http://www.zju.edu.cn/jzus; E-mail: jzus@zju.edu.cn
ISSN 1009 – 3095 *Journal of Zhejiang University SCIENCE* V.4, No.5, P.584 – 590, Sep. – Oct., 2003

Novel boundary element method for resolving plate bending problems*

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Received Dec.29, 2002; revision accepted Mar.18, 2003

Abstract: This paper discusses the application of the boundary contour method for resolving plate bending problems. The exploitation of the integrand divergence free property of the plate bending boundary integral equation based on the Kirchhoff hypothesis and a very useful application of Stokes' Theorem are presented to convert surface integrals on boundary elements to the computation of bending potential functions on the discretized boundary points, even for curved surface elements of arbitrary shape. Singularity and treatment of the discontinued corner point are not needed at all. The evaluation of the physics variant at internal points is also shown in this article. Numerical results are presented for some plate bending problems and compared against analytical and previous solutions.

Key words: Boundary contour method, Plate bending, Boundary element method
Document code: A **CLC number:** O346.1

INTRODUCTION

The boundary element method is an efficient numerical method for solving practical engineering problems, and is particularly attractive for the analysis of elastic mechanics problems, as only the boundary of the problem is required to be discretized. The main advantage of the boundary element method is the reduction of the dimension of boundary value problems. The conventional boundary element method requires numerical evaluation of line integral for two dimensional problems and surface integrals for three dimensional ones (Mukherjee, 1982). Since the boundary integrals involve singular kernels, the boundary element method encounters difficulty in the numerical evaluation of singular integrals, and the overall accuracy of this method is largely dependent on the precision with which the various integrals, especially the singular integrals, are computed. Singular integrals in Cauchy singular integral equations and hypersingular integral equations were treated in previous papers (Tanaka *et al.*, 1994; Toh *et al.*, 1994).

A novel variant of the boundary element method, called the boundary contour method, was first proposed by Nagarajan *et al.* (1994) for linear elasticity. The central idea of the new approach is the exploitation of the divergence free property of the boundary element method integrand. This property of the integrand shows the path independent integral in the boundary integral equation for two-dimensional problems. For three-dimensional problems, surface integrals on boundary elements are converted, through an application of Stokes' Theorem, into line integrals on the boundary contours of these elements. Thus, the boundary contour method requires simply the computation of potential functions at the end points of the boundary elements in two-dimensional problems and only numerical evaluation of line integrals over closed contours in three-dimensional cases. The boundary contour method for two-dimensional problems as presented by Phan *et al.* (1997) used quadratic boundary elements. The boundary method for three-dimensional problems was discussed by Nagarajan *et al.* (1996) and Mukherjee *et al.* (1997). This idea was originally proposed and applied to the Laplace equation by Lutz (1991)

* Project (No: ZE0208) supported by the National Science Foundation of Zhejiang Province, China

and Lutz *et al*. (1992). The divergence free property of the boundary element method integrand is intrinsically linked to the way in which the boundary integral equation is derived for linear problems. Zhou *et al*. (1999a; 1999b) discussed the traction boundary contour method wherein the unknown traction components on the calculated boundary can be uniquely determined in some two-dimensional boundary value problem. Chen *et al*. (1999) and Zhou *et al*. (1998) presented a method to calculate the stress intensity factor and linear elastic crack problems using the boundary contour method.

The boundary element method was further developed plate bending on elastic foundation problems. Jaswon and Maiti (1968) first presented the boundary integral equation for plate bending problems. Altirero and Sikarskie (1978) and Tottenham (1979) discussed indirect boundary integral equation solutions of Kirchhoff plate bending problems. However, in the application of boundary element method to plate, many domain integrals and singularities are included. A special technique (named the dual reciprocity method) to treat the domain integrals was first proposed by Nardini and Brebbia (1982) to solve elastodynamic problems. Wen *et al*. (1998) presented a new direct integral method for transformation of domain integral into boundary integral. Maucher and Hartmann (1999) studied the influence of the corner singularities of Kirchhoff plates on a boundary element solution and split the solution into a regular and singular part.

The present paper is armed to illustrate the effectiveness of the boundary contour method for the analysis of plate bending problems. The plate bending boundary integral equation is derived from the general Reyleigh-Green's identity. The integrand divergence free property of the plate bending boundary integral is shown by a useful application of Stokes' theorem, the boundary contour method requires simply the computation of potential functions at the end points of the boundary elements. This requires the determination of certain potential functions corresponding to the derived boundary integral equation integrand which contains the unknown normal bending moment, effective shear forces, rotations and deflections. Perfect cubic polynomial is employed to describe the deflection. The numerical implementation of the plate bending boundary contour method is presented by using nonlinear element on the plate boundary, resulting in significant condensation of the resulting linear algebraic system. The internal physical variables are then obtained as a post-processing step. Some selected examples are computed to show the effectiveness of this method.

BASIC THEORY OF BOUNDARY CONTOUR METHOD FOR PLATE BENDING

The governing equation for the bending of thin elastic plates under a lateral distributed load $g(x, y)$ is expressed with Kirchhoff's hypothesis by the following biharmonic differential equations:

$$D\nabla^4 w = g(x, y) \qquad (1)$$

where w and D denote the deflection and flexural rigidity, respectively. Assuming that the flexural rigidity $D = Eh^3/(1-\nu^2)$ is constant all over the plate domain. As usual, the generalized internal physics such as the bending moment m_{ij} and the shear force q_i and rotation θ_i can be expressed in the following form:

$$m_{ij} = D[\nu\delta_{ij}w_{,kk} + (1-\nu)w_{,jj}] \qquad (2)$$

$$q_i = -Dw_{,jji} \qquad (3)$$

$$\theta_i = w_{,i} \qquad (4)$$

It is worth remembering that owing to the assumed Kirchhoff hypothesis, only four boundary values are independent. Twisting moments and shear forces along the boundary can be combined to give the well-known effective shear force V_n, given by:

$$V_n = q_n + \frac{\partial m_{ns}}{\partial s} \qquad (5)$$

where (n, s) are the local co-ordinate system, with n and s referred to the boundary normal and tangential directions, respectively, no summation is implied.

By applying the general Reyleigh-Green's identity (Du, 1989), the integral equation is formulated as follows:

$$\int_\Omega [D\nabla^4 wG - Dw\nabla^4 G]\mathrm{d}\Omega = \int_\Gamma [wV(G) -$$

$$\theta(w)m(D) + m(w)\theta(G) - V(w)G]\mathrm{d}\Gamma \tag{6}$$

where Ω is the plate domain and Γ is its boundary. G is fundamental solution:

$$G(r) = r^2(\ln r - 1)/(8\pi) \tag{7}$$

It is very easy to derive the integral representation of displacement without the lateral distributed load, to obtain

$$k(S)w(S) = \int_\Gamma \Big\{V_n(Q)w^*(S,Q) - V_n^*(S,Q)w(Q) + M_{nn}^*(S,Q)\frac{\partial w(Q)}{\partial n} - M_{nn}(Q)\frac{\partial w^*(S,Q)}{\partial n}\Big\}\mathrm{d}\Gamma(Q) \tag{8}$$

where Q is the field point taken along the plate boundary, while S is the source point that can be placed anywhere. The symbol * indicates the corresponding fundamental solutions (which can be computed at the field point owing to a unit load applied at the source point) are expressed as follows:

$$w^*(S,Q) = \frac{r^2(2\ln r - 1)}{16\pi D} \tag{9}$$

$$\frac{\partial w^*(S,Q)}{\partial n} = \frac{n_i r r_{,i}}{4\pi D}\ln r \tag{10}$$

$$M_{ns}^*(S,Q) = -\frac{(1-\nu)(r_{,i}n_i)(r_{,j}s_j)}{4\pi} \tag{11}$$

$$V_n^*(S,Q) = \frac{r_{,i}n_i}{4\pi r}[2(1-\nu)(r_{,j}s_j)^2 - 3 + \nu] + \frac{1-\nu}{4\pi r}[1 - 2(r_{,i}s_i)^2] \tag{12}$$

where r is the displacement between the source and field point, ν is the Poisson's ratio. $k(S)$ represents the free term that is equal to the unit when S is an internal point, 0.5 for the smooth boundary points, zero when outside.

Noticing the term of $\frac{\partial w}{\partial n}$ denotes the rotation θ, rewrite the integrand of Eq.(8) as follows:

$$k(S)w(S) = \int_\Gamma R_n(S,Q)\boldsymbol{e}_n \mathrm{d}\Gamma(Q) \tag{13}$$

where

$$\boldsymbol{R}(S,Q) = R_n\boldsymbol{e}_n = \{V_n(Q)w^*(S,Q) - V_n^*(S,Q)w(Q) + M_{nn}^*(S,Q)\theta_n(Q) - M_{nn}(Q)\theta_n^*(S,Q)\}\boldsymbol{e}_n \tag{14}$$

Taking the divergence of the above at a field point Q

$$\nabla_Q \cdot \boldsymbol{R}_n = R_{n,n} = V_{n,n}w^* - wV_{n,n}^* + (V_n - M_{nn,n})\theta_n^* - (V_n^* - M_{nn,n}^*)\theta_n + M_{nn}^*\theta_{n,n} - M_{nn}\theta_{n,n}^* \tag{15}$$

It is easy to see that the value of $R_{n,n}$ in Eq.(15) vanishes everywhere except at the singular point when the source point coincides with the field point.

The above property indicates the existence of certain vector potential functions $\boldsymbol{WW}_n$ such that

$$\boldsymbol{R}_n = \nabla_Q \times \boldsymbol{W}_n \tag{16}$$

Here, with $W_n = \Phi e_3$ (Φ is called the bending potential), we get:

$$\boldsymbol{R}_n = \frac{\partial\Phi}{\partial y}\boldsymbol{e}_1 - \frac{\partial\Phi}{\partial x}\boldsymbol{e}_2 \tag{17}$$

So that,

$$\int_1^2 \boldsymbol{R}_n \cdot \boldsymbol{n}\mathrm{d}\Gamma = \Phi(x_2,y_2) - \Phi(x_1,y_1) \tag{18}$$

discretizing the plate boundary into N elements yieds

$$k(S)w(S) = \sum_{m=1}^{N}\int_{\Gamma_m}\boldsymbol{R}_n \cdot \boldsymbol{n}\mathrm{d}\Gamma = \sum_{m=1}^{N}\{\Phi(E_2) - \Phi(E_1)\}^m \tag{19}$$

where E_1 and E_2 are the starting and ending points at the m-th curved element on the plate boundary.

From Eq.(19), the singularities and corner tensor can be determined by using the appropriate boundary element model chosen to satisfy the governing equation.

IMPLEMENTATION OF THE BOUNDARY CONTOUR METHOD

Shape functions as perfect cubic polynomial for deflection w are chosen for each element such that

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 \tag{20}$$

Consider a Cartesian system of co-ordinates with axes x and y lying on the middle surface of the flat plate and axis z perpendicular to that plane along the direction of the deflection w.

From Eq.(2), Eq.(3), Eq.(4) and Eq.(5), we get the expressions of the rotations, bending moments and shear forces as follows:

$$\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{bmatrix}0 & 1 & 0 & 2x & y & 0 & 3x^2 & 2xy & y^2 & 0\\ 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 & 2xy & 3y^2\end{bmatrix}\{\alpha\} \tag{21}$$

$$\begin{Bmatrix}M_{11}\\ M_{12}\\ M_{22}\end{Bmatrix} = D\cdot \begin{bmatrix}0 & 0 & 0 & 2 & 0 & 2\nu & 6x & 2y & 2\nu x & 6\nu y\\ 0 & 0 & 0 & 0 & 1-\nu & 0 & 0 & 2(1-\nu)x & 2(1-\nu)y & 0\\ 0 & 0 & 0 & 2\nu & 0 & 2 & 6\nu x & 2\nu y & 2x & 6y\end{bmatrix}\{\alpha\} \tag{22}$$

$$\begin{Bmatrix}V_1\\ V_2\end{Bmatrix} = D\cdot \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 4-2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4-2\nu & 0 & 6\end{bmatrix}\{\alpha\} \tag{23}$$

where $\{\alpha\} = \{\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \ \alpha_5 \ \alpha_6 \ \alpha_7 \ \alpha_8 \ \alpha_9 \ \alpha_{10}\}^T$

Cubic boundary element is selected to establish the relation of the boundary variables and $\{\alpha\}$.

At the m-th curved element on the plate boundary, the variables are employed are as follows:

$$\{L\}^m = \{w^{3m-2} \ \theta_i^{3m-2} \ M_i^{2m-1} \ w^{3m-1} \ \theta_i^{3m-1} \ V_i^m \ M_i^{2m} \ w^{3m} \ \theta_i^{3m} \ w^{3m+1}\}^T \tag{24}$$

So, it is obviously that

$$\{L\}^m = [T(x,y)]_m \{\alpha\}_m \tag{25}$$

The elements of the matrix $[T(x,y)]_m$ depend on the nodal coordinates and material elastic constant. The variables in $\{L\}^m$ are chosen such that the matrix $[T(x,y)]_m$ is invertible and,

$$\{\alpha\}_m = [T(x,y)]_m^{-1}\{L\}^m \tag{26}$$

For convenience, a new co-ordinate system (ξ, η), centered at a source point E_1 on a boundary element, is introduced. The ξ and η axes are parallel to the global x and y axes. The deflection shape function remains in the same form as in Eq.(20), but with new constants $\tilde{\alpha}_i$ $(i=1-10)$ instead of α_i $(i=1-10)$ and ξ, η, instead of x, y, and

$$\{\tilde{\alpha}\}_m = [B]_{m(10\times 10)}\{\alpha\}_m \tag{27}$$

The elements in transformation matrix $[B]_m$ are determined according to the new coordinate system.

The bending potential can be expressed as a linear combination of ten linearly independent potential functions Φ_i $(i=1-10)$.

$$\Phi = [\Phi_i]\{\tilde{\alpha}\} \tag{28}$$

Substituting Eq.(28) into Eq.(19), one has

$$k(S)w(S) = \sum_{m=1}^{N}\sum_{i=1}^{10}[\Phi_i(E_2) - \Phi_i(E_1)]_m[B]_m[T]_m^{-1}\{L\}^m \tag{29}$$

6N algebraic equations are established after Eq.(29) is discretized. In general, the number of unknown variables on the boundary are less than the number of the equations, so the system of equations can be over determined.

Finally, after the usual switching of columns, the result is

$$[F]\{x\} = \{y\} \tag{30}$$

where $\{x\}$ contains the unknown boundary quantities, and $\{y\}$ is known in terms of prescribed boundary quantities and geometrical and material data of the problem.

CALCULATION OF INTERNAL PHYSICAL VARIABLES

For the bending moment M_{ij}, from the governing Eq. (1) and Huang *et al*. (1987), one has:

$$\nabla^2 M_{ij} = g(x,y) \tag{31}$$

Applying Green's second identity

$$\int_\Omega [\Phi\Psi_{,ii} - \Psi\Phi_{,ii}]\mathrm{d}\Omega = \int_\Gamma \left[\Phi\frac{\partial\Psi}{\partial n} - \Psi\frac{\partial\Phi}{\partial n}\right]\mathrm{d}\Gamma \tag{32}$$

In the domain of the plate, $k(S)$ is free. Substituting the fundamental solution $P(r)$ for the Laplacian, the integral equation for Eq.(31) is formulated as follows provided the lateral distributed load is absent:

$$M_{ij}(S) = \int_{\Gamma}\left\{M_{ij}(Q)\frac{\partial P(S,Q)}{\partial n} - \frac{\partial M_{ij}(Q)}{\partial n}P(S,Q)\right\}\mathrm{d}\Gamma \tag{33}$$

where

$$P(r) = \frac{\ln r}{2\pi} \tag{34}$$

The boundary bending moments which match Eq.(29) can be used to compute the internal moment using Eq.(33).

The shear force and rotations inside the plate domain can be evaluated by differentiating Eq.(33) and Eq.(8) to the source point, where $k(S)$ is free.

The plate potential functions Φ_i can be determined using the same implementation procedure of Nagarajan *et al*. (1994) and are presented in the Appendix.

NUMERICAL RESULT

Some selected examples are shown using the boundary contour method based on the present formulation to show the accuracy of boundary element method presented in this article.

Example 1

Consider the simply supported circular plate in Fig.1.

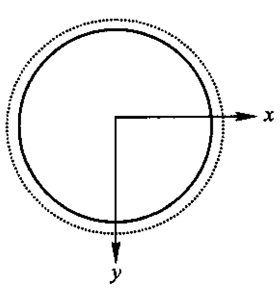

Fig.1 Simply supported circular plate

The radius is $a = 0.1$m, the thickness is 0.0015 m, the elastic constants $\nu = 0.3$ and $E = 210$GPa. The comparison of the present and analytical solutions are shown in Table 1. The plate circular boundary is discretized into six equivalent elements. The distributed load is p.

Table 1 Comparison of the present and analytical solutions

Coefficients	$\kappa_w = \frac{wEh^3}{16pa^4}$	$\kappa_m = \frac{M}{4pa^2}$	$\kappa_\nu = \frac{V}{2pa}$
Coordinates	(0,0)	(0,0)	(a,0)
Analytical solution	0.0434766	0.0515625	0.25
Present solution	0.0434767	0.0515626	0.2500001

Example 2

Simply supported square plate with a center square hole. Each lateral length is a and the lateral side of the square hole is a half, as shown in Fig.2. $a = 0.2$m, the thickness $h = 0.15$m, the elastic constants are $E = 200$GPa, $\nu = 0.25$.

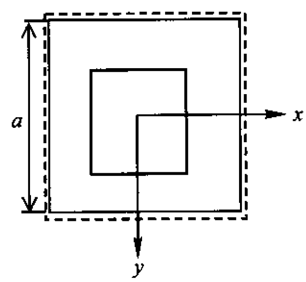

Fig.2 Simply supported square plate with a square hole

Because of symmetry, only a quarter of the plate is considered, and the evaluated boundary is discretized into 36 elements. Comparison of the present solution with Tottenham (1979) and Du (1986) is shown in Table 2. The distributed load is p.

Table 2 Comparison of the present solution with Tottenham (1979) and Du (1986)

	Du (1986) κ_w	Tottenham (1979) κ_w direct BEM	Tottenham (1979) κ_w indirect BEM	Present solution κ_w
Coordinate $(\frac{a}{4}, \frac{a}{4})$	0.2173	0.2188	0.2188	0.2187987
Coordinate $(\frac{a}{4}, 0)$	0.3015	0.3017	0.3141	0.3016935
Coordinate $(\frac{3a}{8}, 0)$	0.1541	0.1558	0.1565	0.1557968

Example 3

Consider three lateral simply supported and one side free square plate with lateral distributed load q, as shown in Fig.3. The Poisson's ratio $\nu = 0.3$. Twenty-four discretized elements along the boundary are employed to solve the problem. Comparison of the present solution with Du (1989) and Timoshenko *et al*. (1959) is shown in Table 3.

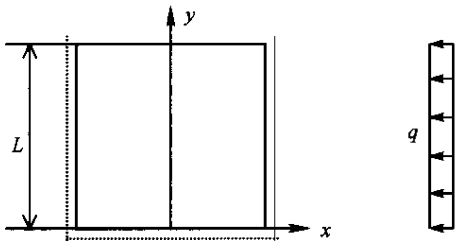

Fig.3 Three lateral simply supported and one side free square plate with lateral distributed load

Table 3 Comparison of the present solution with Du (1989) and Timoshenko *et al*. (1959)

	Coefficients	Du (1989)	Timoshenko (1959)	Present paper
Coordinate(0, L)	κ_w	1.2857	1.286	1.285893
	κ_{mx}	0.115	0.112	0.111986
Coordinate(0, $\frac{L}{2}$)	κ_w	0.7932	--	0.793306
	κ_{mx}	0.07986	0.080	0.079879
	κ_{my}	0.03897	0.039	0.038998

Where $w = \frac{qL^4 \kappa_w}{100D}$ and $M = qL^2 \kappa_m$

CONCLUSIONS

In the present paper, the boundary contour method is extended to cover the plate bending boundary integral equation. Elastic thin plate can be calculated with the boundary contour method based on Kirchhoff's hypothesis. A certain cubic deflection polynomial is employed to establish the implementation. This method requires simply numerical evaluations of plate bending potential functions at the discretized points on the boundary elements. Internal physical variables can be calculated with this method. Numerical results presented in the article show its excellent accuracy.

References

Altirero, N.J. and Sikarskie, D.L., 1978. A boundary integral method applied to plate of arbitrary plane form. *Computer Structure*, **9**: 283 – 305.

Chen, S.Y., Tang, W.X., Zhou, S.J., Leng, X. and Zheng, W., 1999. Evaluations of J integrals and stress intensity factors in a 2D quadratic boundary contour method. *Commum. Numer. Meth. Engng*. **15**(2): 91 – 100.

Du, Q.H. and Lu, X.L., 1986. Boundary element method for aerospace plate. *Chinese Acta of Aerospace*, **7**: 6.

Du, Q. H., 1989. Boundary Integral Equation Method-boundary Element Method. High Education Publishing House, Bejing, p.398.

Huang, K.Z., Xia, Z.X., Xue, M.D. and Ren, W.M., 1987. Theory for Plates and Shells. Qinghua Unversity Press, Beijing.

Jaswon, M.A. and Maiti, M., 1968. An integral equation formulation of plate bending problems. *J Engng Math*, **2**: 83 – 93.

Lutz, E.D., 1991. Numerical Methods for Hypersingular and Near Singular Boundary Integrals in Fracture Mechanics. Ph D. Dissertation, Cornell University, Ithaca, NY.

Lutz, E.D., Back, A., Gray, L. and Ingraffea, A., 1992. Systematic Conversion of Boundary Integral Surface Integrals to Contour Integral. Technical Report, April. Structure Engineering, School of Civil and Environmental Engineering, Cornell University, Ithaca, NY.

Maucher, R. and Hartmann, F., 1999. Corner singularities of Kirchhoff plates and the boundary element method. *Comput. Methods Appl. Engng*., **173**: 273 – 285.

Mukherjee, S., 1982. Boundary Element Methods in Creep and Fracture. Elsevier Applied Science, London.

Mukherjee, Y.X., Mukherjee, S. and Shi, X., 1997. The boundary contour method for three-dimensional linear elasticity with a new quadratic boundary element. *Engineering Analysis with Boundary Elements*, **20**: 35 – 44.

Nagarajan, A., Lutz, E. D. and Mukherjee, S., 1994. A novel boundary element method for linear elasticity with no numerical integration for 2-D and line integrals for 3-D problems. *ASME J. Appl. Mech*., **61**: 264 – 269.

Nagarajan, A., Mukherjee, S. and Lutz, E.D., 1996. The boundary contour method for three dimensional linear elasticity. *ASME J. Appl. Mech*., **63**: 278 – 286.

Nardini, D. and Brebbia, C.A., 1982. A New Approach to Free Vibration Analysis using a Boundary Element Method. *In*: Brebbia C.A., editor. Boundary Element Method in Engineering.

Phan, A.V., Mukherjee, S. and Mayer, J.R.R., 1997. The boundary contour method for two-dimensional linear elasticity with quadratic boundary element. *Computational Mechanics*, **20**: 310 – 319.

Tanaka, M., Sladek, V. and Sladek, J., 1994. Regularization techniques applied to boundary element methods. *Applied Mechanics Reviews*, **47**(10): 457 – 499.

Timoshenko, S. and Wolnowsky-Krieger, S., 1959. Theory of Plate and Shells. McGraw-Hill, New York.

Toh, K.C. and Mukherjee, S., 1994. Hypersingular and finite part integrals in the boundary element method. *International Journal of Solids and Structures*, **31**(17):

2229 – 2312.

Tottenham, H., 1979. The Boundary Element Method for Plates and Shells. *In*: Banerjee PK, Butterfield R, editors. Development in Boundary Element Methods-I, London, Applied Science.

Wen, P.H., Aliabadi, M.H. and Rooke, D.P., 1998. A new method for transformation of domain integrals to boundary integrals in boundary element method. *Commun Numer Mech Engng*, **14**, 1055 – 1065.

Zhou, S.J., Sun, S.X. and Cao, Z.Y., 1998. The dual boundary contour method for two-dimensional crack problems. *International Journal of Fracture*, **92**: 201 – 212.

Zhou, S. J., Sun, S. X. and Cao, Z. Y., 1999a. The boundary contour method based on the equivalent boundary integral equation for 2-D linear elasticity. *Commum. Numer. Meth. Engng*, **15**: 811 – 821.

Zhou, S.J., Cao, Z.Y. and Sun, S.X., 1999b. The traction boundary contour method for linear elasticity. *Int. J. Numer. Meth. Engng*, **46**: 1883 – 1895.

APPENDIX

Plate bending potentials

$$\Phi_1(\xi,\eta) = \frac{1}{2\pi}\mathrm{arctg}\,\frac{\eta}{\xi}$$

$$\Phi_2(\xi,\eta) = \frac{\eta}{4\pi}\left[\frac{1-\nu}{2} - (1+\nu)\ln r\right]$$

$$\Phi_3(\xi,\eta) = -\Phi_2(\xi,\eta)$$

$$\Phi_4(\xi,\eta) = \frac{\nu(\xi^2-\eta^2)}{8\pi}[1+2\ln r]$$

$$\Phi_5(\xi,\eta) = -\frac{\nu\xi\eta}{4\pi}[1+2\ln r]$$

$$\Phi_6(\xi,\eta) = -\Phi_5(\xi,\eta)$$

$$\Phi_7(\xi,\eta) = -\frac{\eta}{8\pi}\left[(2\eta^2+6\nu\xi^2)\ln r + (3\nu-1)\xi^2 - \frac{2}{3}\eta^2\right]$$

$$\Phi_8(\xi,\eta) = -\Phi_7(\xi,\eta)$$

$$\Phi_9(\xi,\eta) = \frac{\xi}{8\pi}\left[(2(1-2\nu)\eta^2+2\nu\xi^2)\ln r + \frac{1}{3}(3\nu-1)\xi^2 - 2\nu\eta^2\right]$$

$$\Phi_{10}(\xi,\eta) = -\Phi_9(\xi,\eta)$$

where ν is the Poisson's ratio.