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Mixed Gl_2/GH_2 multi-channel multi-objective control synthesis for discrete time systems*

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Abstract: This paper proposes a new approach for multi-objective robust control. The approach extends the standard generalized l_2 (Gl_2) and generalized H_2 (GH_2) conditions to a set of new linear matrix inequality (LMI) constraints based on a new stability condition. A technique for variable parameterization is introduced to the multi-objective control problem to preserve the linearity of the synthesis variables. Consequently, the multi-channel multi-objective mixed Gl_2/GH_2 control problem can be solved less conservatively using computationally tractable algorithms developed in the paper.

Key words: Mixed Gl_2/GH_2 synthesis, Multi-objective optimization, Robust control, Discrete linear time-invariant systems, *G*-shaping paradigm

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INTRODUCTION

In the past two decades, a large number of strategies for control systems analysis and synthesis such as H_2, H_∞, l_1 and μ-synthesis had been developed. In H_∞ design, all disturbances are lumped into a single norm rather than bounded separately by the size of each disturbance as $||d||^2=||d_1||^2+\ldots+||d_m||^2$. This certainly leads to some conservatism (D'Andrea, 1999). In contrast, the μ-synthesis technique overcomes the conservatism by introducing structured uncertainty blocks. However, the optimization has to be solved via a so-called D-K iteration, in which the joint convexity is not guaranteed though optimization in an individual step (K-step or D-step) is convex (Skogestad and Postlethwaite, 1996).

Recently, D'Andrea (1999) proposed the generalized-l_2 (Gl_2) formulation. By combining concepts such as H_∞ optimization, linear matrix inequalities (LMI's), and integral quadratic constraints, a convex characterization of the solution to a large class of robust and optimal control problems is presented. The problem can be represented in LMI formulation (Wang and Wilson, 2001) and has the potential to get solutions less conservative than those obtained via H_∞ control synthesis approach. The computations involved in solving a Gl_2-optimization problem is more tractable than those required by the μ-optimization.

Like other LMI-based methodologies, the standard Gl_2 and GH_2 formulations rely on the Lyapunov stability condition and the solution depends on the Lyapunov symmetric matrix $\boldsymbol{P}$. Furthermore, in multi-objective synthesis, the Lyapunov matrices $\boldsymbol{P}$ in different LMI's are always

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assumed to be identical for the sake of solvability (Scherer *et al*., 1997; Masubuchi *et al*., 1998). Hence, it will inevitably introduce some new conservativeness to the whole optimal solution. To reduce such conservativeness, Geromel *et al*.(1998) proposed a new LMI condition with a non-symmetric matrix $\boldsymbol{G}$ to test the Lyapunov stability for precisely known systems and the quadratic stability for uncertain systems. Based on the new stability condition, many different control synthesis problems can be restated so that the resulting controller no longer depends on the symmetric matrix $\boldsymbol{P}$. It implies that some important control problems can benefit from the extended analysis conditions and the extended controller parameterization (de Oliverira *et al*., 2002; Apkarian *et al*., 2002). Consequently, the main source of conservatism, caused by forcing symmetric P_i to be the same in different LMI's in the Lyapunov-shaping paradigm (Scherer *et al*., 1997), can be released by forcing several instrumental variables P_i to be the same in different LMI's. In this way, several Lyapunov matrices can be simultaneously undertaken in the G-shaping paradigm (de Oliverira *et al*., 1999a; 2002).

In this work, the G-shaping paradigm has been further extended to the mixed Gl_2/GH_2 optimization problem with pole placement constraints. GH_2 is a desirable way to keep the peak amplitude of the output below a certain level to avoid actuator saturation if the input is measured in energy (Scherer *et al*., 1997; Rotea, 1993). Meanwhile, suitable pole placement can effectively prevent the rapid system dynamics important for the digital controller implementation.

The paper is organized as follows: In Section 2, analysis conditions corresponding to Gl_2, GH_2 and pole placement constraints are developed based on the new Lyapunov stability condition. Section 3 proposes the synthesis conditions and introduces a new technique of variable parameterization for mixed Gl_2/GH_2 with pole placement constraints. Some features of the new approach are explored. A brief summary of the proposed method is provided in Section 4.

ANALYSIS CONDITION

Consider a linear discrete time-invariant system $\boldsymbol{G}$ in the state space form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \tilde{\boldsymbol{A}}\boldsymbol{x}(k) + \tilde{\boldsymbol{B}}\boldsymbol{w}(k) \\ \boldsymbol{z}(k) &= \tilde{\boldsymbol{C}}\boldsymbol{x}(k) + \tilde{\boldsymbol{D}}\boldsymbol{w}(k) \end{aligned} \tag{1}$$

where the state vector $\boldsymbol{x} \in \mathrm{R}^{n_x}$, the exogenous input $\boldsymbol{w} \in \mathrm{R}^{n_w}$, the controlled output $\boldsymbol{z} \in \mathrm{R}^{n_z}$, and the corresponding matrices have appropriate dimensions. Define the transfer function from the input $\boldsymbol{w}$ to the output $\boldsymbol{z}$ as:

$$\boldsymbol{T}_{zw}(\varsigma) := \left[\begin{array}{c|c} \tilde{\boldsymbol{A}} & \tilde{\boldsymbol{B}} \\ \hline \tilde{\boldsymbol{C}} & \tilde{\boldsymbol{D}} \end{array}\right] \tag{2}$$

In a standard controller synthesis procedure based on LMI's, the following lemma has been used for the asymptotical stability test.

Lemma 1 (Boyd *et al*., 1994) The system $\boldsymbol{G}$ is robustly stable if and only if the fundamental Lyapunov inequality holds

$$\begin{bmatrix} \boldsymbol{P} & \tilde{\boldsymbol{A}}'\boldsymbol{P} \\ (\cdot) & \boldsymbol{P} \end{bmatrix} > 0$$

Recently, an extended stability condition has been proposed where an extra variable G was introduced.

Lemma 2 (de Oliverira *et al*., 1999a) The system $\boldsymbol{G}$ is robustly stable if and only if there exist a symmetric matrix $\boldsymbol{P}$ and a matrix $\boldsymbol{G}$ such that

$$\begin{bmatrix} \boldsymbol{P} & \tilde{\boldsymbol{A}}'\boldsymbol{G}' \\ (\cdot)' & \boldsymbol{G}+\boldsymbol{G}'-\boldsymbol{P} \end{bmatrix} > 0$$

The new condition provides a little advantage in the analysis of precisely known systems as well as more degrees of freedom to build parameter-dependent Lyapunov functions for discrete-time systems with convex polytypic uncertainties. It had also been shown that the new stability condition can provide advantage in synthesis of linear multi-objective controllers (Geromel *et al*., 1998; de Oliverira *et al*., 1999a; 1999b). In addition to these advantages, this condition exhibits a kind of decoupling between the Lyapunov matrix and the system matrices. This feature can be used efficiently for controller synthesis problem.

Remark 1 From Lemma 2,

$$\tilde{G}+\tilde{G}'-\tilde{P}>0 \Rightarrow \tilde{G}+\tilde{G}'>\tilde{P} \Rightarrow \tilde{G} \text{ is nonsingular} \quad (3)$$

Also

$$\tilde{P}>0 \Rightarrow (\tilde{P}-\tilde{G})'\tilde{P}^{-1}(\tilde{P}-\tilde{G}) \geq 0 \Rightarrow \tilde{G}'\tilde{P}^{-1}\tilde{G} \geq \tilde{G}+\tilde{G}'-\tilde{P} \quad (4)$$

Gl_2 Norm

For Gl_2 control problem, the disturbance sets W and criterion sets Z, which are associated with the input from the uncertainty Δ and the cost criterion respectively, will be introduced firstly. Their definitions and properties were described and explored by D'Andrea (1999) in detail. For the sake of simplicity, reconstruct these sets as

$$W := \{w_k \in l_2 : \|w_k\| \leq 1, k \in [1,m]\}$$
$$Z := \{z_l \in l_2 : \|z_l\| \leq 1, l \in [1,n]\}$$

Here w_k and z_l are the elements of $\boldsymbol{w}$ and $\boldsymbol{z}$ in system (1). The Gl_2 analysis theorem (D'Andrea, 1999) is restated for the above specific sets W and Z as follows.

Theorem 1 (Gl_2 analysis condition) Given a system $\boldsymbol{G}$ and sets W and Z, the following are equivalent.

(1) The following inequality is satisfied

$$\|\boldsymbol{T}_{zw}(\varsigma)\|_{Gl_2} := \sup_{w\in W}\sup_{z\in Z}\langle z, \boldsymbol{T}_{zw}(\varsigma)\boldsymbol{w}\rangle < \alpha$$

(2) There exist $x_k, y_l \in \mathrm{R}^+, k\in[1,m], l\in[1,n]$, such that

$$\|\boldsymbol{Y}_z^{-1/2}\boldsymbol{T}_{zw}\boldsymbol{X}_w^{-1/2}\|_\infty < 1 \quad (5)$$

$$\boldsymbol{X}_w := x_1\boldsymbol{I}_{w_1} \oplus x_2\boldsymbol{I}_{w_2} \oplus \cdots \oplus x_m\boldsymbol{I}_{w_m} > 0, \quad \sum_{i=1}^{m} x_i \leq \alpha \quad (6)$$

$$\boldsymbol{Y}_z := y_1\boldsymbol{I}_{z_1} \oplus y_2\boldsymbol{I}_{z_2} \oplus \cdots \oplus y_n\boldsymbol{I}_{z_n} > 0, \quad \sum_{i=1}^{n} y_i \leq \alpha \quad (7)$$

where, ⊕ stands for direct sum of matrices, i.e.

$$\boldsymbol{A} \oplus \boldsymbol{B} := \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{bmatrix}$$

$\boldsymbol{I}_{w_k}$ and $\boldsymbol{I}_{z_l}$ are the identity matrices with the same dimensions as $\boldsymbol{w}_k\boldsymbol{w}_k'$ and $\boldsymbol{z}_l\boldsymbol{z}_l'$ respectively.

To incorporate into the multi-objective synthesis framework, the above analysis condition of Gl_2 needs to be restated in LMI form.

Lemma 3 (LMI-based analysis condition of Gl_2)

Let the system $\boldsymbol{G}$ be a minimal discrete time state space representation, then $\|\boldsymbol{T}_{zw}\|_{Gl_2} \leq \alpha$ if, and only if, there exist $\boldsymbol{X}_w$ and $\boldsymbol{Y}_z$ satisfying Eqs.(6)–(7) and a positive symmetric matrix $\boldsymbol{P}$ such that

$$\begin{bmatrix} \boldsymbol{P} & \tilde{\boldsymbol{A}}\boldsymbol{P} & \tilde{\boldsymbol{B}} & \boldsymbol{0} \\ (\cdot)' & \boldsymbol{P} & \boldsymbol{0} & \boldsymbol{P}\tilde{\boldsymbol{C}}' \\ (\cdot)' & (\cdot)' & \boldsymbol{X}_w & \tilde{\boldsymbol{D}}' \\ (\cdot)' & (\cdot)' & (\cdot)' & \boldsymbol{Y}_z \end{bmatrix} > 0 \quad (8)$$

In the light of the discrete version of Bounded Real Lemma, this lemma can be proven in a way analogous to the proof for continuous systems in (Wang and Wilson, 2001).

On the basis of Lemma 2, the LMI characterizations of Gl_2, GH_2 and pole placement conditions can be extended in the sequel.

Theorem 2 (Gl_2) Let the system $\boldsymbol{G}$ be a minimal discrete time realization, then $\|\boldsymbol{T}_{zw}\|_{Gl_2} \leq \alpha$ holds if, and only if, $\boldsymbol{X}_w$ and $\boldsymbol{Y}_z$ satisfy Eqs.(6) and (7) and there exist a matrix $\boldsymbol{G}$ and a symmetric matrix $\boldsymbol{P}$ such that

$$\begin{bmatrix} \boldsymbol{P} & \tilde{\boldsymbol{A}}\boldsymbol{G} & \tilde{\boldsymbol{B}} & \boldsymbol{0} \\ (\cdot)' & \boldsymbol{G}+\boldsymbol{G}'-\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{G}'\tilde{\boldsymbol{C}}' \\ (\cdot)' & (\cdot)' & \boldsymbol{X}_w & \tilde{\boldsymbol{D}}' \\ (\cdot)' & (\cdot)' & (\cdot)' & \boldsymbol{Y}_z \end{bmatrix} > 0 \quad (9)$$

is feasible.

Proof To prove the necessity, it suffices to choose $\boldsymbol{G}=\boldsymbol{G}'=\boldsymbol{P}$. Then Eq.(9) is equivalent to Eq.(8).

For the sufficiency, assume that Eq.(9) is feasible. Application of lemma 3 to Eq.(9) leads to

$$\begin{bmatrix} \boldsymbol{P} & \tilde{\boldsymbol{A}}\boldsymbol{G} & \tilde{\boldsymbol{B}} & \boldsymbol{0} \\ (\cdot)' & \boldsymbol{G}'\boldsymbol{P}^{-1}\boldsymbol{G} & \boldsymbol{0} & \boldsymbol{G}'\tilde{\boldsymbol{C}}' \\ (\cdot)' & (\cdot)' & \boldsymbol{X}_w & \tilde{\boldsymbol{D}}' \\ (\cdot)' & (\cdot)' & (\cdot)' & \boldsymbol{Y}_z \end{bmatrix} > 0$$

Pre-multiply and post-multiply the above inequality with $\boldsymbol{T}'$ and $\boldsymbol{T}$ respectively, with

$$\boldsymbol{T} := \mathrm{diag}[\boldsymbol{I}, \boldsymbol{G}^{-1}\boldsymbol{P}, \boldsymbol{I}, \boldsymbol{I}]$$

Lemma 3 can be recovered. Hence, the proof is complete.

GH_2 Norm

Assume $\tilde{\boldsymbol{A}}$ is stable and $\tilde{\boldsymbol{D}}=0$. The GH_2 norm of $\boldsymbol{T}_{zw}(\varsigma)$ is defined as follows.

$$\|\boldsymbol{T}_{zw}(\varsigma)\|_{\mathrm{G}H_2} := \lambda_{\max}^{\frac{1}{2}} \left[\frac{1}{2\pi} \int_{-\pi}^{+\pi} \boldsymbol{T}_{zw}(jw)\boldsymbol{T}_{zw}^*(jw)\mathrm{d}w \right]$$

where $\lambda_{\max}$ represents the maximum eigenvalue of a matrix.

The LMI form of the norm in Scherer *et al.* (1997) can also be extended using the new stability condition.

Theorem 3 (GH_2) The inequality $\|\boldsymbol{T}_{zw}(\varsigma)\|_{\mathrm{G}H_2}^2 < \beta$ holds if, and only if, there exist a matrix $\boldsymbol{G}$ and a symmetric matrix $\boldsymbol{P}$ such that

$$\begin{bmatrix} \beta\boldsymbol{I} & \tilde{\boldsymbol{C}}\boldsymbol{G} \\ (\cdot)' & \boldsymbol{G}+\boldsymbol{G}'-\boldsymbol{P} \end{bmatrix} > 0$$

$$\begin{bmatrix} \boldsymbol{P} & \tilde{\boldsymbol{A}}\boldsymbol{G} & \tilde{\boldsymbol{B}} \\ (\cdot)' & \boldsymbol{G}+\boldsymbol{G}'-\boldsymbol{P} & \boldsymbol{0} \\ (\cdot)' & (\cdot)' & \boldsymbol{I} \end{bmatrix} > 0$$

are feasible.

The proof of this theorem is similar to the procedure for H_2 given by de Oliverira *et al.*(2002). Therefore, it is omitted here.

Pole placement constraints

LMI conditions for pole placement in continuous systems were presented in (Chilali and Gahinet, 1996). A uniform condition for continuous and discrete time system can be found in (Masubuchi *et al.*, 1998). For a discrete time system, a simple but useful pole placement subregion, i.e. disc region, can be represented as:

$$\boldsymbol{C}_D(z_0, \rho) := \{\lambda \in \boldsymbol{C}, |\lambda + z_0| < \rho\} \tag{10}$$

According to Yedavalli (1993), $\boldsymbol{G}$ has all its poles in $\boldsymbol{C}_D$ if and only if the following LMI

$$\begin{bmatrix} \rho\tilde{\boldsymbol{P}} & \tilde{\boldsymbol{A}}'\tilde{\boldsymbol{P}} + z_0\tilde{\boldsymbol{P}} \\ (\cdot)' & \rho\tilde{\boldsymbol{P}} \end{bmatrix} > 0 \tag{11}$$

has a positive definite solution $\boldsymbol{P}$. This result can be extended as follows.

Theorem 4 (Root-Clustering Condition) Assuming $\lambda_i \in \boldsymbol{C}_D$ to be the poles of system (1), it is equivalent to

$$\begin{bmatrix} \rho\tilde{\boldsymbol{P}} & \tilde{\boldsymbol{A}}\tilde{\boldsymbol{G}} + z_0\tilde{\boldsymbol{G}} \\ (\cdot)' & \rho(\tilde{\boldsymbol{G}} + \tilde{\boldsymbol{G}}' - \tilde{\boldsymbol{P}}) \end{bmatrix} > 0 \tag{12}$$

for some $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{G}}$.

Proof (Necessity) Choose $\boldsymbol{G}=\boldsymbol{G}'=\boldsymbol{P}$ in order to recover Eq.(11).

(Sufficiency) Assume that Eq.(12) is feasible. Let $\boldsymbol{T} = \mathrm{diag}[\boldsymbol{I}, \tilde{\boldsymbol{P}}(\tilde{\boldsymbol{G}}')^{-1}]$. Due to Eqs.(3) and (4), the following is yielded

$$\begin{bmatrix} \rho\tilde{\boldsymbol{P}} & \tilde{\boldsymbol{A}}\tilde{\boldsymbol{P}} + z_0\tilde{\boldsymbol{P}} \\ (\cdot)' & \rho\tilde{\boldsymbol{P}} \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} \rho\tilde{\boldsymbol{P}} & \tilde{\boldsymbol{A}}\tilde{\boldsymbol{G}} + z_0\tilde{\boldsymbol{G}} \\ (\cdot)' & \rho(\tilde{\boldsymbol{G}}'\tilde{\boldsymbol{P}}^{-1}\tilde{\boldsymbol{G}}) \end{bmatrix} \boldsymbol{T}'$$
$$\geq \boldsymbol{T} \begin{bmatrix} \rho\tilde{\boldsymbol{P}} & \tilde{\boldsymbol{A}}\tilde{\boldsymbol{G}} + z_0\tilde{\boldsymbol{G}} \\ (\cdot)' & \rho(\tilde{\boldsymbol{G}} + \tilde{\boldsymbol{G}}' - \tilde{\boldsymbol{P}}) \end{bmatrix} \boldsymbol{T}' > 0$$

Together with the fact that $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{A}}'$ have the same eigenvalues, the proof is complete.

PARAMETERIZATION AND SYNTHESIS PROCEDURE

Parameterization for Output feedback

For the synthesis purpose, consider the following discrete time system

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}_w\boldsymbol{w}(k) + \boldsymbol{B}_u\boldsymbol{u}(k) \\ \boldsymbol{z}(k) &= \boldsymbol{C}_z\boldsymbol{x}(k) + \boldsymbol{D}_{zw}\boldsymbol{w}(k) + \boldsymbol{D}_{zu}\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= \boldsymbol{C}_y\boldsymbol{x}(k) + \boldsymbol{D}_{yw}\boldsymbol{w}(k) \end{aligned} \tag{13}$$

where, $\boldsymbol{x}(k)\in \mathrm{R}^{n_x}, \boldsymbol{w}(k)\in \mathrm{R}^{n_w}, \boldsymbol{u}(k)\in \mathrm{R}^{n_u}, \boldsymbol{z}(k)\in \mathrm{R}^{n_z}$ and $\boldsymbol{y}(k)\in \mathrm{R}^{n_y}$ are discrete state, exogenous input, control input, controlled output and measurement output respectively.

Define the output feedback controller $K(\zeta)$ in the following state-space form:

$$\begin{aligned} \boldsymbol{x}_c(k+1) &= \boldsymbol{A}_c\boldsymbol{x}(k) + \boldsymbol{B}_c\boldsymbol{y}(k) \\ \boldsymbol{u}(k) &= \boldsymbol{C}_c\boldsymbol{x}(k) + \boldsymbol{D}_c\boldsymbol{y}(k) \end{aligned} \tag{14}$$

Then the closed-loop transfer function $\boldsymbol{T}_{zw}(\varsigma)$ represented in Eq.(2) has the system matrices as follows:

$$\tilde{\boldsymbol{A}} := \begin{bmatrix} \boldsymbol{A}+\boldsymbol{B}_u\boldsymbol{D}_c\boldsymbol{C}_y & \boldsymbol{B}_u\boldsymbol{C}_c \\ \boldsymbol{B}_c\boldsymbol{C}_y & \boldsymbol{A}_c \end{bmatrix} \quad \tilde{\boldsymbol{B}} := \begin{bmatrix} \boldsymbol{B}_w+\boldsymbol{B}_u\boldsymbol{D}_c\boldsymbol{D}_{yw} \\ \boldsymbol{B}_c\boldsymbol{D}_{yw} \end{bmatrix}$$
$$\tilde{\boldsymbol{C}} := \begin{bmatrix} \boldsymbol{C}+\boldsymbol{D}_{zu}\boldsymbol{D}_c\boldsymbol{C}_y & \boldsymbol{D}_{zu}\boldsymbol{C}_c \end{bmatrix} \quad \tilde{\boldsymbol{D}} := \begin{bmatrix} \boldsymbol{D}_{zw}+\boldsymbol{D}_{zu}\boldsymbol{D}_c\boldsymbol{D}_{yw} \end{bmatrix} \tag{15}$$

Obviously, directly taking the above matrices into the analysis conditions would lead to bilinear matrix inequalities (BMI's) rather than LMI's. Therefore it is necessary to find a suitable parameterization to preserve the linearity of the synthesis variables. Define a nonsingular matrix $\boldsymbol{G}$ and its inverse as

$$\tilde{\boldsymbol{G}} := \begin{bmatrix} \boldsymbol{X} & * \\ \boldsymbol{U} & * \end{bmatrix}, \tilde{\boldsymbol{G}}' := \begin{bmatrix} \boldsymbol{Y}' & * \\ \boldsymbol{V}' & * \end{bmatrix} \tag{16}$$

The symbol * denotes blocks in these matrices with no importance in the sequel. Furthermore, a new transforming matrix is introduced

$$\tilde{\boldsymbol{T}} := \begin{bmatrix} \boldsymbol{I} & \boldsymbol{Y}' \\ \boldsymbol{0} & \boldsymbol{V}' \end{bmatrix}, \tilde{\boldsymbol{T}}' := \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{T} & \boldsymbol{V} \end{bmatrix} \tag{17}$$

then it is readily verified that

$$\begin{aligned} &\tilde{\boldsymbol{T}}'\tilde{\boldsymbol{A}}\tilde{\boldsymbol{G}}\tilde{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{AX}+\boldsymbol{B}_u\boldsymbol{L} & \boldsymbol{A}+\boldsymbol{B}_u\boldsymbol{RC}_y \\ \boldsymbol{Q} & \boldsymbol{YA}+\boldsymbol{FC}_y \end{bmatrix} \\ &\tilde{\boldsymbol{T}}'\tilde{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B}_w+\boldsymbol{B}_u\boldsymbol{RD}_{yw} \\ \boldsymbol{YB}_w+\boldsymbol{FD}_{yw} \end{bmatrix} \\ &\tilde{\boldsymbol{C}}\tilde{\boldsymbol{G}}\tilde{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{C}_z\boldsymbol{X}+\boldsymbol{D}_{zu}\boldsymbol{L} & \boldsymbol{C}_z+\boldsymbol{D}_{zu}\boldsymbol{RC}_y \end{bmatrix} \\ &\tilde{\boldsymbol{T}}'\tilde{\boldsymbol{G}}\tilde{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{I} \\ \boldsymbol{S} & \boldsymbol{Y} \end{bmatrix} \quad \tilde{\boldsymbol{T}}'\tilde{\boldsymbol{G}}'\tilde{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{X}' & \boldsymbol{S}' \\ \boldsymbol{I} & \boldsymbol{Y}' \end{bmatrix} \end{aligned} \tag{18}$$

where

$$\begin{aligned} &\boldsymbol{Q} := \boldsymbol{Y}(\boldsymbol{A}+\boldsymbol{B}_u\boldsymbol{D}_c\boldsymbol{C}_y)\boldsymbol{X}+\boldsymbol{VB}_c\boldsymbol{C}_y\boldsymbol{X}+\boldsymbol{YB}_u\boldsymbol{C}_c\boldsymbol{U}+\boldsymbol{VA}_c\boldsymbol{U} \\ &\boldsymbol{F} := \boldsymbol{YB}_u\boldsymbol{D}_c+\boldsymbol{VB}_c;\ \boldsymbol{L} := \boldsymbol{D}_c\boldsymbol{C}_y\boldsymbol{X}+\boldsymbol{C}_c\boldsymbol{U}; \\ &\boldsymbol{R} := \boldsymbol{D}_c;\ \boldsymbol{S} := \boldsymbol{YX}+\boldsymbol{VU} \end{aligned} \tag{19}$$

The same transformation is applied to the Lyapunov matrix to get

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{J} \\ (\cdot)' & \boldsymbol{H} \end{bmatrix} := \tilde{\boldsymbol{T}}'\tilde{\boldsymbol{P}}\tilde{\boldsymbol{T}} \tag{20}$$

where $\tilde{\boldsymbol{P}}$ denotes the Lyapunov matrix $\boldsymbol{P}$ in analysis condition in Section 2.

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{J} & \boldsymbol{AX}+\boldsymbol{B}_u\boldsymbol{L} & \boldsymbol{A}+\boldsymbol{B}_u\boldsymbol{RC}_y & \boldsymbol{B}_w+\boldsymbol{B}_u\boldsymbol{RD}_{yw} & \boldsymbol{0} \\ (\cdot)' & \boldsymbol{H} & \boldsymbol{Q} & \boldsymbol{YA}+\boldsymbol{FC}_y & \boldsymbol{YB}_w+\boldsymbol{FD}_{yw} & \boldsymbol{0} \\ (\cdot)' & (\cdot)' & \boldsymbol{X}+\boldsymbol{X}'-\boldsymbol{P} & \boldsymbol{I}+\boldsymbol{S}'-\boldsymbol{J} & \boldsymbol{0} & \boldsymbol{X}'\boldsymbol{C}'_z+\boldsymbol{L}'\boldsymbol{D}'_{zu} \\ (\cdot)' & (\cdot)' & (\cdot)' & \boldsymbol{Y}+\boldsymbol{Y}'-\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{C}'_z+\boldsymbol{C}'_y\boldsymbol{R}'\boldsymbol{D}'_{zu} \\ (\cdot)' & (\cdot)' & (\cdot)' & (\cdot)' & \boldsymbol{X}_w & \boldsymbol{D}'_{zw}+\boldsymbol{D}'_{yw}\boldsymbol{R}'\boldsymbol{D}'_{zu} \\ (\cdot)' & (\cdot)' & (\cdot)' & (\cdot)' & (\cdot)' & \boldsymbol{Y}_z \end{bmatrix} > 0 \tag{21}$$

Then, the following central results for control synthesis can be obtained.

Theorem 5 (Gl_2 Output Feedback) All controllers in the form Eq.(14), such that the inequality $\|\boldsymbol{T}_{zw}\|_{Gl_2} \le \alpha$ holds, are parameterized in Eq.(21) (shown at the top of this page), where the matrices $\boldsymbol{X}$, $\boldsymbol{L}$, $\boldsymbol{Y}$, $\boldsymbol{F}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{J}$ and the symmetric matrices $\boldsymbol{P}$, $\boldsymbol{H}$ and $\boldsymbol{W}$ are the synthesis variables.

Theorem 6 (GH_2 Output Feedback) All controllers in the form Eq.(14) such that the inequality holds are parameterized by the LMI

$$\begin{bmatrix} \beta \boldsymbol{I} & \boldsymbol{C}_z\boldsymbol{X}+\boldsymbol{D}_{zu}\boldsymbol{L} & \boldsymbol{C}_z+\boldsymbol{D}_{zu}\boldsymbol{R}\boldsymbol{C}_y \\ (\cdot)' & \boldsymbol{X}+\boldsymbol{X}'-\boldsymbol{P} & \boldsymbol{I}+\boldsymbol{S}'-\boldsymbol{J} \\ (\cdot)' & (\cdot)' & \boldsymbol{Y}+\boldsymbol{Y}'-\boldsymbol{H} \end{bmatrix} > 0$$

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{J} & \boldsymbol{A}\boldsymbol{X}+\boldsymbol{B}_u\boldsymbol{L} & \boldsymbol{A}+\boldsymbol{B}_u\boldsymbol{R}\boldsymbol{C}_y & \boldsymbol{B}_w+\boldsymbol{B}_u\boldsymbol{R}\boldsymbol{D}_{yw} \\ (\cdot)' & \boldsymbol{H} & \boldsymbol{Q} & \boldsymbol{Y}\boldsymbol{A}+\boldsymbol{F}\boldsymbol{C}_y & \boldsymbol{Y}\boldsymbol{B}_w+\boldsymbol{F}\boldsymbol{D}_{yw} \\ (\cdot)' & (\cdot)' & \boldsymbol{X}+\boldsymbol{X}'-\boldsymbol{P} & \boldsymbol{I}+\boldsymbol{S}'-\boldsymbol{J} & \boldsymbol{0} \\ (\cdot)' & (\cdot)' & (\cdot)' & \boldsymbol{Y}+\boldsymbol{Y}'-\boldsymbol{H} & \boldsymbol{0} \\ (\cdot)' & (\cdot)' & (\cdot)' & (\cdot)' & \boldsymbol{I} \end{bmatrix} > 0$$

$$\boldsymbol{D}_{zw}+\boldsymbol{D}_{zu}\boldsymbol{R}\boldsymbol{D}_{yw}=0$$

Theorem 7 (Pole Placement) All controllers in the form Eq.(14) such that poles of the closed-loop system satisfying Eq.(10) are parameterized by the following LMI

$$\begin{bmatrix} \rho\boldsymbol{P} & \rho\boldsymbol{J} & (\boldsymbol{A}+z_0\boldsymbol{I})\boldsymbol{X}+\boldsymbol{B}_u\boldsymbol{L} & \boldsymbol{A}+\boldsymbol{B}_u\boldsymbol{R}\boldsymbol{C}_y+z_0\boldsymbol{I} \\ (\cdot)' & \rho\boldsymbol{H} & \boldsymbol{Q}+z_0\boldsymbol{S} & \boldsymbol{Y}(\boldsymbol{A}+z_0\boldsymbol{I})+\boldsymbol{F}\boldsymbol{C}_y \\ (\cdot)' & (\cdot)' & \rho\boldsymbol{X}+\rho\boldsymbol{X}'-\rho\boldsymbol{P} & \rho\boldsymbol{I}+\rho\boldsymbol{S}'-\rho\boldsymbol{J} \\ (\cdot)' & (\cdot)' & (\cdot)' & \rho\boldsymbol{Y}+\rho\boldsymbol{Y}'-\rho\boldsymbol{H} \end{bmatrix} > 0$$

These three theorems can be obtained by substituting for closed-loop matrices Eq.(15) and appropriately applying congruence transformations based on the transformation matrix $\tilde{\boldsymbol{T}}$ to the extended analysis conditions contributed in Section 2. See (Yan and Cao, 2002a) for details of the proof.

It is necessary to emphasize that although the recipe to obtain the inequalities given in Theorems 5, 6 and 7 follows from Scherer *et al.*(1997) and Masubuchi *et al.*(1998), the constructive issues addressed above do not appear in those papers.

Remark 2 The matrix $\tilde{\boldsymbol{T}}$ is very similar to the transformation matrix used in (Scherer *et al.*, 1997, Masubuchi *et al.*, 1998) differences are: (1) $\boldsymbol{Y}$ is no longer symmetric, but it keeps its linearity even though the basic matrix $\tilde{\boldsymbol{G}}$ is no longer restricted to be symmetric. (2) Most importantly, the following identity

$$\tilde{\boldsymbol{T}}'(\tilde{\boldsymbol{G}}'+\tilde{\boldsymbol{G}}')\tilde{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{X}+\boldsymbol{X}' & \boldsymbol{I}+\boldsymbol{S}' \\ (\cdot)' & \boldsymbol{Y}+\boldsymbol{Y}' \end{bmatrix}$$

involves the calculation of matrix $\boldsymbol{S}$ whilst it does not appear in the standard formulations.

Remark 3 Given matrices $\boldsymbol{X}$, $\boldsymbol{L}$, $\boldsymbol{Y}$, $\boldsymbol{F}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{J}$ from the above theorems, a feasible controller is available by choosing $\boldsymbol{V}$ and $\boldsymbol{U}$ nonsingular such that $\boldsymbol{VU}=\boldsymbol{S}-\boldsymbol{YX}$ and calculating the following matrices $\boldsymbol{D}_c$, $\boldsymbol{C}_c$, $\boldsymbol{B}_c$, $\boldsymbol{A}_c$ in that order in Eq.(19).

Remark 4 One interesting feature of this new framework is that the feasible controller Eq.(14) does not depend on any of the Lyapunov matrices $\boldsymbol{P}$, $\boldsymbol{J}$ or $\boldsymbol{H}$. It can reduce the conservatism involved in standard multi-objective optimization problems and allows more flexible and accurate specifications to restrict closed-loop behavior. Particularly, if $\boldsymbol{P}=\boldsymbol{X}=\boldsymbol{X}'$, $\boldsymbol{J}=\boldsymbol{S}=\boldsymbol{I}$, $\boldsymbol{H}=\boldsymbol{Y}=\boldsymbol{Y}'$, it obviously encompasses the results of Gl_2 and GH_2 obtained in (D'Andrea, 1999; Scherer *et al.*, 1997). Moreover, if the matrices $\boldsymbol{X}_w$ and $\boldsymbol{Y}_z$ in Eq.(21) are both set to αI, then the extended Gl_2 synthesis theorem is reduced to the extended H_∞ theorem in de Oliverira *et al.*(2002; 1999b). Therefore, the new framework is indeed an extension of the standard optimization framework.

Multi-objective controller synthesis

The multi-objective optimization problem based on the synthesis results will be discussed here. The goal is to compute a dynamical output feedback controller Eq.(14) that meets various constraints on the closed-loop behavior. Typically, these specifications are defined for particular channels or channel combinations. More precisely, by defining the associated pairs of signals as:

$$w_j := \boldsymbol{R}_j w \qquad z_j := \boldsymbol{L}_j z$$

where the matrices $\boldsymbol{L}_j$, $\boldsymbol{R}_j$ select the appropriate input/output channels or channel combinations, each specification or objective is formulated relative to some closed transfer functions of the form

$$\boldsymbol{T}_{z_j w_j}(\varsigma) := \boldsymbol{L}_j \boldsymbol{T}_{zw}(\varsigma) \boldsymbol{R}_j$$

The closed-loop subsystems $\boldsymbol{T}_{z_j w_j}$ for different channels are given by:

$$\boldsymbol{T}_{z_j w_j} := \left[\begin{array}{c|c} \tilde{\boldsymbol{A}} & \tilde{\boldsymbol{B}}_j \\ \hline \tilde{\boldsymbol{C}}_j & \tilde{\boldsymbol{D}}_j \end{array}\right] = \left[\begin{array}{cc|c} \boldsymbol{A}+\boldsymbol{B}_u\boldsymbol{D}_c\boldsymbol{C}_y & \boldsymbol{B}_u\boldsymbol{C}_c & \boldsymbol{B}_j+\boldsymbol{B}_u\boldsymbol{D}_c\boldsymbol{F}_j \\ \boldsymbol{B}_c\boldsymbol{C}_y & \boldsymbol{A}_c & \boldsymbol{B}_c\boldsymbol{F}_j \\ \hline \boldsymbol{C}_j+\boldsymbol{E}_j\boldsymbol{D}_c\boldsymbol{C}_y & \boldsymbol{E}_j\boldsymbol{C}_c & \boldsymbol{D}_j+\boldsymbol{E}_j\boldsymbol{D}_c\boldsymbol{F}_j \end{array}\right] \tag{22}$$

with the dynamical matrices of relevant subsystems $\boldsymbol{B}_j:=\boldsymbol{B}_w\boldsymbol{R}_j$, $\boldsymbol{C}_j:=\boldsymbol{L}_j\boldsymbol{C}_z$, $\boldsymbol{D}_j:=\boldsymbol{L}_j\boldsymbol{D}_{zw}\boldsymbol{R}_j$, $\boldsymbol{E}_j:=\boldsymbol{L}_j\boldsymbol{D}_{zu}$, $\boldsymbol{F}_j:=\boldsymbol{D}_{yw}\boldsymbol{R}_j$. The performance and robustness can be ensured by constraining the GH_2 and Gl_2 norms of transfer functions from exogenous input w_j to regulated output z_j.

In this synthesis problem, variables to be tackled are of two types: the controller parameters and some auxiliary variables such as $\boldsymbol{G}_j$ and $\boldsymbol{P}_j$. Because there exist different $\boldsymbol{G}_j$ in different LMI, directly applying the results of Theorems 5, 6 and 7 to each closed-loop system will lead to different controllers. Therefore, to force controllers of different channels to be identical, extra constraints are imposed:

$$\boldsymbol{G}_j := \boldsymbol{G}, j \in \{1, 2, \cdots, n\}$$

This constraint also restricts the order of the controller. Meanwhile, it guarantees that the joint problem is an LMI optimization readily to be solved efficiently by a convex optimization program. This method is denoted the G-shaping paradigm by de Oliverira (1999b; 2002) analogous to the Lyapunov-shaping paradigm used for the standard synthesis problems in (Scherer *et al.*, 1997). Comparing with the standard multi-objective control synthesis problems presented in (Scherer *et al.*, 1997; Masubuchi *et al.*, 1998), some conservativeness is still present in the new approach due to the extra constraints (23). However, it has been reduced because there is no symmetric restriction on $\boldsymbol{G}_j$, and $\boldsymbol{P}_j$, $\boldsymbol{H}_j$ or $\boldsymbol{J}_j$ could be different in various LMI constraints. Additionally, these extra Lyapunov matrices provide more degrees of freedom in search of a controller in the new framework.

Specifically, a two-channel mixed Gl_2/GH_2 problem is considered as follows.

Proposition 1 (Mixed Gl_2/GH_2 Synthesis) The two-channel multi-objective synthesis problem is to minimize

$$k_1 \left\|\boldsymbol{T}_{z_1 w_1}(\varsigma)\right\|_{\mathrm{G}l_2}^2 + k_2 \left\|\boldsymbol{T}_{z_2 w_2}(\varsigma)\right\|_{\mathrm{G}H_2}^2$$

subject to

$$\left\|\boldsymbol{T}_{z_1 w_1}(\varsigma)\right\|_{\mathrm{G}l_2} < \alpha, \left\|\boldsymbol{T}_{z_2 w_2}(\varsigma)\right\|_{\mathrm{G}H_2} < \beta$$

and the closed loop poles located inside the sub-region $\boldsymbol{C}_D$.

In this proposition, Gl_2 plays the same stabilizing role as H_∞ in a mixed H_∞/H_2 problem. This synthesis framework achieves the optimal performance of channel $\boldsymbol{T}_{z_2 w_2}(\varsigma)$ while guaranteeing a certain level of robust stability and the system transient behavior. Selecting appropriate coefficients k_1 and k_2, one can compromise the performance of two channels. Given some β, if the performance level α is predefined, then the optimization problem is to find the feasible solution under constraints stated in Theorems 5 to 7. Otherwise, by absorbing α into Eqs.(6) and (7) and denoting it another decision variable, the problem then becomes a minimization problem with constraints. Programs for this new multi-channel multi-objective synthesis framework have been developed in MATLAB using LMI Toolbox (Gahinet *et al.*, 1995). Both the 'pure' Gl_2 and the mixed Gl_2/GH_2 problems have been successfully applied to the controller synthesis for an evaporator process (Yan and Cao, 2002a; 2002b).

Remark 5 Comparing to the standard multi-objective optimization problem with Lyapunov-shaping (Scherer *et al*., 1997), the total number of decision variables in G-shaping mixed Gl_2/GH_2 synthesis increases by $6n_x^2$. In order to eliminate the decision variables from the LMI's, especially for the high order system, it is recommended to let $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{S}$ be symmetric so as to reduce the total number of decision variables by $1.5n_x^2$.

Remark 6 The multi-objective optimization problem proposed here is only an application of the new stability condition and parameterization. Actually, many multi-objective problems arise from advanced control issues, such as some combinations of H_2, GH_2, H_∞, Gl_2, l_1 and so on, can be developed based on the G-shaping paradigm as well. For example, de Oliverira *et al*.(1999b; 2002) employed the same concept to a multi-channel multi-objective H_∞ and H_2 optimization problem.

CONCLUSION

A new mixed Gl_2/GH_2 multi-channel multi-objective robust control synthesis approach has been proposed. It is jointly based on the work of Gl_2 by D'Andrea (1999) and the G-shaping paradigm by de Oliverira (1999b; 2002). The new method can deal with disturbance and uncertainty simultaneously in a unified LMI form rather than D-K iteration in μ synthesis. Based on the G-shaping paradigm, the controller parameterization does not depend on the Lyapunov symmetric matrix $\boldsymbol{P}$, which remains in the inequality just as an extra optimization variable. This results in a reduction of the conservativeness involved in a standard design framework. The design freedom is improved as well. Although the approach presented in this paper is for discrete time systems, it can be easily extended to continuous time systems as well.

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