to zero. Moreover, $a_j(t)$'s are Lifschitz continuous for $t \in [0, T]$ when the initial data satisfy

$$2E(n_0^\epsilon) = n 2\pi \log \frac{1}{\epsilon} + W(a(0)) + o_\epsilon(1),$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t} a_j(t) = J \nabla_{a_j} W(a(t)), \text{ for } t \in [0, T].$$

We have to point out that neither Eq.(7) nor Eq.(8) have a general well-posed theory for large finite energy data. However, there are natural ways to approximate systems Eq.(7) and Eq.(8) that lead to the existence of weak-solutions. One way of approximating Eq.(8) is to consider the following system instead

$$\epsilon^2 \frac{\partial^2}{\partial t^2} U_\epsilon^\delta - \Delta U_\epsilon^\delta + \frac{U_{\epsilon,e}^\delta \boldsymbol{e}_3}{\epsilon^2} + \boldsymbol{e}_3 \wedge \left(\frac{\partial}{\partial t} U_\epsilon^\delta\right) = \frac{(1 - |U_\epsilon^\delta|^2)}{\delta^2} U_\epsilon^\delta, \quad U_\epsilon^\delta(x, t) \in \mathbb{R}^3, \quad (11)$$

which has global classical solution for suitable initial data on $\mathbb{R}^2$. Note that various conservation laws for U_ϵ^δ analogous to those for Eq.(8) remain valid. Moreover, as $\delta \to 0^+$, $u_\epsilon^\delta \to n$ a weak solution of Eq.(9)(see Shatah, 1988). One then applies our proofs to the solutions $\{U_\epsilon^\delta\}$ and let $\delta \to 0^+$, $\epsilon \to 0^+$ at the end to obtain the desired conclusion. We must point out that the dynamical law for these topological solitons are independent of how one approximates these systems. Similar process can be carried out for the system Eq.(7); we won't elaborate on this here. Moreover instead of presenting proofs for both Theorem 1 and Theorem 2, we shall simply sketch the proof for the case of Gross-Pitaevskii equation and make remarks how those arguments can be modified for ferromagnets and anti-ferromagnets systems.

SKETCH OF PROOFS

The proofs are very similar to those presented in Lin and Xin (1999) for the case of Gross-Pitaevskii equation, which we shall recall below. Consider

$$\begin{cases} iU_t^\epsilon = \Delta U^\epsilon + \frac{U^\epsilon}{\epsilon^2}(1 - |U^\epsilon|^2), x \in \mathbb{R}^2 \times \mathbb{R} \\ u^\epsilon(x, 0) = U_0^\epsilon(x). \end{cases} \quad (12)$$

where $U_0^\epsilon(x)$ is a complex valued function such that U_0^ϵ is a constant of magnitude one as $|x| \to +\infty$. Assume U_0^ϵ has n degree 1 vortices and n degree -1 vortices, and

$$(*)\ E_*(U_0^\epsilon) = \int_{\mathbb{R}^2} \left[\frac{1}{2} |\nabla U_0^\epsilon|^2 + \frac{1}{4\epsilon^2} \cdot (1 - |U_0^\epsilon|^2)^2\right] dx = 2\pi n \log \frac{1}{\epsilon} + O(1).$$

The key conservation law used in Lin and Xin (1999) is the conservation of linear momentum

$$\frac{\partial}{\partial t} p(U_\epsilon) = 2\mathrm{div}(\nabla U_\epsilon \otimes \nabla U_\epsilon) + \nabla P_\epsilon, \quad (13)$$

for scalar valued function. Here $p(U_\epsilon) = I_m(U \nabla \bar{U}) = U_\epsilon \wedge \nabla U_\epsilon$. From Eq.(13), one derives also conservation of topological vorticity:

$$\frac{\partial}{\partial t} \mathrm{Jac}(U_\epsilon) = \mathrm{curl} \cdot \mathrm{div}(\nabla U_\epsilon \otimes \nabla U_\epsilon). \quad (14)$$

Under the assumption that U_0^ϵ has $2n$-vortices satisfying (*), one immediately derives (see Lin *et al*. (1999) for details):

(i) From the structure of vortices and conservation of energy

(a) $\frac{e_\epsilon(U_\epsilon)}{\pi \log \frac{1}{\epsilon}} dx \rightharpoonup \sum_{j=1}^{2n} \delta_{a_j(t)}$, as $\epsilon = \epsilon_k \to 0^+$,

as Radon measures, for all $0 \leq t \leq T$ (Note that if one considers Eq.(12) on a bounded smooth domain Ω with $U^\epsilon(x, t) \equiv g(x)$ on $\partial\Omega$, where $g: \partial\Omega \to \mathbb{S}^1$ is a degree d smooth map, and if $E_*(U_0^\epsilon) = \pi d \log \frac{1}{\epsilon} + O(1)$, then T can be chosen to be infinity).

(b) $\int_{\mathbb{R}^2} \frac{(1 - |U_\epsilon|^2)^2}{\epsilon^2} dx + \int_{\mathbb{R}^2} |\nabla |U_\epsilon||^2 \leq C$,

$\|\nabla U^\epsilon\|_{L^p(\mathbb{R}^2)}(t) \leq C(p)$, $1 \leq p < 2$, $0 \leq t \leq T$.

(ii) Using conservation of vorticity Eq.(14), one obtains

(c) $a_j(t)$ is continuous in $t \in [0, T]$.

(iii) Using conservation of momentum Eq.(13), one obtains

(d) $p(U_\epsilon) \rightharpoonup V$ in $L^1(0, T; L^1_{\mathrm{loc}}(\mathbb{R}^2_a))$.

$V_t = 2\mathrm{div}(V \otimes V) + \nabla P$, div $V = 0$ in $\mathbb{R}^2_a$.

where $L^1_{\mathrm{loc}}(\mathbb{R}^2_a)$ denotes those L^1-vectors on $\mathbb{R}^2 \setminus \bigcup_{j=1}^{2n} B_\rho(a_j(t))$, for every $\rho > 0$.

(e) P is a single valued function on $\mathbb{R}^2_a$ which is smooth away from $\{a_1(t), \ldots, a_{2n}(t)\}$. Moreover, $\nabla U_\epsilon \otimes \nabla U_\epsilon \rightharpoonup V \otimes V + \mu$, the matrix-valued Radon measure μ satisfies

$$\operatorname{div}\mu = \nabla P_\mu \quad \text{in} \quad \mathbb{R}_a^2.$$

The measure μ is called defect-measure that is responsible for the so-called sound wave effect, and P_μ can be viewed as sound wave pressure. The motion law for vortices under the assumption of (*) follows from Eq.(14) to give

$$\frac{d}{dt}a_j(t) = J\nabla_{a_j}W(a(t)) + f_j(\mu), \tag{15}$$

where W is simply the Logrithium interaction potential between vortices, and all $f_j(\mu)$ are bounded in t, for $j = 1, 2, \ldots, 2n$. Under the additional assumption on the initial data

$$E_*(U_0^\epsilon) = 2\pi n\log\frac{1}{\epsilon} + W(a(0)) + o_\epsilon(1) \tag{16}$$

one can show

$$\frac{d}{dt}a_j(t) = J\nabla_{a_j}W\, a(t), \quad \text{for } 0 \leqslant t \leqslant T.$$

Turning our attention to Eq.(7) we can make similar conclusions. Under the assumption that m_0^ϵ has $2n$, ± 1 degree vortices such that $m_0^\epsilon(x) \simeq$ a constant unit vector for $|x| \simeq +\infty$, and $E(m_0^\epsilon) = n2\pi\log\frac{1}{\epsilon} + O(1)$, we see that the function $U \stackrel{\text{def}}{=} m_1^\epsilon + im_2^\epsilon$ satisfies

(i) $U(x, t) \simeq$ a constant unit vector at $|x| \simeq +\infty$;

(ii) $E_*(U) = \int_{\mathbb{R}^2}\left[\frac{1}{2}|\nabla U|^2 + \frac{1}{4\epsilon^2}(1 - |U|^2)^2\right]dx \leqslant \int_{\mathbb{R}^2}\left(\frac{1}{2}|\nabla m^\epsilon|^2 + \frac{1}{4\epsilon^2}|m_3^\epsilon|^2\right)dx \leqslant \int_{\mathbb{R}^2}\left(\frac{1}{2}|\nabla m^\epsilon|^2 + \frac{|m_3^\epsilon|^2}{2\epsilon^2}\right)dx = E(m_0^\epsilon) = 2\pi n\log\frac{1}{\epsilon} + O(1)$;

(iii) $U(x, 0)$ has $2n$, degree ± 1 vortices.

Thus from Lin and Xin (1999), we see that (a) and (b) are valid for $m_1^\epsilon(t) + im_2^\epsilon(t)$, $0 \leqslant t \leqslant T$. In fact, (b) can be strengthened further, via (ii) above to give

(b′) $\int_{\mathbb{R}^2}\frac{|m_3^\epsilon|^2}{\epsilon^2}dx + \int_{\mathbb{R}^2}|\nabla|m_1^\epsilon + im_2^\epsilon||^2 dx + \int_{\mathbb{R}^2}|\nabla m_3^\epsilon|^2 \leqslant C$, and $\|\nabla m^\epsilon\|_{L^p(\mathbb{R}^2)}(t) \leqslant C(p)$, $1 \leqslant p < 2$, $0 \leqslant t \leqslant T$,

while (e) should be rewritten as

(e′) $\nabla m_\epsilon \otimes \nabla m_\epsilon \rightharpoonup \begin{pmatrix} V\otimes V & 0 \\ 0 & 0 \end{pmatrix} + \tilde{\mu}$, for a matrix-valued Radon measure $\tilde{\mu}$.

Again in a manner similar to that in Lin and Xin (1999) we conclude (c), i.e. continuity (or Lifschitz continuity) of $a_j(t)$, by using Eq.(6b) to obtain the following identity

$$\frac{d}{dt}\int_{B_R(a_j(t))} x\,\omega^\epsilon(x, t)dx = \int_{B_R(a_j(t))} -x\operatorname{curl}\operatorname{div}\sigma^\epsilon(x, t)dx. \tag{17}$$

where $\omega^\epsilon(x, t) = m^\epsilon \cdot m_{x_1}^\epsilon \wedge m_{x_2}^\epsilon$,

$$\sigma_{ij}^\epsilon(x, t) = \frac{1}{2}\left(|\nabla m^\epsilon|^2 + \frac{|m_3^\epsilon|^2}{\epsilon^2}\right)\delta_{ij} - m_{x_i}^\epsilon \cdot m_{x_j}^\epsilon,$$

for t close to a given time $\tau \in [0, T]$, and $R > 0$ is a small but fixed constant. Integrating by parts on the right-hand side of Eq.(17) we obtain as $\epsilon \to 0^+$, as in Lin and Xin (1999).

$$\frac{d}{dt}a_j(t) = J\nabla_{a_j}W(a_j(t)) + f_j(\tilde{\mu}), \quad \text{for } 0 \leqslant t \leqslant T. \tag{18}$$

Here we used the fact that

$$\omega^\epsilon(x, t)dx \rightharpoonup \sum_{j=1}^{2n} \pm\delta_{a_j(t)}$$

in the sense of distributions, which follows from the structure of the vortices.

If the initial data satisfy the additional assumption:

$$E(m_0^\epsilon) = 2\pi n\log\frac{1}{\epsilon} + W(a(t)) + o_\epsilon(1), \tag{19}$$

then as in Lin and Xin (1999), we denote by $b_j(t)$ the solution of

$$\begin{cases} b_j'(t) = J\nabla_{b_j}W(b(t)), \ j = 1, \cdots, 2n, \\ b(0) = a(0), \end{cases}$$

and let $\xi(t) = \sum_{j=1}^{2n} | a_j(t) - b_j(t) |$. Then using Eq.(18) it is easy to verify that

$$\frac{d}{dt}\xi(t) = \sum_{j=1}^{2n} | a_j'(t) - b_j'(t) | \leqslant C\xi(t). \tag{20}$$

The last estimate follows from the following observation:

$$E(m^{\epsilon}(t)) \equiv E(m^{\epsilon}(0)) = 2\pi n \log \frac{1}{\epsilon} + W(a(0)) + 0_{\epsilon}(1) \equiv 2\pi n \log \frac{1}{\epsilon} + W(b(t)) + o_{\epsilon}(1), \tag{21}$$

and

$$E(m^{\epsilon}(t)) \geqslant 2\pi n \log \frac{1}{\epsilon} + W(a(t)) + C_0 \sum_{j=1}^{2n} |f_j(\tilde{\mu})|. \tag{22}$$

Hence $\sum_{j=1}^{2n} \left| f(\tilde{\mu}) \right| \leqslant C \left| W(a(t)) - W(b(t)) \right| \leqslant C\xi(t)$. Via the uniqueness of the O.D.E. solution of

$$\begin{cases} \frac{d}{dt}\xi(t) \leqslant C\xi(t) \\ \xi(0) = 0 \end{cases} \tag{23}$$

we obtain $a_j(t) \equiv b_j(t)$, and Theorem 1 is proved.

Remarks: (i) When $\tilde{\mu}(0) \neq 0$ (defect measure is nonzero) we do not know if Kirchhoff dynamical law

$$\frac{d}{dt}a_j(t) = J \nabla_{a_j} W(a(t))$$

holds. In the case of the Gross- Pitaevskii equations, if either div $\mu = \nabla P_{\mu}$ in the whole of $\mathbb{R}^2$ (instead of $\mathbb{R}^2_a$) or spt $\mu \subseteq \mathbb{R}^2_a$, then we see that the same dynamical law is valid. The same remark also applies to solutions of Eq.(7).

(ii) We assumed that the initial data have exactly $2n$ vortices of degree ± 1 and that the topological degree at $|x| = +\infty$ is zero. If the winding number at $|x| = +\infty$ is nonzero, then we may have the so-called radiation effect from the infinity. In the latter case, one always has $E(m_0^{\epsilon}) = +\infty$.

For solution of the antiferromagnetic equations Eq.(9) the proofs are again very similar. The assumption on the initial energy

$$E(n_0^{\epsilon}) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\epsilon^2 |n_t|^2 + |\nabla n|^2 + \frac{n_3^2}{\epsilon^2} \right) dx = 2\pi n \log \frac{1}{\epsilon} + O(1),$$

implies that the function $U = n_1 + in_2$ satisfies (*). The dynamical law for vortices follows from the invariance of the symplectic form Eq.(10) which implies

$$\frac{d}{dt}\int_{B_R(a_j(z))} x\left[e_3 \cdot (n_{x_1} \wedge n_{x_2}) + \epsilon^2 \epsilon_{ij} \partial x_j \left(n_t \cdot \frac{\partial}{x_j} n \right) \right] dx = \int_{B_R(a_j(z))} x \,\mathrm{curl}\ \mathrm{div}\ \sigma_{\epsilon} dx \tag{24}$$

with $(\sigma_{\epsilon})_{ij} = \frac{1}{2}\left(\epsilon^2 | n_t |^2 + | \nabla n |^2 + \frac{n_3^2}{\epsilon^2} \right) \delta_{ij} - n_{x_i} n_{x_j}$,

and the complex function (or 2-vector) $U = n_1 + in_2$ satisfies

$$\frac{1}{2}(U \wedge \nabla U)_t = \mathrm{div}(\nabla n \otimes \nabla n) + \nabla P + \epsilon^2 (n_t \cdot \nabla n)_t. \tag{25}$$

Note that $e_3 \cdot n_{x_1}^{\epsilon} \wedge n_{x_2}^{\epsilon} \rightharpoonup \sum_{j=1}^{2n} \pm \delta_{a_j(t)}$ as $\epsilon \to 0^+$. Thus under the assumption of Theorem 2, one may employ the arguments in Lin and Xin (1999) to establish the incompressible fluid limit from Eq.(25).

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