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Extreme value distributions of mixing two sequences with different MDA's

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Abstract: Suppose $\{X_i, i\geq 1\}$ and $\{Y_i, i\geq 1\}$ are two independent sequences with distribution functions $F_X(x)$ and $F_Y(x)$, respectively. $Z_{i,n}$ is the combination of X_i and Y_i with a probability p_n for each i with $1\leq i\leq n$. The extreme value distribution $G_Z(x)$ of this particular triangular array of the i.i.d. random variables $Z_{1,n}$, $Z_{2,n}$,…, $Z_{n,n}$ is discussed. We found a new form of the extreme value distribution $\Lambda^A(\rho x)\Lambda(x)(0<\rho<1)$, which is not max-stable. It occurs if $F_X(x)$ and $F_Y(x)$ belong to the same $MDA(\Lambda)$. $G_Z(x)$ does not exist as mixture forms of the different types of extreme value distributions.

Key words: Extreme value distribution, Maximum domain of attraction(MDA), Mixed distribution functions
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INTRODUCTION

Let $\{X_i, i \geq 1\}$ and $\{Y_i, i \geq 1\}$ be two independent sequences of independent and identically distributed random variables with distribution functions $F_X(x)\in MDA(G_X)$ and $F_Y(x)\in MDA(G_Y)$, respectively. We deal with the case when $\{Z_{i,n}, 1\leq i\leq n\}$ is a mixture of two independent sequences $\{X_i, i\geq 1\}$ and $\{Y_i, i\geq 1\}$, for $p_n\in[0, 1)$ which is defined by:

$$Z_{i,n}=\begin{cases}X_i & \text{with probability } p_n\\ Y_i & \text{with probability } 1-p_n.\end{cases}$$

We consider the extreme value distribution $G_Z(x)$ of $\{Z_{i,n}, 1\leq i\leq n\}$. The cases i) F_X and $F_Y\in MDA(\Phi_\alpha)$ and ii) F_X and $F_Y\in MDA(\Psi_\alpha)$ were dealt with in Jiang (2004). In this paper we continue to discuss the other cases:

i) F_X and $F_Y \in MDA(\Lambda)$;

ii) F_X and $F_Y \in MDA(\Phi_\alpha)$ or $MDA(\Lambda)$;

iii) F_X and $F_Y \in MDA(\Psi_\alpha)$ or $MDA(\Lambda)$.

$F_X(x)$ AND $F_Y(x)\in MDA(\Lambda)$

Firstly, we discuss the situation with $x_{F_X}=x_{F_Y}=\infty$, then we deal with the situation with

$$x_{F_X}=x_{F_Y}=x_F<\infty .$$

Theorem 1 Suppose F_X, $F_Y \in MDA(\Lambda)$ have the auxiliary functions $f_1(x)$ and $f_2(x)$ with $x_{F_X}=x_{F_Y}=\infty$. Let $np_n\overline{F}_X(\beta_{2,n})\to A\in[0,\infty]$ and

$$\lim_{x\to\infty}\frac{f_2(x)}{f_1(x)}=\rho\in[0,\infty].$$

i) If $0<\rho\leq 1$ and $A\in[0, \infty]$, then with the normalizing sequences $\alpha_n=\alpha_{2,n}=f_2(\beta_{2,n})$ and $\beta_n=\beta_{2,n}$

$$G_Z(x)=\Lambda^A(\rho x)\Lambda(x).$$

ii) Otherwise, with the normalizing sequences

$\alpha_n = \alpha_{2,n} = f_2(\beta_{2,n})$, $\beta_n = \beta_{2,n}$ or $\alpha_n = \alpha'_{1,n}$, $\beta_n = \beta'_{1,n}$

$$G_Z(x) = \Lambda(x).$$

Proof We consider the different cases as follows.

1) If $\rho>1$, by Lemma 1 in Jiang (2004) we have $\lim_{x\to\infty}\frac{\overline{F}_Y(x)}{\overline{F}_X(x)} = \infty$. By Proposition 1 in Jiang (2004), the claim follows.

2) If $0 < \rho \le 1$, we have

$$n\overline{F}_{Z,n}(\alpha_{2,n}x+\beta_{2,n}) \sim np_n\overline{F}_X(\alpha_{2,n}x+\beta_{2,n}) + \mathrm{e}^{-x}$$
$$= np_n\overline{F}_X(\beta_{2,n})\frac{\overline{F}_X(\alpha_{2,n}x+\beta_{2,n})}{\overline{F}_X(\beta_{2,n})} + \mathrm{e}^{-x}.$$

We derive $\lim_{n\to\infty}\frac{\overline{F}_X(\alpha_{2,n}x+\beta_{2,n})}{\overline{F}_X(\beta_{2,n})}$. In fact,

$$\frac{\overline{F}_X(\alpha_{2,n}x+\beta_{2,n})}{\overline{F}_X(\beta_{2,n})} \sim \exp\left\{-\int_{\beta_{2,n}}^{f_2(\beta_{2,n})x+\beta_{2,n}}\frac{1}{f_1(t)}\mathrm{d}t\right\}$$
$$= \exp\left\{-\int_0^x \frac{f_2(\beta_{2,n})}{f_1(f_2(\beta_{2,n})s+\beta_{2,n})}\mathrm{d}s\right\}$$
$$= \exp\left\{-\int_0^x \frac{f_2(\beta_{2,n})\ f_2(f_2(\beta_{2,n})s+\beta_{2,n})}{f_2(f_2(\beta_{2,n})s+\beta_{2,n})\ f_1(f_2(\beta_{2,n})s+\beta_{2,n})}\mathrm{d}s\right\}$$
$$\to \mathrm{e}^{-\rho x}, \tag{1}$$

by Lemma 1, 3 in Resnick (1987) and since $\lim_{x\to\infty}\frac{f_2(x)}{f_1(x)} = \rho \in (0,1]$. Hence,

$$n\overline{F}_{Z,n}(\alpha_{2,n}x+\beta_{2,n}) \sim np_n\overline{F}_X(\beta_{2,n})\mathrm{e}^{-\rho x} + \mathrm{e}^{-x}. \tag{2}$$

i) If $np_n\overline{F}_X(\beta_{2,n}) \to A \ge 0$, then with $\alpha_n = f_2(\beta_{2,n})$ and $\beta_n = \beta_{2,n}$

$$G_Z(x) = \Lambda^A(\rho x)\Lambda(x).$$

ii) If $np_n\overline{F}_X(\beta_{2,n}) \to \infty$, then also $np_n\to\infty$. We have $np_n\overline{F}_X(\beta'_{1,n}) \to 1$ and

$$n\overline{F}_{Z,n}(\alpha'_{1,n}x+\beta'_{1,n}) = np_n\overline{F}_X(\alpha'_{1,n}x+\beta'_{1,n})$$
$$+n(1-p_n)\overline{F}_Y(\alpha'_{1,n}x+\beta'_{1,n})$$
$$\sim \mathrm{e}^{-x} + n\overline{F}_Y(\alpha'_{1,n}x+\beta'_{1,n}). \tag{3}$$

If we show that $n\overline{F}_Y(\alpha'_{1,n}x+\beta'_{1,n}) \to 0$, then with $\alpha_n = \alpha'_{1,n} = f_1(\beta'_{1,n})$ and $\beta_n = \beta'_{1,n}$,

$$G_Z(x) = \Lambda(x).$$

In fact, we have

$$\lim_{x\to\infty}\frac{\overline{F}_X(\beta'_{1,n})}{\overline{F}_X(\beta_{2,n})} = \lim_{x\to\infty}\frac{np_n\overline{F}_X(\beta'_{1,n})}{np_n\overline{F}_X(\beta_{2,n})} = 0. \tag{4}$$

Hence, as the derivation in Eq.(1) for large n we have

$$0 \le n\overline{F}_Y(\alpha'_{1,n}x+\beta'_{1,n})$$
$$\sim \frac{\overline{F}_Y(\alpha'_{1,n}x+\beta'_{1,n})}{\overline{F}_Y(\beta_{2,n})} = \frac{\overline{F}_Y(\alpha'_{1,n}x+\beta'_{1,n})}{\overline{F}_Y(\beta'_{1,n})}\frac{\overline{F}_Y(\beta'_{1,n})}{\overline{F}_Y(\beta_{2,n})}$$
$$\sim \mathrm{e}^{-\rho^{-1}x}\frac{\overline{F}_Y(\beta'_{1,n})}{\overline{F}_Y(\beta_{2,n})} \to 0, \tag{5}$$

by Lemma 2 in Jiang (2004).

iii) If $\rho = 0$, by Eq.(2) we have

$$n\overline{F}_{Z,n}(\alpha_{2,n}x+\beta_{2,n}) \sim np_n\overline{F}_X(\beta_{2,n}) + \mathrm{e}^{-x}.$$

a) If $np_n\overline{F}_X(\beta_{2,n}) \to 0$, then $n\overline{F}_{Z,n}(\alpha_{2,n}x+\beta_{2,n}) \to \mathrm{e}^{-x}$, which implies with $\alpha_n = f_2(\beta_{2,n})$ and $\beta_n = \beta_{2,n}$,

$$G_Z(x) = \Lambda(x).$$

b) If $np_n\overline{F}_X(\beta_{2,n}) \to A \in (0,\infty]$, then also $np_n \to \infty$ and $np_n\overline{F}_X(\beta'_{1,n}) \to 1$, As in the derivation of Eq.(4)

$$\lim_{x\to\infty}\exp\left\{-\int_{\beta_{2,n}}^{\beta'_{1,n}}\frac{1}{f_1(t)}\mathrm{d}t\right\}=\frac{1}{A}. \tag{6}$$

Furthermore, by Eq.(3)

$$n\overline{F}_{Z,n}(\alpha'_{1,n}x+\beta'_{1,n})\sim\mathrm{e}^{-x}+n\overline{F}_Y(\alpha'_{1,n}x+\beta'_{1,n}),$$

For large n as in derivation of Eq.(5) we have

$$\begin{aligned}0\le n\overline{F}_Y(\alpha'_{1,n}x+\beta'_{1,n})&\sim\frac{\overline{F}_Y(\alpha'_{1,n}x+\beta'_{1,n})}{\overline{F}_Y(\beta_{2,n})}\\ &\sim\exp\left\{-\int_0^x\frac{f_1(f_1(\beta'_{1,n})s+\beta'_{1,n})}{f_2(f_1(\beta'_{1,n})s+\beta'_{1,n})}\mathrm{d}s\right\}\\ &\quad\exp\left\{-\int_{\beta_{2,n}}^{\beta'_{1,n}}\frac{1}{f_2(t)}\mathrm{d}t\right\}\to 0,\end{aligned} \tag{7}$$

by using Eq.(6) with $\rho=0$. Hence with $\alpha_n=\alpha'_{1,n}=f_1(\beta'_{1,n})$ and $\beta_n=\beta'_{1,n}$, we get $G_Z(x)=\Lambda(x)$.

Remark 1 When $\rho=1$ with $np_n\overline{F}_X(\alpha_{2,n})\to A>0$, we can change the normalizing sequences: $\alpha_n=\alpha_{2,n}$ and $\beta_n=\log(A+1)\alpha_{2,n}+\beta_{2,n}$ to get

$$G_Z(x)=\Lambda(x).$$

Thus only if $0<\rho<1$, $G_Z(x)$ exists as a mixture form $\Lambda^A(\rho x)\Lambda(x)$.

Now we deal with the situation with $x_{F_X}=x_{F_Y}=x_F<\infty$. In order to establish the theorem, the following lemmas are necessary.

Lemma 1 Suppose $F_x\in MDA(\Lambda)$ have $x_F<\infty$ with absolutely continuous auxiliary function $f(t)$. Then $F^*(x)=F(x_F-x^{-1})\in MDA(\Lambda)$ with the right end-point $x_F^*=\infty$ and absolutely continuous auxiliary function $f^*(t)=t^2f(x_F-t^{-1})$.

Proof It is easy to check.

Lemma 2 Suppose F_x and $F_Y\in MDA(\Lambda)$ have $x_{F_X}=x_{F_Y}=x_F<\infty$

i) If $\lim_{t\to x_F}\frac{f_2(t)}{f_1(t)}>1$, then $\lim_{x\to x_F}\frac{\overline{F}_Y(x)}{\overline{F}_X(x)}=\infty$.

ii) If $\lim_{x\to x_F}\frac{\overline{F}_Y(x)}{\overline{F}_X(x)}=d>0$, then $\lim_{t\to x_F}\frac{f_2(t)}{f_1(t)}=1$.

Proof i) By Lemma 1 $F_X^*(x)=F_X(x_F-x^{-1})$ and $F_Y^*(x)=F_Y(x_F-x^{-1})\in MDA(\Lambda)$ with $x_{F_X^*}=x_{F_Y^*}=\infty$ and their absolutely continuous auxiliary functions are $f_1^*(t)=t^2f_1(x_F-t^{-1})$ and $f_2^*(t)=t^2f_2(x_F-t^{-1})$, respectively. Furthermore,

$$\lim_{t\to\infty}\frac{f_1^*(t)}{f_2^*(t)}=\lim_{t\to\infty}\frac{f_2(x_F-t^{-1})}{f_1(x_F-t^{-1})}>1.$$

Lemma 1 in Jiang (2004) implies

$$\lim_{x\to\infty}\frac{\overline{F^*}_Y(x)}{\overline{F^*}_X(x)}=\infty,$$

while

$$\lim_{x\to x_F}\frac{\overline{F}_Y(x)}{\overline{F}_X(x)}=\lim_{x\to\infty}\frac{\overline{F}_Y(x_F-t^{-1})}{\overline{F}_X(x_F-t^{-1})}=\lim_{x\to\infty}\frac{\overline{F^*}_Y(x)}{\overline{F^*}_X(x)},$$

the statement follows.

ii) By using L'Hospital's rule and the method as in Lemma 1 in Jiang (2004), we have

$$\left(1-\lim_{x\to x_F}\frac{f_1(x)}{f_2(x)}\right)\lim_{x\to x_F}\frac{\overline{F}_Y(x)}{\overline{F}_X(x)}=0,$$

the result then follows.

Theorem 2 Suppose F_x and $F_Y\in MDA(\Lambda)$ have the auxiliary functions $f_1(x)$ and $f_2(x)$, respectively and $x_{F_X}=x_{F_Y}=x_F<\infty$. Suppose $\lim_{x\to x_F}\frac{f_2(x)}{f_1(x)}=\rho\in[0,\infty]$ and $\overline{F}_Y(\beta_{2,n})\sim\frac{1}{n}$. If $np_n\to\infty$, let

$$\overline{F}_X(\beta'_{1,n})\sim\frac{1}{np_n}. \tag{8}$$

Assume $np_n\overline{F}_X(\beta_{2,n})\to A\in[0,\infty]$.

i) If $0<\rho\le 1$ and

a) if $A\in(0,\infty)$, then with the normalizing sequences $\alpha_n = f_2(\beta_{2,n})$ and $\beta_n = \beta_{2,n}$

$$G_Z(x) = \Lambda^A(\rho x)\Lambda(x).$$

b) If A=∞, then with the normalizing sequences $\alpha_n = f_1(\beta'_{1,n})$ and $\beta_n = \beta'_{1,n}$

$$G_Z(x) = \Lambda(x).$$

ii) Otherwise, with the normalizing sequences $\alpha_n = \alpha_{2,n} = f_2(\beta_{2,n})$, $\beta_n = \beta_{2,n}$ or $\alpha_n = \alpha'_{1,n}$, $\beta_n = \beta'_{1,n}$

$$G_Z(x) = \Lambda(x).$$

Proof By Lemma 1 $F_X^*(x) = F_X(x_F - x^{-1}) \in MDA(\Lambda)$ and $F_Y^*(x) = F_Y(x_F - x^{-1}) \in MDA(\Lambda)$ with $x_{F_X^*} = x_{F_Y^*} = \infty$ and their absolutely continuous auxiliary functions are $f_1^*(t) = t^2 f_1(x_F - t^{-1})$ and

$$f_2^*(t) = t^2 f_2(x_F - t^{-1}), \tag{9}$$

respectively. Furthermore,

$$\rho^* = \lim_{x\to\infty}\frac{f_1^*(t)}{f_2^*(t)} = \lim_{t\to\infty}\frac{f_2(x_F - t^{-1})}{f_1(x_F - t^{-1})} = \rho \in [0,\infty]$$

Hence we can apply Theorem 1 to these two functions. Let

$$F_{Z,n}^*(x) = p_n F_X^*(x) + (1-p_n)F_Y^*(x)$$

Suppose $\alpha_{2,n}^*$ and $\beta_{2,n}^*$ are the normalizing sequences from $\overline{F^*}_Y(\beta_{2,n}^*) \sim n^{-1}$. We have

$$\beta_{2,n} = x_F - \frac{1}{\beta_{2,n}^*}. \tag{10}$$

Hence by Eq.(8)

$$np_n\overline{F^*}_X(\beta_{2,n}^*) = np_n\overline{F}_X(x_F - (\beta_{2,n}^*)^{-1})$$

$$= np_n\overline{F}_X(\beta_{2,n}) \to A. \tag{11}$$

1) If $0<\rho\le 1$ and $A>0$, then $0<\rho^*\le 1$, by Theorem 1 there exist normalizing sequences $\alpha_{2,n}^*$ and $\beta_{2,n}^*$, such that

$$n\overline{F_{Z,n}^*}(x)(\alpha_{2,n}^* x + \beta_{2,n}^*) \to A\mathrm{e}^{-\rho x} + \mathrm{e}^{-x}, \tag{12}$$

and furthermore, by Eq.(10) and since $\lim_{n\to\infty}\frac{\alpha_{2,n}^*}{\beta_{2,n}^*} = 0$, for any x

$$\begin{aligned}
&n\overline{F_{Z,n}^*}(\alpha_{2,n}^* x + \beta_{2,n}^*)\\
&= np_n\overline{F}_X\left(x_F - \frac{1}{\alpha_{2,n}^* x + \beta_{2,n}^*}\right) + n(1-p_n)\overline{F}_Y\left(x_F - \frac{1}{\alpha_{2,n}^* x + \beta_{2,n}^*}\right)\\
&= np_n\overline{F}_X\left(x_F - \frac{1}{\beta_{2,n}^*} + \frac{1}{\beta_{2,n}^*} - \frac{1}{\alpha_{2,n}^* x + \beta_{2,n}^*}\right) + n(1-p_n)\overline{F}_Y\left(x_F - \frac{1}{\beta_{2,n}^*} + \frac{1}{\beta_{2,n}^*} - \frac{1}{\alpha_{2,n}^* x + \beta_{2,n}^*}\right)\\
&= np_n\overline{F}_X\left(\beta_{2,n} + \left(1 + \frac{\alpha_{2,n}^*}{\beta_{2,n}^*}x\right)^{-1}\frac{\alpha_{2,n}^*}{(\beta_{2,n}^*)^2}x\right) + n(1-p_n)\overline{F}_Y\left(\beta_{2,n} + \left(1 + \frac{\alpha_{2,n}^*}{\beta_{2,n}^*}x\right)^{-1}\frac{\alpha_{2,n}^*}{(\beta_{2,n}^*)^2}x\right)\\
&= np_n\overline{F}_X\left(\beta_{2,n} + \frac{\alpha_{2,n}^*}{(\beta_{2,n}^*)^2}x(1+o(1))\right) + n(1-p_n)\overline{F}_Y\left(\beta_{2,n} + \frac{\alpha_{2,n}^*}{(\beta_{2,n}^*)^2}x(1+o(1))\right)\\
&= n\overline{F}_{Z,n}\left(\beta_{2,n} + \frac{\alpha_{2,n}^*}{(\beta_{2,n}^*)^2}x(1+o(1))\right).
\end{aligned} \tag{13}$$

which implies by Eqs.(12) and (11),

$$n\overline{F}_{Z,n}\left(\beta_{2,n} + \frac{\alpha_{2,n}^*}{(\beta_{2,n}^*)^2}x(1+o(1))\right) \to A\mathrm{e}^{-\rho x} + \mathrm{e}^{-x}.$$

Hence with normalizing sequences

$$\alpha_n = \frac{\alpha_{2,n}^*}{(\beta_{2,n}^*)^2},\ \beta_n = \beta_{2,n}, \quad (14)$$

the first statement a) in i) follows.

Now we examine the normalizing sequence α_n. In fact, by Eqs.(9) and (10) we have

$$\alpha_n = \frac{\alpha_{2,n}^*}{(\beta_{2,n}^*)^2} = \frac{f_2^*(\beta_{2,n}^*)}{(\beta_{2,n}^*)^2} = f_2(x_F - (\beta_{2,n}^*)^{-1}) = f_2(\beta_{2,n}) \quad (15)$$

2) If $0<\rho\le 1$ and $A=\infty$, by Theorem 1 there exist normalizing sequences $\alpha'^*_{1,n}$ and $\beta'^*_{1,n}$, such that

$$n\overline{F^*_{Z,n}}(\alpha'^*_{1,n}x + \beta'^*_{1,n}) \to \mathrm{e}^{-x}$$

Where $\beta'_{1,n} = x_F - \frac{1}{\beta'^*_{1,n}}$. Furthermore, by the similar derivation as in Eqs.(13), (14) and (15), the statement b) in ii) follows.

3) By using Lemma 2 and Theorem 1, the proofs of the other situations are similar.

DIFFERENT MDA'S

In this section we consider the extreme value distribution $G_Z(x)$ and its corresponding normalizing sequences when F_X and F_Y belong to different MDA's.

F_X and $F_Y \in MDA(\Phi_\alpha)$ or $MDA(\Lambda)$

The case $F_X \in MDA(\Lambda)$ and $F_Y \in MDA(\Phi_\alpha)$ is simple, since by Lemma 1 in Jiang (2004) $\lim_{x\to\infty} \frac{\overline{F}_X(x)}{\overline{F}_Y(x)} = 0$ and by Proposition 1 in Jiang (2004), $G_Z(x) = \Phi_\alpha$ with the normalizing sequences $\alpha_n = \alpha_{2,n}$ and $\beta_n = 0$. X_i has no influences on $M_n(Z)$. The other case is more interesting.

Theorem 3 Suppose $F_X \in MDA(\Phi_\alpha)$ and $F_Y \in MDA(\Lambda)$ with $x_{F_Y} = \infty$ and let

$$np_n \overline{F}_X(\beta_{2,n}) \to A \in [0,\infty].$$

i) If $A=0$, then with the normalizing sequences $\alpha_n = \alpha_{2,n}$, $\beta_n = \beta_{2,n}$

$$G_Z(x) = \Lambda(x).$$

ii) If $A \in (0, \infty]$, then with the normalizing sequences $\alpha_n = \alpha'_{1,n}$, $\beta_n = 0$

$$G_Z(x) = \begin{cases} \Phi_\alpha(x) & \text{if } x \ge A^{-1/\alpha}\ (x \ge 0 \text{ if } A = \infty) \\ 0 & \text{otherwise.} \end{cases}$$

In order to prove Theorem 3, the following lemma is necessary for $F \in MDA(\Lambda)$.

Lemma 3 Suppose $F \in MDA(\Lambda)$, then for any large $T \in \mathbb{R}$, there exists t_0 such that, for every $x \ge x_1 \ge t_0$, we have

$$\frac{\overline{F}(x)}{\overline{F}(x_1)} \le \left(\frac{1+T^{-1}}{1-T^{-1}}\right)\left(1 + \frac{(x-x_1)}{Tf(x_1)}\right)^{-T}. \quad (16)$$

Proof Since f is absolutely continuous with $f'(t) \to 0$, as $t \to \infty$ and $\lim_{x\to\infty} c(x) = c$, for large $T>1$, there exists t_0 such that for any $t>t_0$,

$$-T^{-1} \le f'(t) \le T^{-1}, \quad (17)$$

and $(1-T^{-1})c \le c(t) \le (1+T^{-1})c$, hence for any $x \ge x_1 \ge t_0$

$$\frac{c(x)}{c(x_1)} \le \frac{1+T^{-1}}{1-T^{-1}}. \quad (18)$$

Integrating the above inequality Eq.(17) over (x_1, t) for any $t \ge x_1 \ge t_0$, we have

$$-T^{-1}(t-x_1) \le f(t) - f(x_1) \le T^{-1}(t-x_1)$$

so that

$$\frac{1}{f(t)} \geq \frac{1}{f(x_1)+T^{-1}(t-x_1)}$$

and hence for $x \geq x_1$ such that

$$\frac{\overline{F}(x)}{\overline{F}(x_1)} = \frac{c(x)}{c(x_1)}\exp\left\{-\int_{x_1}^{x}\frac{1}{f_1(t)}\mathrm{d}t\right\}$$
$$\leq \left(\frac{1+T^{-1}}{1-T^{-1}}\right)\exp\left\{-\int_{x_1}^{x}\frac{1}{f(x_1)+T^{-1}(t-x_1)}\mathrm{d}t\right\}$$
$$= \left(\frac{1+T^{-1}}{1-T^{-1}}\right)\exp\left\{-T\int_{f(x_1)}^{f(x_1)+T^{-1}(x-x_1)} s^{-1}\mathrm{d}s\right\}$$
$$= \left(\frac{1+T^{-1}}{1-T^{-1}}\right)\left(\frac{f(x_1)+T^{-1}(x-x_1)}{f(x_1)}\right)^{-T}$$
$$= \left(\frac{1+T^{-1}}{1-T^{-1}}\right)\left(1+\frac{(x-x_1)}{Tf(x_1)}\right)^{-T}.$$

Proof of Theorem 3 Setting $\alpha_n = \alpha_{2,n} = f_2(\beta_{2,n})$ and $\beta_n = \beta_{2,n}$, we get

$$n\overline{F}_{Z,n}(\alpha_{2,n}x+\beta_{2,n})$$
$$\sim np_n\overline{F}_X(\beta_{2,n})\frac{\overline{F}_X(\alpha_{2,n}x+\beta_{2,n})}{\overline{F}_X(\beta_{2,n})}+\mathrm{e}^{-x}$$
$$\sim np_n\overline{F}_X(\beta_{2,n})+\mathrm{e}^{-x} \tag{19}$$

since

$$\lim_{n\to\infty}\frac{\overline{F}_X(\alpha_{2,n}x+\beta_{2,n})}{\overline{F}_X(\beta_{2,n})} = \lim_{n\to\infty}\frac{\overline{F}_X(\beta_{2,n}(\frac{\alpha_{2,n}}{\beta_{2,n}}x+1))}{\overline{F}_X(\beta_{2,n})}$$
$$= (x\lim_{n\to\infty}\frac{\alpha_{2,n}}{\beta_{2,n}}+1)^{-\alpha} = 1$$

i) If A=0, by Eq.(19) the statement follows.

ii) If A>0, then $np_n \to \infty$ and

$$n\overline{F}_{Z,n}(\alpha'_{1,n}x) \sim x^{-\alpha}+n(1-p_n)\overline{F}_Y(\alpha'_{1,n}x).$$

If we can prove that

$$n(1-p_n)\overline{F}_Y(\alpha'_{1,n}x) \to \begin{cases} 0 & \text{if } x \geq A^{-1/\alpha} (x \geq 0 \text{ if } A=\infty) \\ \infty & \text{otherwise.}\end{cases}$$

then the statement follows.

In fact, we have $np_n\overline{F}_X(\alpha'_{1,n}) \to 1$ and

$$\lim_{n\to\infty}\exp\left\{-\int_{\beta_{2,n}}^{\alpha'_{1,n}} t^{-1}\alpha(t)\mathrm{d}t\right\} = \lim_{n\to\infty}\frac{np_n\overline{F}_X(\alpha'_{1,n})}{np_n\overline{F}_X(\beta_{2,n})} \to A^{-1}$$

Now

$$\int_{\beta_{2,n}}^{\alpha'_{1,n}} t^{-1}\alpha(t)\mathrm{d}t = \int_1^{\alpha'_{1,n}/\beta_{2,n}} s^{-1}\alpha(\beta_{2,n}s)\mathrm{d}s$$

implying

$$\lim_{n\to\infty}\int_1^{\alpha'_{1,n}/\beta_{2,n}} s^{-1}\alpha(\beta_{2,n}s)\mathrm{d}s = \log A. \tag{20}$$

Since $\lim_{n\to\infty}\alpha(\beta_{2,n}s)=\alpha$ for any s>1, there exists N>0 for given ε>0, such that for any n>N and s>1

$$(1-\varepsilon)\alpha < \alpha(\beta_{2,n}s) < (1+\varepsilon)\alpha.$$

Hence for $\alpha'_{1,n} \geq \beta_{2,n}$

$$(1-\varepsilon)\alpha\log\left(\frac{\alpha'_{1,n}}{\beta_{2,n}}\right) \leq \int_1^{\alpha'_{1,n}/\beta_{2,n}} s^{-1}\alpha(\beta_{2,n}s)\mathrm{d}s$$
$$\leq (1+\varepsilon)\alpha\log\left(\frac{\alpha'_{1,n}}{\beta_{2,n}}\right)$$

by Eq.(20) and similar for $\alpha'_{1,n} < \beta_{2,n}$, which implies

$$\lim_{n\to\infty}\frac{\alpha'_{1,n}}{\beta_{2,n}} = A^{1/\alpha} \tag{21}$$

We consider the different cases as follows:

a) If $x > A^{-1/\alpha}$, then by Eq.(21) for large n

$$\alpha'_{1,n}x \geq \beta_{2,n}$$

Hence by Lemma 3

$$n(1-p_n)\overline{F}_Y(\alpha'_{1,n}x) \sim \frac{\overline{F}_Y(\alpha'_{1,n}x)}{\overline{F}_Y(\beta_{2,n})}$$
$$\leq \left(\frac{1+T^{-1}}{1-T^{-1}}\right)\left(1+\frac{\alpha'_{1,n}x-\beta_{2,n}}{f_2(\beta_{2,n})T}\right)^{-T}$$
$$= \left(\frac{1+T^{-1}}{1-T^{-1}}\right)\left(1+\frac{1}{T}\frac{\beta_{2,n}}{f_2(\beta_{2,n})}\left(\frac{\alpha'_{1,n}}{\beta_{2,n}}x-1\right)\right)^{-T} \to 0 \tag{22}$$

Since $\frac{x}{f_2(x)} \to \infty$ for $x \to \infty$ and $\frac{\alpha'_{1,n}}{\beta_{2,n}}$ converges.

b) If $x < A^{-1/\alpha}$, then by Eq.(21) for large n,

$$\alpha'_{1,n} x \le \beta_{2,n}.$$

As the derivation in Eq.(22) we get

$$\frac{\overline{F}_Y(\beta_{2,n})}{\overline{F}_Y(\alpha'_{1,n}x)} \to 0.$$

Hence,

$$n(1-p_n)\overline{F}_Y(\alpha'_{1,n}x) \sim \frac{\overline{F}_Y(\alpha'_{1,n}x)}{\overline{F}_Y(\beta_{2,n})} \to \infty \tag{23}$$

the statement follows.

iii) If $A = \infty$ then

$$\lim_{n\to\infty} \exp\left\{-\int_{\beta_{2,n}}^{\alpha'_{1,n}} t^{-1}\alpha(t)\mathrm{d}t\right\} = \lim_{n\to\infty} \frac{np_n\overline{F}_X(\alpha'_{1,n})}{np_n\overline{F}_X(\beta_{2,n})} \to 0$$

which implies $\lim_{n\to\infty}\int_1^{\alpha'_{1,n}/\beta_{2,n}} s^{-1}\alpha(\beta_{2,n}s)\mathrm{d}s = \infty$ and moreover, $\lim_{n\to\infty}\frac{\alpha'_{1,n}}{\beta_{2,n}} = \infty$. If $x \ge 0$ or $x<0$, then as the derivation in Eqs.(22) and (23) the results follow.

F_X and $F_Y \in MDA(\Psi_\alpha)$ or $MDA(\Lambda)$

The case $F_X \in MDA(\Lambda)$ and $F_Y \in MDA(\Psi_\alpha)$ ($\alpha>0$) with $x_{F_X} = x_{F_Y} = x_F$ is simple again, since $F_X(x_F - x^{-1}) \in MDA(\Lambda)$ and $F_Y(x_F - x^{-1}) \in MDA(\Phi_\alpha)$. Lemma 1 in Jiang (2004) gives

$$\lim_{x\to x_F} \frac{\overline{F}_X(x)}{\overline{F}_Y(x)} = \lim_{x\to\infty}\frac{\overline{F}_X(x_F - x^{-1})}{\overline{F}_Y(x_F - x^{-1})} = 0,$$

Proposition 1 in Jiang (2004) implies $G_Z(x) = \Psi_\alpha(x)$. The other case is again more interesting.

Theorem 4 Suppose $F_X(x) \in MDA(\Psi_\alpha)$ and $F_Y \in MDA(\Lambda)$. Let $\overline{F}_Y(\beta_{2,n}) \sim \frac{1}{n}$ and if $np_n \to \infty$, let $\overline{F}_X(\gamma'_{1,n}) \sim \frac{1}{np_n}$. Assume

$$np_n\overline{F}_X(\beta_{2,n}) \to A \in [0,\infty]. \tag{24}$$

i) If $A=0$, then with the normalizing sequences $\alpha_n = \alpha_{2,n} = f_2(\beta_{2,n})$ and $\beta_n = \beta_{2,n}$

$$G_Z(x) = \Lambda(x)$$

ii) If $A \in (0, \infty]$, then with the normalizing sequences $\alpha_n = \alpha'_{1,n} = x_F - \gamma'_{1,n}$ and $\beta_n = \beta'_{1,n} = x_F$

$$G_Z(x) = \begin{cases} \Psi_\alpha(x) & \text{if } 0>x \ge -A^{-1/\alpha} \ (x<0 \text{ if } A=\infty) \\ 0 & \text{otherwise.} \end{cases}$$

Proof Since $x_{F_X} = x_{F_Y} = x_F < \infty$, $F^*_X(x) = F_X(x_F - x^{-1}) \in MDA(\Phi_\alpha)$, and by Lemma 1, $F^*_Y(x) = F_Y(x_F - x^{-1}) \in MDA(\Lambda)$ with $x_{F^*_X} = x_{F^*_Y} = \infty$. Hence we can apply Theorem 1 to these two functions. Let

$$F^*_{Z,n}(x) = p_nF^*_X(x) + (1-p_n)F^*_Y(x)$$

Suppose $\alpha^*_{2,n}$ and $\beta^*_{2,n}$ are the normalizing sequences from $F^*_Y(x)$. We have $\beta_{2,n} = x_F - \frac{1}{\beta^*_{2,n}}$, and by Eq.(24)

$$\begin{aligned} np_n\overline{F}_X(\beta^*_{2,n}) &= np_n\overline{F}_X(x_F - (\beta^*_{2,n})^{-1}) \\ &= np_n\overline{F}_X(\beta_{2,n}) \to A. \end{aligned}$$

i) If $A=0$, by Theorem 3 we have

$$n\overline{F^*_{Z,n}}(\alpha^*_{2,n}x + \beta^*_{2,n}) \to \mathrm{e}^{-x}. \tag{25}$$

Furthermore, by using the derivation in Eq.(13) we have

$$n\overline{F^*_{Z,n}}(\alpha^*_{2,n}x + \beta^*_{2,n})$$

$$= np_n \overline{F}_X\left(\beta_{2,n} + \frac{\alpha^*_{2,n}}{(\beta^*_{2,n})^2}x\left(1+o(1)\right)\right)$$

$$+n(1-p_n)\overline{F}_Y\left(\beta_{2,n} + \frac{\alpha^*_{2,n}}{(\beta^*_{2,n})^2}x\left(1+o(1)\right)\right)$$

$$= n\overline{F}_{Z,n}\left(\beta_{2,n} + \frac{\alpha^*_{2,n}}{(\beta^*_{2,n})^2}x\left(1+o(1)\right)\right)$$

Hence by Eq.(25) with normalizing sequences

$$\alpha_n = \frac{\alpha^*_{2,n}}{(\beta^*_{2,n})^2},\ \beta_n = \beta_{2,n}$$

implying by Eq.(15)

$$\alpha_n = f_2(\beta_{2,n}),\ \beta_n = \beta_{2,n},$$

the statement follows.

ii) If $A \in (0, \infty]$, by Theorem 3 there exists a normalizing sequence $\alpha'^*_{1,n}$ from $F^*_X(x)$, such that

$$n\overline{F^*_{Z,n}}(\alpha'^*_{1,n}x) \to \begin{cases} x^{-\alpha} & \text{if } x \ge A^{-1/\alpha}\ (x \ge 0 \text{ if } A=\infty); \\ 0 & \text{otherwise.} \end{cases} \quad (26)$$

We have $\gamma'_{1,n} = x_F - \frac{1}{\alpha'^*_{1,n}}$ and

$$n\overline{F^*_{Z,n}}(\alpha'^*_{1,n}x)$$

$$= np_n\overline{F}_X(x_F - \frac{1}{\alpha'^*_{1,n}x}) + n(1-p_n)\overline{F}_Y(x_F - \frac{1}{\alpha'^*_{1,n}x})$$

$$= np_n\overline{F}_X(x_F - \frac{x_F-\gamma'_{1,n}}{x})$$

$$+n(1-p_n)\overline{F}_Y(x_F - \frac{x_F-\gamma'_{1,n}}{x})$$

$$= n\overline{F}_{Z,n}(x_F - \frac{x_F-\gamma'_{1,n}}{x})$$

Hence by Eq.(26) we get

$$n\overline{F}_{Z,n}((x_F-\gamma'_{1,n})y + x_F) \to$$

$$\begin{cases} (-y)^{\alpha} & \text{if } 0 > y \ge -A^{-1/\alpha}\ (y<0 \text{ if } A=\infty); \\ 0 & \text{otherwise.} \end{cases}$$

Hence with the normalizing sequences

$$\alpha_n = \alpha'_{1,n} = x_F - \gamma'_{1,n} \quad \text{and} \quad \beta_n = \beta'_{1,n} = x_F,$$

the statement follows.

CONCLUSION

Now we show that $\Lambda^A(\rho x)\Lambda(x)(\rho<1)$ is not max-stable distribution, and hence it does not belong to the three types of extreme value distributions.

Theorem 5 If $A\in(0, \infty)$, then the mixture extreme value distribution $\Lambda^A(\rho x)\Lambda(x)$ ($\rho<1$) is not max-stable distribution function.

Proof Suppose that $G_Z(x) = \Lambda^A(\rho x)\Lambda(x)$ is a max-stable distribution, it means that there exist constants $a_k>0$ and b_k such that

$$\Lambda^{kA}(\rho(a_kx+b_k))\Lambda^k(a_kx+b_k) = \Lambda^A(\rho x)\Lambda(x)$$

By taking logarithms, this is equivalent to

$$Ak\mathrm{e}^{-\rho(a_kx+b_k)} + k\mathrm{e}^{-(a_kx+b_k)} = A\mathrm{e}^{-\rho x} + \mathrm{e}^{-x} \quad (27)$$

Thus,

$$Ak\mathrm{e}^{-\rho(a_kx+b_k)}(1+\mathrm{e}^{-(1-\rho)(a_kx+b_k)}) = A\mathrm{e}^{-\rho x}(1+\mathrm{e}^{-(1-\rho)x})$$

For fixed k, let $x\to\infty$, we get $Ak\mathrm{e}^{-\rho(a_kx+b_k)} \sim A\mathrm{e}^{-\rho x}$, and

$$a_kx + b_k - \frac{\log k}{\rho} = x + o(1)$$

This results in

$$a_k + \left(b_k - \frac{\log k}{\rho}\right)x^{-1} = 1 + o(x^{-1})$$

again let $x\to\infty$, we get $a_k = 1$ and $b_k = \frac{\log k}{\rho}$.

Putting it into Eq.(27) gives

$$A\mathrm{e}^{-\rho x} + k^{1-\rho^{-1}}\mathrm{e}^{-x} = A\mathrm{e}^{-\rho x} + \mathrm{e}^{-x}$$

implying for any k we get $k^{1-\rho^{-1}} = 1$. Hence, ρ =1. This is a contradiction.

By using Eq.(1) in Jiang (2004), these three types of mixture forms can be uniformly expressed as mixed generalized extreme value distribution MGEV

$$H_{\rho_1,\rho_2}(x) = \begin{cases} \exp\left\{-A(1+\rho_2 x)^{-1/\rho_1}\right\}\exp\left\{-(1+\rho_2 x)^{-1/\rho_2}\right\} \\ \qquad \text{if } \rho_1\rho_2 \neq 0 \\ \exp\left\{-A\exp\left\{-\lambda x\right\}\right\}\exp\left\{-\exp\left\{-x\right\}\right\} \\ \qquad \text{if } \rho_1 = \rho_2 = 0 \end{cases}$$

Where ρ_1 and ρ_2 correspond to F_X and F_Y, respectively, $A \in (0, \infty)$, $\rho_1 \geq \rho_2 > 0$ or $\rho_1 \leq \rho_2 < 0$, $1+\rho_2 x > 0$ and $\lambda \in (0, 1)$.

In this family of the distribution functions, the parameter ρ_1, ρ_2 which can be usually called the extreme value indices(EVI) determine the type and shape of generalized mixture extreme value distributions. Distributions with negative EVI $\rho_i < 0$ for i = 1,2 correspond to Weibull mixture with $\alpha_i = -\rho_i^{-1}$ for i =1,2; its support is $(-\infty, -\rho_2^{-1})$ with a finite endpoint. Distributions with the EVI $\rho_1 = \rho_2 = 0$ correspond to Gumbel mixture with its support R having tails that diminish exponentially fast. Distributions with the EVI $\rho_i > 0$ correspond to Fréchet mixture with $\alpha_i = \rho_i^{-1}$ for i = 1,2; its support is $(-\rho_2^{-1}, \infty)$ and every term in the mixture has so-called heavy-tails.

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