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Analytical solutions of simply supported magnetoelectroelastic circular plate under uniform loads*

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Abstract: In this paper, the axisymmetric general solutions of transversely isotropic magnetoelectroelastic media are expressed with four harmonic displacement functions at first. Then, based on the solutions, the analytical three-dimensional solutions are provided for a simply supported magnetoelectroelastic circular plate subjected to uniform loads. Finally, the example of circular plate is presented.

Key words: Simply supported, Magnetoelectroelastic circular plate, General solution, Analytical solution
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INTRODUCTION

In elasticity, the problem of a simply supported circular plate subjected to uniform loads is a classic one. Timoshenko *et al*. (1970) presented a solution for an isotropic circular plate. Ding *et al*. (2000; 2001) obtained the analytical three-dimensional solution for a transversely isotropic piezoelectric circular plate. Recently Pan (2000) obtained a solution for a magneto-electroelastic rectangular plate. To the author's knowledge, no literature about the corresponding solution of magnetoelectroelastic circular plate had been published yet.

In this paper, the general solutions are expressed by four harmonic displacement functions for the axisymmetric problem of transversely isotropic magnetoelectroelastic media at first. For the problem of magnetoelectroelastic circular plate, the displacement functions are constructed by a linear composition of harmonic functions. The constants in displacement functions are obtained from a group of equations, and are determined by the boundary conditions. Then, substitution of the displacement functions in the general solutions yields the exact three-dimensional solutions for a simply supported magnetoelectroelastic circular plate under uniformly distributed loads. At last, the dimensionless deflections and bending moments at the center of purely elastic transversely isotropic, piezoelectric and magnetoelectroelastic circular plates are given for comparison.

AXISYMMETRIC GENERAL SOLUTIONS FOR TRANSVERSELY ISOTROPIC MAGNETOELECTROELASTIC MEDIA

1. Basic equations of axisymmetric problems

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \quad (1)$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0, \quad (2)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{\partial D_z}{\partial z} = 0, \quad (3)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0, \quad (4)$$

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$$\sigma_r = c_{11}\frac{\partial u_r}{\partial r} + c_{12}\frac{u_r}{r} + c_{13}\frac{\partial u_z}{\partial z} + e_{31}\frac{\partial \Phi}{\partial z} + d_{31}\frac{\partial \Psi}{\partial z}, \quad (5)$$

$$\sigma_\theta = c_{12}\frac{\partial u_r}{\partial r} + c_{11}\frac{u_r}{r} + c_{13}\frac{\partial u_z}{\partial z} + e_{31}\frac{\partial \Phi}{\partial z} + d_{31}\frac{\partial \Psi}{\partial z}, \quad (6)$$

$$\sigma_z = c_{13}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) + c_{33}\frac{\partial u_z}{\partial z} + e_{33}\frac{\partial \Phi}{\partial z} + d_{33}\frac{\partial \Psi}{\partial z}, \quad (7)$$

$$\tau_{rz} = c_{44}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) + e_{15}\frac{\partial \Phi}{\partial r} + d_{15}\frac{\partial \Psi}{\partial r}, \quad (8)$$

$$D_r = e_{15}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) - \varepsilon_{11}\frac{\partial \Phi}{\partial r} - g_{11}\frac{\partial \Psi}{\partial r}, \quad (9)$$

$$D_z = e_{31}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) + e_{33}\frac{\partial u_z}{\partial z} - \varepsilon_{33}\frac{\partial \Phi}{\partial z} - g_{33}\frac{\partial \Psi}{\partial z}, \quad (10)$$

$$B_r = d_{15}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) - g_{11}\frac{\partial \Phi}{\partial r} - \mu_{11}\frac{\partial \Psi}{\partial r}, \quad (11)$$

$$B_z = d_{31}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) + d_{33}\frac{\partial u_z}{\partial z} - g_{33}\frac{\partial \Phi}{\partial z} - \mu_{33}\frac{\partial \Psi}{\partial z}, \quad (12)$$

where σ_{ij}, u_i, D_i and B_i are the components of stress, displacement, electric displacement and magnetic induction, respectively; Φ and Ψ are the electric potential and magnetic potential, and c_{ij}, e_{ij}, d_{ij}, ε_{ij}, g_{ij} and μ_{ij} are elastic, piezoelectric, piezomagnetic, dielectric, electromagnetic and magnetic constants, respectively.

2. Axisymmetric general solutions

Ding *et al*. (1996) derived general solutions of a transversely isotropic piezoelectric media. In case the characteristic roots of Eq. (14) are distinct, the axisymmetric general solutions can be obtained by virtue of the methods of Ding *et al*. (1996) as follows

$$u_r = \sum_{j=1}^{4}\frac{\partial \Psi_j}{\partial r}, \quad u_z = \sum_{j=1}^{4} s_j k_{1j}\frac{\partial \Psi_j}{\partial z_j},$$

$$\Phi = \sum_{j=1}^{4} s_j k_{2j}\frac{\partial \Psi_j}{\partial z_j}, \quad \Psi = \sum_{j=1}^{4} s_j k_{3j}\frac{\partial \Psi_j}{\partial z_j},$$

$$\sigma_r = -2c_{66}\sum_{j=1}^{4}\frac{\partial \Psi_j}{r\partial r} - \sum_{j=1}^{4}\omega_{1j}s_j^2\frac{\partial^2 \Psi_j}{\partial z_j^2},$$

$$\sigma_\theta = -2c_{66}\sum_{j=1}^{4}\frac{\partial^2 \Psi_j}{\partial r^2} - \sum_{j=1}^{4}\omega_{1j}s_j^2\frac{\partial^2 \Psi_j}{\partial z_j^2},$$

$$\sigma_z = \sum_{j=1}^{4}\omega_{1j}\frac{\partial^2 \Psi_j}{\partial z_j^2},$$

$$\tau_{rz} = \sum_{j=1}^{4}\omega_{1j}s_j\frac{\partial^2 \Psi_j}{\partial r\partial z_j};$$

$$D_r = \sum_{j=1}^{4}\omega_{2j}s_j\frac{\partial^2 \Psi_j}{\partial r\partial z_j},$$

$$D_z = \sum_{j=1}^{4}\omega_{2j}\frac{\partial^2 \Psi_j}{\partial z_j^2},$$

$$B_r = \sum_{j=1}^{4}\omega_{3j}s_j\frac{\partial^2 \Psi_j}{\partial r\partial z_j},$$

$$B_z = \sum_{j=1}^{4}\omega_{3j}\frac{\partial^2 \Psi_j}{\partial z_j^2}, \quad (13)$$

where $c_{66} = (c_{11} - c_{12})/2$, $z_j = s_j z$, and s_j ($\mathrm{Re}[s_j] > 0$, if s_j are complex number; $\mathrm{Im}[s_j] > 0$, if s_j are imaginary, $j = 1, 2, 3, 4$) are the four distinct characteristic roots of the following equation:

$$a_1 s^8 - a_2 s^6 + a_3 s^4 - a_4 s^2 + a_5 = 0, \quad (14)$$

the displacement functions Ψ_j satisfy

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{\partial z_j^2}\right)\Psi_j = 0, \ (j = 1,2,3,4), \quad (15)$$

and the other material constants appearing in the above equations are listed in the Appendix.

SIMPLY SUPPORTED CIRCULAR PLATE UNDER UNIFORM LOADS

Consider a circular plate with thickness h and radius r_1. The plane ($z = 0$) is identical with the middle plane of the plate. For a simply supported circular plate, the boundary conditions are as follows:

$$z = \frac{h}{2}: \begin{cases} \tau_{rz} = 0 \\ D_z = 0 \\ \sigma_z = -p \\ B_z = 0 \end{cases}, \quad z = -\frac{h}{2}: \begin{cases} \tau_{rz} = 0 \\ D_z = 0 \\ \sigma_z = 0 \\ B_z = 0 \end{cases},$$

$$r = r_1: \begin{cases} \int_{-h/2}^{h/2} D_r \mathrm{d}z = 0 \\ \int_{-h/2}^{h/2} B_r \mathrm{d}z = 0 \\ \int_{-h/2}^{h/2} \sigma_r \begin{Bmatrix} z \\ 1 \end{Bmatrix} \mathrm{d}z = 0. \\ u_z |_{z=0} = 0 \end{cases} \quad (16)$$

Similar to Ding *et al*. (2000; 2001), the displacement functions Ψ_j can be taken as:

$$\Psi_j = F_{1j}z_j + F_{2j}(z_j^2 - \frac{1}{2}r^2) + F_{3j}(z_j^3 - \frac{3}{2}r^2 z_j) + F_{5j}(z_j^5 - 5r^2 z_j^3 + \frac{15}{8}r^4 z_j), \ (j = 1, 2, 3, 4), \tag{17}$$

where F_{1j}, F_{2j}, F_{3j} and F_{5j} are undetermined constants.

Let $F_{12} = F_{13} = F_{14} = 0$ and substituting Eq. (17) into Eq. (13), the displacements, stresses, electric potential, electric displacements, magnetic potential and magnetic inductions can be obtained. By virtue of the boundary conditions in Eq. (16). We have

$$2\sum_{j=1}^{4}\omega_{1j}s_jF_{3j} + 5h^2\sum_{j=1}^{4}\omega_{1j}s_j^3F_{5j} = 0, \tag{18}$$

$$\sum_{j=1}^{4}\omega_{1j}s_jF_{5j} = 0, \tag{19}$$

$$\sum_{j=1}^{4}\omega_{2j}s_jF_{5j} = 0, \tag{20}$$

$$\sum_{j=1}^{4}\omega_{2j}F_{2j} = 0, \tag{21}$$

$$3\sum_{j=1}^{4}\omega_{2j}s_jF_{3j} + \frac{5h^2}{2}\sum_{j=1}^{4}\omega_{2j}s_j^3F_{5j} = 0, \tag{22}$$

$$\sum_{j=1}^{4}\omega_{1j}F_{2j} = -\frac{p}{4}, \tag{23}$$

$$3\sum_{j=1}^{4}\omega_{1j}s_jF_{3j} + \frac{5h^2}{2}\sum_{j=1}^{4}\omega_{1j}s_j^3F_{5j} = -\frac{p}{2h}, \tag{24}$$

$$\sum_{j=1}^{4}\omega_{3j}s_jF_{5j} = 0, \tag{25}$$

$$\sum_{j=1}^{4}\omega_{3j}F_{2j} = 0, \tag{26}$$

$$3\sum_{j=1}^{4}\omega_{3j}s_jF_{3j} + \frac{5h^2}{2}\sum_{j=1}^{4}\omega_{3j}s_j^3F_{5j} = 0, \tag{27}$$

$$6r_1^2\sum_{j=1}^{4}(c_{66} - \omega_{1j}s_j^2)s_jF_{3j} + 3r_1^2\sum_{j=1}^{4}s_j[(c_{66} - \omega_{1j}s_j^2)s_j^2h^2 - 5(c_{66} - 2\omega_{1j}s_j^2)r_1^2]F_{5j} = 0 \tag{28}$$

$$\sum_{j=1}^{4}(c_{66} - \omega_{1j}s_j^2)F_{2j} = 0 \tag{29}$$

$$s_1k_{11}F_{11} - \frac{3r_1^2}{2}\sum_{j=1}^{4}s_jk_{1j}F_{3j} + \frac{15r_1^4}{8}\sum_{j=1}^{4}s_jk_{1j}F_{5j} = 0 \tag{30}$$

Thus, the displacement functions Ψ_j are determined by substituting constants F_{11}, F_{2j}, F_{3j} and F_{5j} ($j = 1, 2, 3, 4$), which are determined by Eqs. (18) to (30), into Eq. (17). Then from Eq. (13), the corresponding magnetoelectroelastic fields can be presented for a simply supported magnetoelectroelastic circular plate subjected to uniform loads on the upper surface.

EXAMPLE

Let $r_1 = 1$ m, $h = 0.1$ m and $p = 10^5$ Pa. The material constants are shown in Table 1. The dimensionless deflection $\bar{u}_z$ ($\bar{u}_z = \frac{u_z}{r_1}$; $r = 0$, $z = 0$) and bending moment $\bar{M}$ ($\bar{M} = \frac{M}{c_{33}r_1^2}$, $M = \int_{-h/2}^{h/2} z\sigma_r \mathrm{d}z$; $r = 0$) of the magnetoelectroelastic circular plate are shown in Table 2. For comparison, we also consider a piezoelectric plate and a purely elastic one with the same corresponding constants and boundary conditions. The calculated results are listed in Table 2. It is seen that the dimensionless deflections and bending moments at center caused by a uniform load p are not noticeably different.

Table 1 Material constants of magnetoelectroelastic media (Li, 2000)

c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	g_{11}
2.86×10^{11}	1.73×10^{11}	1.70×10^{11}	2.695×10^{11}	4.53×10^{11}	5.0×10^{-12}
e_{15}	e_{31}	e_{33}	ε_{11}	ε_{33}	g_{33}
11.6	−4.4	18.6	8.0×10^{-11}	9.3×10^{-11}	3.0×10^{-12}
d_{15}	d_{31}	d_{33}	μ_{11}	μ_{33}	
550	580.3	699.7	-5.90×10^{-4}	1.57×10^{-4}	

Units: elastic constants: Nm^{-2}; piezoelectric constants: Cm^{-2}; piezomagnetic constants: $NA^{-1}m^{-1}$; dielectric constants: $C^2N^{-1}m^{-2}$; electromagnetic constants: $NsV^{-1}C^{-1}$; magnetic constants: Ns^2C^{-2}.

Table 2 Dimensionless deflections and bending moments at center

Type of material	Deflection	Bending moment
Magnetoelectroelastic media	-1.7066×10^{-4}	-8.5700×10^{-8}
Piezoelectric media	-1.7211×10^{-4}	-8.5640×10^{-8}
Transversely isotropic media	-4.1694×10^{-4}	-7.8105×10^{-8}

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APPENDIX

$$a_1 = c_{44}[c_{33}(\varepsilon_{33}\mu_{33} - g_{33}^2) - 2e_{33}g_{33}d_{33} + \mu_{33}e_{33}^2 + \varepsilon_{33}d_{33}^2],$$

$$a_2 = c_{11}[c_{33}(\varepsilon_{33}\mu_{33} - g_{33}^2) - 2e_{33}g_{33}d_{33} + \mu_{33}e_{33}^2 + \varepsilon_{33}d_{33}^2] + c_{44}[c_{44}(\varepsilon_{33}\mu_{33} - g_{33}^2) + c_{33}(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) - 2e_{15}g_{33}d_{33} - 2e_{33}(g_{11}d_{33} + g_{33}d_{15}) + (\mu_{11}e_{33}^2 + 2\mu_{33}e_{15}e_{33}) + (\varepsilon_{11}d_{33}^2 + 2\varepsilon_{33}d_{15}d_{33})] - (c_{13} + c_{44})[(c_{13} + c_{44})(e_{33}\mu_{33} - g_{33}^2) + (e_{15} + e_{31})(e_{33}\mu_{33} - d_{33}g_{33}) - (d_{15} + d_{31})(e_{33}g_{33} - d_{33}\varepsilon_{33})] - (e_{15} + e_{31})[(c_{13} + c_{44})(e_{33}\mu_{33} - g_{33}d_{33}) - (e_{15} + e_{31})\cdot(c_{33}\mu_{33} + d_{33}^2) + (d_{15} + d_{31})(c_{33}g_{33} + d_{33}e_{33})] - (d_{15} + d_{31})[(c_{13} + c_{44})(-e_{33}g_{33} + \varepsilon_{33}d_{33}) + (e_{15} + e_{31})(c_{33}g_{33} + e_{33}d_{33}) - (d_{15} + d_{31})(c_{33}\varepsilon_{33} + e_{33}^2)],$$

$$a_3 = c_{11}[c_{44}(\varepsilon_{33}\mu_{33} - g_{33}^2) + c_{33}(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) - 2e_{15}g_{33}d_{33} - 2e_{33}(g_{11}d_{33} + g_{33}d_{15}) + (\mu_{11}e_{33}^2 + 2\mu_{33}e_{15}e_{33}) + (\varepsilon_{11}d_{33}^2 + 2\varepsilon_{33}d_{15}d_{33})] + c_{44}[c_{44}(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) + c_{33}(\varepsilon_{11}\mu_{11} - g_{11}^2) - 2e_{15}(g_{11}d_{33} + g_{33}d_{15}) - 2e_{33}g_{11}d_{15} + 2\mu_{11}e_{15}e_{33} + \mu_{33}e_{15}^2 + 2\varepsilon_{11}d_{15}d_{33} + \varepsilon_{33}d_{15}^2] - (c_{13} + c_{44})[(c_{13} + c_{44})(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) + (e_{15} + e_{31})\cdot(e_{15}\mu_{33} + e_{33}\mu_{11} - d_{15}g_{33} - d_{33}g_{11}) - (d_{15} + d_{31})(e_{15}g_{33} + e_{33}g_{11} - d_{15}\varepsilon_{33} - d_{33}\varepsilon_{11})] - (e_{15} + e_{31})[(c_{13} + c_{44})(e_{15}\mu_{33} + e_{33}\mu_{11} - g_{11}d_{33} - g_{33}d_{15}) - (e_{15} + e_{31})(c_{44}\mu_{33} + c_{33}\mu_{11} + 2d_{15}d_{33}) + (d_{15} + d_{31})(c_{44}g_{33} + c_{33}g_{11} + d_{15}e_{33} + d_{33}e_{15})] - (d_{15} + d_{31})[(c_{13} + c_{44})\cdot(-e_{15}g_{33} - e_{33}g_{11} + \varepsilon_{11}d_{33} + \varepsilon_{33}d_{15}) + (e_{15} + e_{31})(c_{44}g_{33} + c_{33}g_{11} + e_{15}d_{33} + e_{33}d_{15}) - (d_{15} + d_{31})(c_{44}\varepsilon_{33} + c_{33}\varepsilon_{11} + 2e_{15}e_{33})],$$

$$a_4 = c_{11}[c_{44}(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) + c_{33}(\varepsilon_{11}\mu_{11} - g_{11}^2) - 2e_{15}(g_{11}d_{33} + g_{33}d_{15}) - 2e_{33}g_{11}d_{15} + 2\mu_{11}e_{15}e_{33} + \mu_{33}e_{15}^2 + 2\varepsilon_{11}d_{15}d_{33} + \varepsilon_{33}d_{15}^2] + c_{44}[c_{44}(\varepsilon_{11}\mu_{11} - g_{11}^2) - 2e_{15}g_{11}d_{15} + \mu_{11}e_{15}^2 + \varepsilon_{11}d_{15}^2 - (c_{13} + c_{44})[(c_{13} + c_{44})(\varepsilon_{11}\mu_{11} - g_{11}^2) + (e_{15} + e_{31})(e_{15}\mu_{11} - d_{15}g_{11}) - (d_{15} + d_{31})\cdot(e_{15}g_{11} - d_{15}\varepsilon_{11}) - (e_{15} + e_{31})[(c_{13} + c_{44})(e_{15}\mu_{11} - g_{11}d_{15}) - (e_{15} + e_{31})(c_{44}\mu_{11} + d_{15}^2) + (d_{15} + d_{31})(c_{44}g_{11} + d_{15}e_{15})] - (d_{15} + d_{31})[(c_{13} + c_{44})(-e_{15}g_{11} + \varepsilon_{11}d_{15}) + (e_{15} + e_{31})\cdot(c_{44}g_{11} + e_{15}d_{15}) - (d_{15} + d_{31})(c_{44}\varepsilon_{33} + e_{15}^2)],$$

$$a_5 = c_{11}[c_{44}(\varepsilon_{11}\mu_{11} - g_{11}^2) - 2e_{15}g_{11}d_{15} + \mu_{11}e_{15}^2 + \varepsilon_{11}d_{15}^2], \tag{A.1}$$

$$k_{mj} = \beta_{mj}/(\alpha_j s_j^2), \ (m = 1, 2, 3) \tag{A.2}$$

$$\alpha_j = -n_1 + n_2 s_j^2 - n_3 s_j^4, \ \beta_{mj} = -n_{4m} + n_{5m}s_j^2 - n_{6m}s_j^4 + n_{7m}s_j^6, \ (m = 1, 2, 3) \tag{A.3}$$

$$n_1 = (c_{13} + c_{44})(\varepsilon_{11}\mu_{11} - g_{11}^2) + (e_{15} + e_{31})(e_{15}\mu_{11} - g_{11}d_{15}) - (d_{15} + d_{31})(e_{15}g_{11} - \varepsilon_{11}d_{15}),$$

$$n_2 = (c_{13} + c_{44})(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) + (e_{15} + e_{31})(e_{15}\mu_{33} + e_{33}\mu_{11} - g_{11}d_{33} - g_{33}d_{15}) - (d_{15} + d_{31})(e_{15}g_{33} + e_{33}g_{11} - \varepsilon_{11}d_{33} - \varepsilon_{33}d_{15}),$$

$$n_3 = (c_{13} + c_{44})(\varepsilon_{33}\mu_{33} - g_{33}^2) + (e_{15} + e_{31})(e_{33}\mu_{33} - g_{33}d_{33}) - (d_{15} + d_{31})(e_{33}g_{33} - \varepsilon_{33}d_{33}),$$

$$n_{41} = c_{11}(\varepsilon_{11}\mu_{11} - g_{11}^2),$$

$$n_{51} = c_{11}(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) + c_{44}(\varepsilon_{11}\mu_{11} - g_{11}^2) + \mu_{11}(e_{15} + e_{31})^2 + \varepsilon_{11}(d_{15} + d_{31})^2 - 2g_{11}(e_{15} + e_{31})(d_{15} + d_{31}),$$

$$n_{61} = c_{11}(\varepsilon_{33}\mu_{33} - g_{33}^2) + c_{44}(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) + \mu_{33}(e_{15} + e_{31})^2 + \varepsilon_{33}(d_{15} + d_{31})^2 - 2g_{33}(e_{15} + e_{31})(d_{15} + d_{31}),$$

$$n_{71} = c_{44}(\varepsilon_{33}\mu_{33} - g_{33}^2), \ n_{42} = c_{11}(e_{15}\mu_{11} - g_{11}d_{15}),$$

$$n_{52} = c_{11}(e_{15}\mu_{33} + e_{33}\mu_{11} - g_{11}d_{33} - g_{33}d_{15}) + c_{44}(e_{15}\mu_{11} - g_{11}d_{15}) - (e_{15} + e_{31})[\mu_{11}(c_{13} + c_{44}) + d_{15}(d_{15} + d_{31})] + (d_{15} + d_{31})[g_{11}(c_{13} + c_{44}) + e_{15}(d_{15} + d_{31})],$$

$$n_{62} = c_{11}(e_{33}\mu_{33} - g_{33}d_{33}) + c_{44}(e_{15}\mu_{33} + e_{33}\mu_{11} - g_{11}d_{33} - g_{33}d_{15}) - (e_{15} + e_{31})[\mu_{33}(c_{13} + c_{44}) + d_{33}(d_{15} + d_{31})] + (d_{15} + d_{31})[g_{33}(c_{13} + c_{44}) + e_{33}(d_{15} + d_{31})],$$

$$n_{72} = c_{44}(e_{33}\mu_{33} - g_{33}d_{33}); \ n_{43} = c_{11}(-e_{15}g_{11} + \varepsilon_{11}d_{15}),$$

$$n_{53} = c_{11}(-e_{15}g_{33} - e_{33}g_{11} + \varepsilon_{11}d_{33} + \varepsilon_{33}d_{15}) + c_{44}(-e_{15}g_{11} + \varepsilon_{11}d_{15}) + (e_{15} + e_{31})[g_{11}(c_{13} + c_{44}) + d_{15}(d_{15} + d_{31})] - (d_{15} + d_{31})[\varepsilon_{11}(c_{13} + c_{44}) + e_{15}(e_{15} + e_{31})],$$

$$n_{63} = c_{11}(-e_{33}g_{33} + \varepsilon_{33}d_{33}) + c_{44}(-e_{15}g_{33} - e_{33}g_{11} + \varepsilon_{11}d_{33} + \varepsilon_{33}d_{15}) + (e_{15} + e_{31})[g_{33}(c_{13} + c_{44}) + d_{33}(e_{15} + e_{31})] - (d_{15} + d_{31})[\varepsilon_{33}(c_{13} + c_{44}) + e_{33}(e_{15} + e_{31})],$$

$$n_{73} = c_{44}(-e_{33}g_{33} + \varepsilon_{33}d_{33}), \tag{A.4}$$

$$\omega_{1j} = c_{44}(1 + k_{1j}) + e_{15}k_{2j} + d_{15}k_{3j}, \ \omega_{2j} = e_{15}(1 + k_{1j}) - \varepsilon_{11}k_{2j} - g_{11}k_{3j},$$

$$\omega_{3j} = d_{15}(1 + k_{1j}) - g_{11}k_{2j} - \mu_{11}k_{3j} \tag{A.5}$$