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Image segmentation based on Mumford-Shah functional

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Abstract: In this paper, the authors propose a new model for active contours segmentation in a given image, based on Mumford-Shah functional (Mumford and Shah, 1989). The model is composed of a system of differential and integral equations. By the experimental results we can keep the advantages of Chan and Vese's model (Chan and Vese, 2001) and avoid the regularization for Dirac function. More importantly, in theory we prove that the system has a unique viscosity solution.

Key words: Image segmentational, Mumford-Shah functional, Viscosity solution, Level set method
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INTRODUCTION

In the fields of Computer Vision and Image Processing, the theory of PDE has been applied very successfully to image segmentation and image smoothing (see Ambrosio *et al.*, 1992; March, 1992; Weickert, 1998; Bourdin, 1999; IEEE Trans. on Image Processing, 1998). In the classical snakes and active contour models, as pointed out in the Chan and Vese's work (Chan and Vese, 2001), to stop the evolving curve on the boundary of the desired object, an edge-detector is used, depending on the gradient of the initial image u_0. For the problems of curve evolution, the level set method is also widely used. The present model considered is based on a level set formulation, in which the boundary of the objects segmented with our model is modelled as the zero set of a smooth function ϕ defined on the entire domain. The boundary is then updated by solving a nonlinear equation in the whole domain. This level set formulation of the moving interface was introduced by Osher and Sethian (1988) and was capable of computing geometric properties of highly complicated boundaries without explicitly tracking the interface. Hence the moving boundary can develop corners, cusps, and undergo topological changes quite naturally. Various results using this level set method have been published, in particular by Malladi *et al.* (1995), Chen *et al.* (2000), Chang *et al.* (1996), Sussman *et al.* (1994) and Whitaker (2000). Moreover, the level set formulation generalizes easily to three-dimensional problems.

Anyway, there are two kinds of PDE models: one depends on the gradient of the initial image u_0, or its approximation $\nabla G_\sigma(x) * u_0(x)$ (Alvarez *et al.*, 1992; Barcelos and Chen, 2000), and the other is independent of them, such as Chan and Vese's model (Chan and Vese, 2001). Many of them have a common point: their corresponding PDE's are Euler equations of some specified functional. In this paper for segmentation we begin to work from the famous Mumford-Shah functional, i.e., the functional with area constraint as follows (see Chan and Vese, 2001 or Mumford and Shah, 1989):

$$F^{MS}(u, C) = \mu Length(C) + \gamma\, Area(C) + \lambda\int_\Omega \left|u_0(x,y) - u(x,y)\right|^2 \mathrm{d}x\mathrm{d}y + \int_{\Omega\setminus C}\left|\nabla u(x,y)\right|^2 \mathrm{d}x\mathrm{d}y \quad (1)$$

where $u_0: \Omega = [0,1]^2 \to R$ is a given image, μ, γ and λ are positive parameters, and the *Length*(C) and *Area*(C) represent the length of the curve C and the area of the region inside C respectively. The solution $u(x,y)$, the minimizer of this functional is formed by finite smooth regions with boundary C. Some applications of the Mumford and Shah functional can be found in chambolle(1995, 1999), Chambolle and Maso (1999), and Bourdin and Chambolle(2000).

Using the level set method and Heaviside function H(Chan and Vese, 2001), we have

$$Length(C) = Length\{\phi = 0\} = \int_\Omega \left|\nabla H(\phi(x,y))\right| dxdy = \int_\Omega \delta_0(\phi(x,y))\left|\nabla\phi(x,y)\right| dxdy \quad (2)$$

$$Area(C) = Area\{\phi \geqslant 0\} = \int_\Omega H(\phi(x,y))dxdy \quad (3)$$

where $\{\phi = 0\}$ is the level set of the curve C and $\delta_0(x)$ is the Dirac function centered at zero point.

If we assume for all $(x,y)\in\Omega$

$$u(x,y) = c_1 H(\phi(x,y)) + c_2(1 - H(\phi(x,y))), \quad (4)$$

then the Mumford-Shah functional becomes the following form:

$$F(c_1,c_2,\phi) = \mu\int_\Omega \delta_0(\phi(x,y))\left|\nabla\phi(x,y)\right| dxdy + \gamma\int_\Omega H(\phi(x,y))dxdy + \lambda_1\int_\Omega \left|u_0(x,y) - c_1\right|^2 H(\phi(x,y))dxdy + \lambda_2\int_\Omega \left|u_0(x,y) - c_2\right|^2 (1 - H(\phi(x,y)))dxdy \quad (5)$$

where we use different constants λ_1 and λ_2 instead of λ.

The Euler equations of this functional are as follows:

$$\delta_0(\phi)\left[\mu div\left(\frac{\nabla\phi(x,y)}{\left|\nabla\phi(x,y)\right|}\right) - \gamma - \lambda_1(u_0(x,y) - c_1)^2 + \lambda_2(u_0(x,y) - c_2)^2\right] = 0 \quad (6)$$

$$c_1(\phi) = \frac{\int_\Omega u_0(x,y)H(\phi(x,y))dxdy}{\int_\Omega H(\phi(x,y))dxdy} \quad (7)$$

$$c_2(\phi) = \frac{\int_\Omega u_0(x,y)(1 - H(\phi(x,y)))dxdy}{\int_\Omega (1 - H(\phi(x,y)))dxdy} \quad (8)$$

The above system can be used to detect the objects in a given image u_0, but at first, one has to regularize the Dirac function δ_0, just as done in Chan and Vese's work(Chan and Vese, 2001). In the next section we modify this system, i.e., by $|\nabla\phi(x,y)|$ instead of δ_0; hence the steady solutions of them are same, and furthermore, we consider a coupled system consisting of an evolution equation, Eqs.(7) and (8). Then we regularize it and prove that the regularized system has at least one classical solution satisfying some prior estimates. In Section 3, we prove this system has a unique viscosity solution. Some experimental results are presented in last section to show that our model keeps the advantages of Chan and Vese's model(Chan and Vese, 2001), the locations of boundaries are very well detected and preserved.

NEW MODEL AND REGULARIZATION PROBLEM

As mentioned in Section 1, for the fixed $T > 0$, our new model called problem (NP) is described as follows:

$$\frac{\partial\phi}{\partial t} = \left|\nabla\phi\right|\left[\mu \mathrm{div}\left(\frac{\nabla\phi}{\left|\nabla\phi\right|}\right) - \gamma - \lambda_1(u_0(x,y) - c_1(t))^2 + \lambda_2(u_0(x,y) - c_2(t))^2\right],$$
$$\text{in } Q_T = R^2 \times (0,T] \quad (9)$$

$$c_1(t) = \frac{\int_\Omega u_0(x,y)H(\phi(x,y,t))dxdy}{\int_\Omega H(\phi(x,y,t))dxdy} \quad (10)$$

$$c_2(t) = \frac{\int_\Omega u_0(x,y)(1 - H(\phi(x,y,t)))dxdy}{\int_\Omega (1 - H(\phi(x,y,t)))dxdy} \quad (11)$$

$$\phi(x,y,0) = \phi_0(x,y) \quad (12)$$

where the unknown function $\phi(x,y,t)$ and its initial value $\phi_0(x,y)$ and the given data $u_0(x,y)$ are 2-period for x and y respectively as usually done in Guichard and Morel's paper(Guichard and Morel, 1998). Let $C\subset\Omega$ be a closed Jordan curve in R^2, the zero level set of ϕ_0 with

$\phi_0 > 0$ in the inside of C and $\phi_0 < 0$ in the outside of C, t is the artificial time or scale parameter.

We assume that

(A1) $u_0 \geqslant 0$ is Lipschitz continuous, i.e., $u_0 \in C(R^2) \cap W^{1,\infty}(R^2)$.

(A2) ϕ_0 is smooth, i.e., $\phi_0 \in C^\infty(R^2)$.

Remark 1 The assumption for $u_0 \geqslant 0$ is natural and we can choose ϕ_0 to satisfy its assumption.

Remark 2 The principal part $div\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$ in Eq.(9) just is the mean curvature of level set of $\phi = 0$.

Remark 3 Let $p = (p_1, p_2, \cdots, p_n) \in R^n$ and $a_{ij} = \delta_{ij} - \frac{p_i p_j}{|p|^2}$; then we know the matrix $(a_{ij}(p))$ is semi-positive.

For the problem (NP), we first regularize it as follows:

$$\frac{\partial \phi^\varepsilon}{\partial t} - \mu a_{ij}^\varepsilon(\nabla\phi^\varepsilon)\phi_{ij}^\varepsilon - \alpha(u_0^\varepsilon, c_1^\varepsilon, c_2^\varepsilon)h_\varepsilon \cdot (\nabla\phi^\varepsilon) = 0, \tag{13}$$

$$\phi^\varepsilon(x, y, 0) = \phi_0(x, y), \tag{14}$$

$$c_1^\varepsilon = \frac{\int_\Omega u_0^\varepsilon(x, y)H_\varepsilon(\phi^\varepsilon(x, y, t))\mathrm{d}x\mathrm{d}y}{\int_\Omega H_\varepsilon(\phi^\varepsilon(x, y, t))\mathrm{d}x\mathrm{d}y + \varepsilon}, \tag{15}$$

$$c_2^\varepsilon = \frac{\int_\Omega u_0^\varepsilon(x, y)(1 - H_\varepsilon(\phi^\varepsilon(x, y, t)))\mathrm{d}x\mathrm{d}y}{\int_\Omega (1 - H_\varepsilon(\phi^\varepsilon(x, y, t)))\mathrm{d}x\mathrm{d}y + \varepsilon}, \tag{16}$$

where

$$a_{ij}^\varepsilon(p) = \delta_{ij} - \frac{p_i p_j}{h_\varepsilon^2(p)}, \tag{17}$$

$$h_\varepsilon(p) = \sqrt{|p|^2 + \varepsilon}, \tag{18}$$

$$\alpha(u_0^\varepsilon, c_1^\varepsilon, c_2^\varepsilon) = -[\gamma + \lambda_1(u_0^\varepsilon - c_1^\varepsilon)^2 - \lambda_2(u_0^\varepsilon - c_2^\varepsilon)^2], \tag{19}$$

$$H_\varepsilon(z) = \begin{cases} 1 & \text{if } \varepsilon \leqslant z \\ smooth, & \text{if } 0 \leqslant z \leqslant \varepsilon \\ 0, & \text{if } z \leqslant 0 \end{cases} \tag{20}$$

with $H_\varepsilon \in C^\infty(R)$ and $0 \leqslant H_\varepsilon' \leqslant \frac{2}{\varepsilon}$, and $0 \leqslant u_0^\varepsilon \in C^\infty(R^2)$ with

$$\lim_{\varepsilon\to 0} u_0^\varepsilon = u_0, \quad \text{in } C(R^2) \cap W^{1,\infty}(R^2). \tag{21}$$

We call the regularized problem (13) – (16) as problem (RP).

Next we will use Schauder fixed point theorem to prove that there exists at least one classical solution to problem (RP). Indeed, we choose a closed and convex subset E in Banach space $C([0, T] \times [0, T])$, that is

$$E = \{c(t) = (c_1(t), c_2(t)) \in C([0, T] \times [0, T]) : 0 \leqslant c_1, c_2 \leqslant \max u_0\}, \tag{22}$$

with the norm $\| c \| = \max_{[0,T]}(c_1^2(t) + c_2^2(t))^{1/2}$. For given $c = (c_1, c_2) \in E$ instead of $(c_1^\varepsilon, c_2^\varepsilon)$ in Eq.(13), there exists a unique classical solution ϕ^ε to the problem (13) and (14) (Ladyzhenskaya *et al.*, 1968). Moreover, we have

Lemma 1 The following uniform estimates of ϕ^ε with respect to ε hold:

$$\min_{R^2}\phi_0 \leqslant \phi^\varepsilon \leqslant \max_{R^2}\phi_0, \tag{23}$$

and

$$\| \nabla\phi^\varepsilon(\cdot, t) \|_{L^\infty(R^2)} \leqslant \exp(KT) \| \nabla\phi_0 \|_{L^\infty(R^2)} \leqslant K_T. \tag{24}$$

where the positive constants K and K_T depend only on given data u_0 and T.

Proof This lemma can be proved by maximum principle (Ladyzhenskaya *et al.*, 1968) and similar method can be found in Chen's paper (Chen *et al.*, 2000), we omit this proof here.

For all (x, y, t), $(x', y', t') \in R^2 \times [0, T]$ and $0 < \eta < 1/4$, we also have

Lemma 2 The following relationships hold:

$$\left|\phi^\varepsilon(x, y, t) - \phi^\varepsilon(x', y', t')\right| \leqslant K\left(\left|x - x'\right| + \left|y - y'\right| + \left|t - t'\right|^{1/4}\right), \tag{25}$$

$$\lim_{\varepsilon\to 0}\phi^\varepsilon = \phi, \text{ in } C^{1-\eta, 1/4-\eta}(R^2 \times [0, T]),$$
$$\phi \in C(R^2 \times [0, T]) \cap L^\infty(0, T; W^{1,\infty}(R^2)), \tag{26}$$

where Eq.(26) holds for some subsequence of ε, which we denote by ε itself here.

Proof From Eq.(13) and by the part of integral we have

$$\int_{Q_T} \frac{1}{h_\varepsilon(\nabla \phi^\varepsilon)}\left(\frac{\partial \phi^\varepsilon}{\partial t}\right)^2 = \int_{Q_T} \alpha(u_0^\varepsilon, c_1^\varepsilon, c_2^\varepsilon)\frac{\partial \phi^\varepsilon}{\partial t} - \mu\int_{Q_T} \frac{\partial h_\varepsilon(\nabla \phi^\varepsilon)}{\partial t}.$$

Hence, using Eq.(24) and noting that the function α is bounded, we get

$$\left\| \frac{\partial \phi^\varepsilon}{\partial t} \right\|_{L^2(Q_T)} \leqslant K.$$

By the above estimate, Eq.(24) and Sobolev imbedding theorem, for all (x, y, t), $(x', y', t') \in R^2 \times [0, T]$ we obtain the following estimation:

$$|\phi^\varepsilon(x, y, t) - \phi^\varepsilon(x, y, t')| \leqslant K|t - t'|^{1/4}.$$

Hence Eq.(25) holds, and by the Ascoli-Arzela Theorem, we get Eq.(26).

Using this ϕ^ε to the right hands of Eqs.(15) and (16), we obtain a new $c = (c_1, c_2)$, in this way, we can define an operator

$$F: E \to C([0,T] \times [0,T]), \ i.e., \ \bar{c} = F(c). \tag{27}$$

Obviously, the fixed points of F are the solutions to problem (RP). In order to prove that F has a fixed point, we need

Lemma 3 $F(E) \subset E$ and F is compact.

Proof It is clear that $F(E) \subset E$ by assumption (A1) and the definitions of Eqs.(15) and (16). The compactness of F can be obtained by Lemma 2.

Lemma 4 F is continuous.

Proof For simplicity written, we omit the superscripts (or subscripts) ε. Let c and d belong to E and corresponding to them solutions of problem (15)–(16) are ϕ_c and ϕ_d. Setting $\phi = \phi_c - \phi_d$, we get the following equation

$$\frac{\partial \phi}{\partial t} - \mu a_{ij}(\nabla \phi)\phi_{ij} - G_1 = G_2, \tag{28}$$

where

$$G_1 = \mu(a_{ij}(\nabla \phi_c) - a_{ij}(\nabla \phi_d))\phi_{d,ij} + \alpha(u_0, c_1, c_2)(h(\nabla \phi_c) - h(\nabla \phi_d)), \tag{29}$$

$$G_2 = -[\lambda_1(u_0 - c_1)^2 - \lambda_1(u_0 - d_1)^2 - \lambda_2(u_0 - c_2)^2 + \lambda_2(u_0 - d_2)^2]h(\nabla \phi_d). \tag{30}$$

From Eqs.(29), (30) and the Maximum Principle, we get

$$\| \phi \|_{L^\infty(R^2)} \leqslant K_\varepsilon \| c - d \|, \tag{31}$$

where the positive constant K_ε depends only on ε and the given data u_0.

On the other hand, by Eqs.(15) and (16), we can easily obtain

$$\| c - d \| \leqslant K_\varepsilon \| \phi \|_{L^\infty(R^2)}. \tag{32}$$

Then by Eqs.(31) and (32), we have proved this lemma.

By means of the above lemmas and Schauder fixed point theorem, we have

Theorem 1 The problem (RP) has at least one classical solution.

Remark 4 In fact, similar to the proof of continuity of F, we can prove the uniqueness of solutions to problem (RP).

PROBLEM (NP) AND MAIN THEOREM

We use the notations with a little modification for c_1 and c_2 in Section 2, and the problem (NP) is described as follows:

$$\frac{\partial \phi}{\partial t} - \mu a_{ij}(\nabla \phi)\phi_{ij} - \alpha(u_0, c_1(\phi), c_2(\phi)) \cdot \left|\nabla \phi\right| = 0. \tag{33}$$

$$\phi(x, y, 0) = \phi_0(x, y). \tag{34}$$

$$c_1(\phi) = \frac{\int_\Omega u_0(x, y)H(\phi(x, y, t))\mathrm{d}x\mathrm{d}y}{\int_\Omega H(\phi(x, y, t))\mathrm{d}x\mathrm{d}y} \tag{35}$$

$$c_2(\phi) = \frac{\int_\Omega u_0(x, y)(1 - H(\phi(x, y, t)))\mathrm{d}x\mathrm{d}y}{\int_\Omega (1 - H(\phi(x, y, t)))\mathrm{d}x\mathrm{d}y} \tag{36}$$

We begin by a brief recall of the definition of viscosity solutions to problem (NP) periodic on Ω.

Definition 1 Let $\phi \in C(R^2 \times [0, T]) \cap W^{1,\infty}(R^2 \times [0, T])$ be a viscosity subsolution (supersolution) of problem (NP) if for all $\xi \in C^2(R^2 \times R)$, $(\xi - \phi)$ attains a local maximum at $(x_0, t_0) \in \overline{Q}_T = R^2 \times [0, T]$; then the following conditions hold

(a) $\frac{\partial \xi}{\partial t} - \mu a_{ij}(\nabla \xi)\xi_{ij} - \alpha(u_0, c_1(\phi), c_2(\phi))\left|\nabla \xi\right| \leqslant (\geqslant) 0$, if $\left|\nabla \xi\right| \neq 0$,

(b) $\frac{\partial \xi}{\partial t} - \mu \limsup_{|\nabla \xi| \to 0} (\liminf_{|\nabla \xi| \to 0}) a_{ij}(\nabla \xi)\xi_{ij} \leqslant (\geqslant) 0$, if $\left|\nabla \xi\right| = 0$.

Definition 2 ϕ is a viscosity solution of problem (NP) if it is both a viscosity subsolution and a viscosity supersolution.

We have the following theorem:

Theorem 2 There exists a unique viscosity solution to Problem (NP).

Proof Let ϕ be the limit of $\{\phi^{\varepsilon}\}$ in Section 2 and $\xi \in C^2(R^2 \times R)$, $(\xi - \phi)$ attains a strict local maximum at $(x_0, t_0) \in \bar{Q}_T = R^2 \times [0, T]$; by Lemma 2, $(\phi^{\varepsilon} - \xi)$ has a local maximum at point $(x_{\varepsilon}, t_{\varepsilon})$ with

$$(x_{\varepsilon}, t_{\varepsilon}) \to (x_0, t_0), \quad as\ \varepsilon \to 0. \tag{37}$$

Then at $(x_{\varepsilon}, t_{\varepsilon})$, we have

$$\nabla \phi^{\varepsilon} = \nabla \xi, \quad \phi_t^{\varepsilon} \geqslant \xi_t,$$

and $\alpha_{ij}^{\varepsilon}(\nabla \phi^{\varepsilon})\phi_{ij}^{\varepsilon} \leqslant a_{ij}^{\varepsilon}(\nabla \xi)\xi_{ij}$. (38)

Hence,

$$\xi_t - \mu\alpha_{ij}^{\varepsilon}(\nabla \xi)\xi_{ij} - \alpha(u_0, c_1(\phi^{\varepsilon}), c_2(\phi^{\varepsilon}))h_{\varepsilon}(\nabla \xi) \leqslant \phi_t^{\varepsilon} - \mu\alpha_{ij}^{\varepsilon}(\nabla \phi^{\varepsilon})\phi_{ij}^{\varepsilon} - \alpha(u_0, c_1(\phi^{\varepsilon}), c_2(\phi^{\varepsilon}))h_{\varepsilon}(\nabla \phi^{\varepsilon}) = 0 \tag{39}$$

Let $\varepsilon \to 0$ on the left hand of (39) and by the uniform convergence of ϕ^{ε} in (26), we conclude ϕ is a subsolution of problem (NP). Similarly, it also is a supersolution. Hence, existence of solution of problem (NP) is proved. The uniqueness can be proved by similar method in Wang's paper (Chen *et al*., 2000) only noting that $\alpha(u_0, c_1(\phi), c_2(\phi))$ is bounded and the λ of Eq. (25) is chosen large enough.

Remark 5 Strictly speaking, the convergent interval of t depended on that the denominations in Eqs. (35) and (36) are not zero, but due to Lemma 2, this interval exists, here we omit it for simplicity.

EXPERIMENTAL RESULTS

For the problem (NP), we use the finite difference method to calculate ϕ. In a variety of contours in the images, our experiments showed that the treated images had good segmentation by our model. Moreover, we can treat any samples of the images such as segments and open curves, and the initial closed curve $C(\{\phi_0 = 0\})$ can be in a place anywhere which can contain the objects or not, even if far away from them. In the experiments, we rescaled the domain $\bar{\Omega}$ and fixed the parameters $\lambda_1 = \lambda_2 = \gamma = 1$ for simplicity of calculation and only modified μ.

There are four images in Fig. 1: (a) represents the original image, (b) and (c) are the processing images, and (d) is the resulting image, where $\mu = 0.001$ and $\phi_0 = 100 - \sqrt{(x-100)^2 + (y-100)^2}$. In Fig. 2, we treat the image with different grey levels: (a) represents the original one which contain a circle (grey

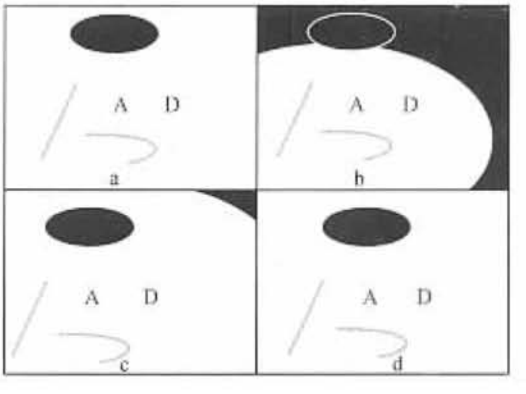

Fig. 1 The detection of multi-objects (image size 400 × 300). (a) represents the original image, (b) and (c) are the processing images; and (d) is the resulting image. $\mu = 0.001$ and $\phi_0 = 100 - \sqrt{(x-100)^2 + (y-100)^2}$

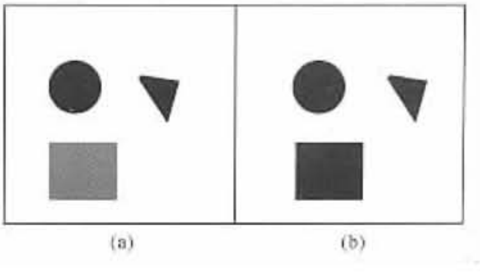

Fig. 2 The detection of bjects with different grey levels (image size 256 × 256). (a) Original with different grey levels, containing a circle (grey level is zero), a rectangle (grey level is 128); and (b) is the resulting image

level is zero), a rectangle(grey level is 128) and a quadrilateral(grey level is 50); (b) is the resulting image. These show that our model keeps the advantages of Chan and Vese's model (Chan and Vese, 2001) – the locations of boundaries are detected very well and preserved.

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