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Mathematical model of cylindrical form tolerance*

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Abstract: Tolerance is essential for integration of CAD and CAM. Unfortunately, the meaning of tolerances in the national standard is expressed in graphical and language forms and is not adaptable for expression, processing and data transferring with computers. How to interpret its semantics is becoming a focus of relevant studies. This work based on the mathematical definition of form tolerance in ANSI Y14.5.1M-1994, established the mathematical model of form tolerance for cylindrical feature. First, each tolerance in the national standard was established by vector equation. Then on the foundation of tolerance's mathematical definition theory, each tolerance zone's mathematical model was established by inequality based on degrees of feature. At last the variance area of each tolerance zone is derived. This model can interpret the semantics of form tolerance exactly and completely.

Key words: Tolerance zone, Form tolerance, Mathematical definition, Mathematical model
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INTRODUCTION

Tolerance is defined by graphic example and text description in the national standard and is not adaptable for expression, processing and transferring in computers. Tolerance is essential for integration of CAD and CAM and how to interpret its semantics is becoming a focus (Wu and Yang, 1999; Liu *et al.*, 2001). The domain of mathematical model of planer feature has been the topic of active research in recent years (Cai *et al.*, 2000; 2002). Gou *et al.*(1999) developed a geometric theory which unifies the formulation and computation of form, profile and orientation tolerances stipulated in ANSI Y14.5M standard. Roy and Li (1998; 1999) presented a solid modeler scheme for representing form tolerances for polyhedral objects, and developed a complete form tolerance zone definition based on rigorous mathematical formulation. Liu *et al.*(2001) gave a generic mathematical model of size tolerance for plane feature based on mathematical definition. This work is aimed at developing a mathematical model of form tolerance for cylindrical feature (other than plane feature), which consists of circularity, cylindricity and straightness.

SPACTIAL MOVEMENT POINT

Tolerance zone means the region or area which limits the real feature's movement. Real feature can also be called feature with error, and is composed of points in the nominal feature after some ways of movement. The spatial movement of a point is defined by the product of its transfer and rotation in measuring geometry and can be described by a homogeneous matrix. Suppose a point in the three-dimensional Euclidean space. The translation range along the axle is δ_x, δ_y, δ_z and the rotation

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range round the axle is δ_θ, δ_ϕ, δ_ψ respectively. $\boldsymbol{p}$ is the point after the movement. Then the point movement in the space expressed by the 4×4 homogeneous matrix is as follows:

$$\boldsymbol{p} = \boldsymbol{D} \cdot \boldsymbol{p}_0 \tag{1}$$

where: $\boldsymbol{D}(\delta_x,\delta_y,\delta_z,\delta_\theta,\delta_\phi,\delta_\psi) =$

$$\begin{bmatrix} \cos\delta_\psi\cos\delta_\phi & -\sin\delta_\phi\cos\delta_\theta+\cos\delta_\psi\sin\delta_\phi\sin\delta_\theta & \sin\delta_\psi\sin\delta_\theta+\cos\delta_\psi\sin\delta_\phi\cos\delta_\theta & \delta_x \\ \sin\delta_\psi\cos\delta_\phi & \cos\delta_\psi\cos\delta_\theta+\sin\delta_\psi\sin\delta_\phi\sin\delta_\theta & -\cos\delta_\psi\sin\delta_\theta+\sin\delta_\psi\sin\delta_\phi\cos\delta_\theta & \delta_y \\ -\sin\delta_\phi & \cos\delta_\phi\sin\delta_\theta & \cos\delta_\phi\cos\delta_\theta & \delta_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Suppose δ_θ, δ_ϕ, δ_ψ is very small and these trigonometric functions can be linearized as follows:

$$\lim_{\delta\to 0}\sin\delta_\theta=\delta_\theta,\ \lim_{\delta\to 0}\sin\delta_\phi=\delta_\phi,\ \lim_{\delta\to 0}\sin\delta_\psi=\delta_\psi;$$

$$\lim_{\delta\to 0}\cos\delta_\theta=1,\ \lim_{\delta\to 0}\cos\delta_\phi=1,\ \lim_{\delta\to 0}\cos\delta_\psi=1.$$

Regarding Eq.(2), the result is:

$$\boldsymbol{D}(\delta_x,\delta_y,\delta_z,\delta_\theta,\delta_\phi,\delta_\psi)=\begin{bmatrix} 1 & -\delta_\psi & \delta_\phi & \delta_x \\ \delta_\psi & 1 & -\delta_\theta & \delta_y \\ -\delta_\phi & \delta_\theta & 1 & \delta_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

Tolerance zone is the permittable movement range of the nominal point. If $\boldsymbol{p}_0$ is a nominal point and $\boldsymbol{D}(\delta_x,\delta_y,\delta_z,\delta_\theta,\delta_\phi,\delta_\psi)$ is the permittable movement range of the nominal point then $\boldsymbol{p}$ is the point in the tolerance zone.

This work aimed at studying the tolerance of the cylindrical feature. Tolerance of cylindrical feature divides into the constraint on the cylinder surface and constraint on the cylindrical axis. The degree of cylinder surface is the same as that of the cylindrical axis, so that the spatial movement of the cylinder surface and of the cylindrical axis can be expressed as follows in unison:

$$\boldsymbol{D}_{\text{Cylinder}}(0,\delta_y,\delta_z,0,\delta_\phi,\delta_\psi)\cdot\boldsymbol{p}_0 = \begin{bmatrix} 1 & -\delta_\psi & \delta_\phi & 0 \\ \delta_\psi & 1 & 0 & \delta_y \\ -\delta_\phi & 0 & 1 & \delta_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix} \tag{4}$$

$\boldsymbol{D}_{\text{Cylinder}}$ was abbreviated as $\boldsymbol{D}$ since the main study object is cylinder. For the same feature, the matrix equation of different tolerance can be the same. But the movement ranges of degree variables in the equations are generally different. So the most important thing is to establish the constraint model of each degree variable according to tolerance definition. First, based on the mathematical definition of form tolerance in ANSI Y14.5.1M-1994, each of the tolerances in the national standard was established by vector equations. Then on the foundation of tolerance mathematical definition theory, each tolerance zone's mathematical model was established by inequalities based on the degree of the feature. Because space is limited, this paper gives only mathematical modeling of circularity and cylindricity.

Each of the symbols are defined as follows:

$\boldsymbol{C}_{\text{d}}$ is the direction vector of tolerance zone; $\boldsymbol{C}_{\text{P}}$ is the position vector of tolerance zone; $\boldsymbol{p}$ is a point in the tolerance zone; r is the normal radius of the cylinder; es, ei are the upper and lower specification limits of the cylinder's diameter; T is the form tolerance; d_x, d_y, d_z are the translation range of the coordinate origin along the axle x, y, z respectively; d_θ, d_ϕ, d_ψ are the rotation range of the coordinate origin round the axle x, y z respectively.

MATHEMATICAL MODEL OF CIRCULARITY TOLERANCE

In the ANSI Y14.5.1M-1994, circularity tolerance specifies that all points of each circular element of the surface must lie in some zone bounded by two concentric circles whose radii differ by the specified tolerance.

A circularity tolerance controls the form error of a real circle relative to the nominal circle. As shown in Fig.1, the circularity tolerance zone consists of an annular area, or the area between two concentric circles that are centered on the spine. The difference in radius between these circles is the circularity tolerance T_{Cir}. The zone can be written in the following vector form:

$$\begin{cases} \boldsymbol{C}_{\mathrm{d}} \cdot (\boldsymbol{p} - \boldsymbol{C}_{\mathrm{p}}) = 0 \\ \left| \left\| \boldsymbol{p} - \boldsymbol{C}_{\mathrm{p}} \right| - r \right| \le T_{\mathrm{Cir}} / 2 \end{cases} \quad (5)$$

Consider Fig.1 where the origin of the Local Coordinate System (LCS) is the center of the circle. We have $\boldsymbol{C}_{\mathrm{p}}=0$ and $\boldsymbol{C}_{\mathrm{d}}=[1\ 0\ 0]^{\mathrm{T}}$, then Eq.(5) can be rewritten as:

$$-T_{\mathrm{Cir}} / 2 \le \sqrt{(y+\delta_y)^2 + (z+\delta_z)^2} - r \le T_{\mathrm{Cir}} / 2 \quad (6)$$

where: $-T_{\mathrm{Cir}}/2 \le \delta_y \le T_{\mathrm{Cir}}/2$, $-T_{\mathrm{Cir}}/2 \le \delta_z \le T_{\mathrm{Cir}}/2$.

Usually, the radius of the circularity tolerance zone is not equal to the radius of the size tolerance and the only link between the size tolerance and the circularity tolerance is the classical equation $T_{\mathrm{s}}>T_{\mathrm{cir}}$. The position and direction of the circularity tolerance zone are indefinite and the only restriction is that there exists at least one circle in the intersection area of the size tolerance zone and the circularity tolerance zone. Fig.2 illustrates eight possible extreme configurations of the resultant zone in *yoz* plane. T_{s} is the size tolerance in Fig.2. Fig.2a–2f satisfy the Eq.(5) while Fig.2g and Fig.2h do not.

Fig.3 shows the situation when the circularity tolerance zone has the maximum transfer relative to the size tolerance zone along axle *y*. The circle $\boldsymbol{C}_{\mathrm{Cir,i}}(y,z)$ and circle $\boldsymbol{C}_{\mathrm{Cir,o}}(y,z)$ shown in Fig.3 are the inner and outer boundary of the circularity tolerance zone respectively; the circle $\boldsymbol{C}_{\mathrm{S,i}}(y,z)$ and circle $\boldsymbol{C}_{\mathrm{S,o}}(y,z)$ are the intersection lines of the inner/outer boundary of the size tolerance zone and the cross section. Thus the following equations can be derived:

$$\boldsymbol{C}_{\mathrm{Cir,o}}(y,z): (y-d_y)^2 + (z-d_z)^2 = (r + T_{\mathrm{Cir}}/2)^2 \quad (7)$$

$$\boldsymbol{C}_{\mathrm{Cir,i}}(y,z): (y-d_y)^2 + (z-d_z)^2 = (r - T_{\mathrm{Cir}}/2)^2 \quad (8)$$

$$\boldsymbol{C}_{\mathrm{S,o}}(y,z): y^2 + z^2 = (r + es/2)^2 \quad (9)$$

$$\boldsymbol{C}_{\mathrm{S,i}}(y,z): y^2 + z^2 = (r + ei/2)^2 \quad (10)$$

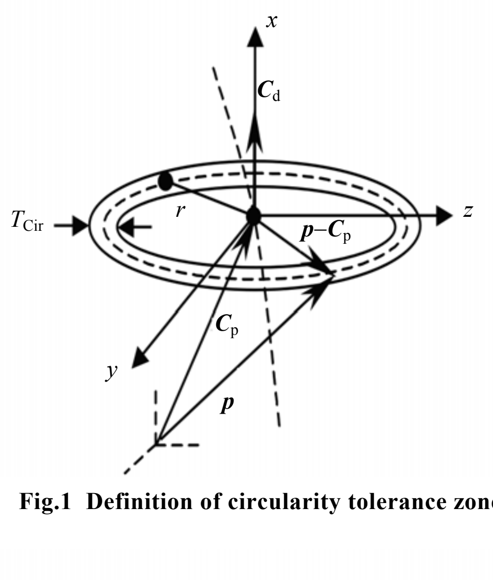

Fig.1 Definition of circularity tolerance zone

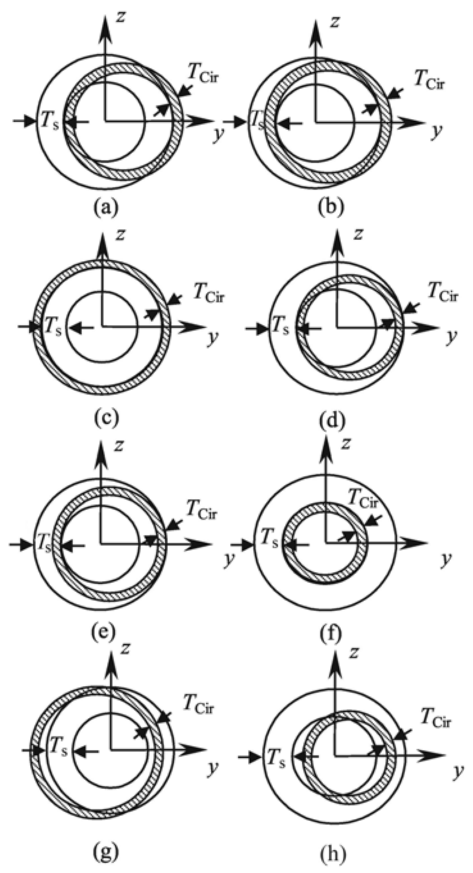

Fig.2 Some examples of size and circularity tolerance zone

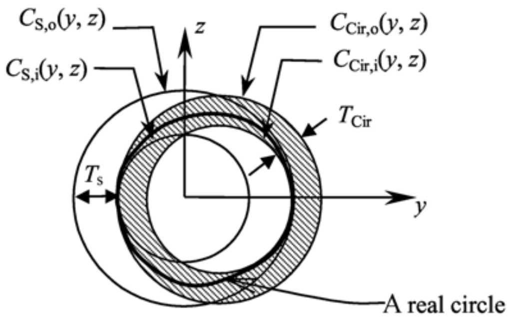

Fig.3 Maximum shift of circularity tolerance

When d_y has the maximum value $d_{y,\max}$, point $(-r-ei/2,0)$ is on the circle $\boldsymbol{C}_{\text{Cir,o}}(y,z)$, and point $(r+es/2,0)$ is on the circle $\boldsymbol{C}_{\text{Cir,i}}(y,z)$. Therefore:

$$(-r-ei/2-d_{y,\max})^2+(0-d_z)^2=(r+T_{\text{Cir}}/2)^2 \quad (11)$$

$$(r+es/2-d_{y,\max})^2+(0-d_z)^2=(r-T_{\text{Cir}}/2)^2 \quad (12)$$

From Eq.(11) and Eq.(12), the maximum transfer along axle y is:

$$d_{y,\max}=\frac{(es^2-ei^2)/4+(es-ei+2T_{\text{Cir}})r}{4r+es+ei}$$

And the maximum transfer along axle z can also be derived following the same procedure as above:

$$d_{z,\max}=\frac{(es^2-ei^2)/4+(es-ei+2T_{\text{Cir}})r}{4r+es+ei}$$

Thus, the degree variables' variation zone of circularity tolerance zone's axles is as follows:

$$\begin{cases} -\dfrac{(es^2-ei^2)/4+(es-ei+2T_{\text{Cir}})r}{4r+es+ei}\le d_y \\ \qquad\le\dfrac{(es^2-ei^2)/4+(es-ei+2T_{\text{Cir}})r}{4r+es+ei} \\ -\dfrac{(es^2-ei^2)/4+(es-ei+2T_{\text{Cir}})r}{4r+es+ei}\le d_z \\ \qquad\le\dfrac{(es^2-ei^2)/4+(es-ei+2T_{\text{Cir}})r}{4r+es+ei} \end{cases} \quad (13)$$

MATHEMATICAL MODEL OF CYLINDERICITY TOLERANCE

A cylindricity tolerance controls the form error of cylindrically shaped features. The cylindricity tolerance zone consists of a set of points between a pair of coaxial cylinders. The difference in radii equals the cylindricity tolerance T_{Cy}. We mathematically define a cylindricity tolerance zone as follows.

$$\left|\left|\boldsymbol{C}_{\text{d}}\times(\boldsymbol{p}-\boldsymbol{C}_{\text{p}})\right|-r\right|\le T_{\text{Cy}}/2 \quad (14)$$

If we consider the local coordinate system as illustrated in Fig.4, $\boldsymbol{C}_{\text{p}}=0$, $\boldsymbol{C}_{\text{d}}=[1\ 0\ 0]^{\text{T}}$. Therefore:

$$-T_{\text{Cy}}/2\le\sqrt{(y+\delta_y+x\cdot\delta_\psi)^2+(z+\delta_z+x\cdot\delta_\phi)^2}-r \le T_{\text{Cy}}/2 \quad (15)$$

where:

$$-T_{\text{Cy}}/2\le\delta_y\le T_{\text{Cy}}/2\,,\ -T_{\text{Cy}}/2\le\delta_z\le T_{\text{Cy}}/2\,,$$

$$-T_{\text{Cy}}/2l\le\delta_\phi\le T_{\text{Cy}}/2l\,,\ -\frac{T_{\text{Cy}}}{2l}\le\delta_\psi\le\frac{T_{\text{Cy}}}{2l}\,.$$

The position and direction of the cylindricity tolerance zone are not determined. And the only restriction is that there exist at least one cylinder in intersection area of the size tolerance zone and the cylindricity tolerance zone. Fig.5 illustrates eight possible extreme configurations of the intersection zone in the *yoz* plane. Fig.5a–5f satisfy the Eq.(14) while Fig.5g and Fig.5h do not.

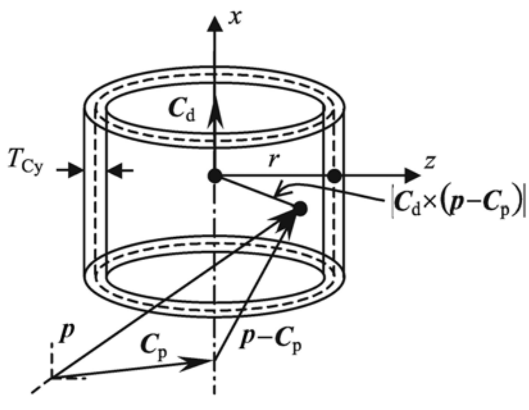

Fig.4 Cylindricity tolerance zone definition

Fig.6 shows the situation when the cylindricity tolerance zone has the maximum orientation angle relative to the size tolerance. The following equations can be derived from Fig.6:

$$l_{\mathrm{Cy,i}}(x,z): z = r + x \cdot d_\phi + d_z \tag{16}$$

$$l_{\mathrm{Cy,o}}(x,z): z = r + x \cdot d_\phi + d_z + \frac{T_{\mathrm{Cy}}}{\cos d_\phi} \tag{17}$$

$$l_{\mathrm{S,i}}(x,z): z = r + ei/2 \tag{18}$$

$$l_{\mathrm{S,o}}(x,z): z = r + es/2 \tag{19}$$

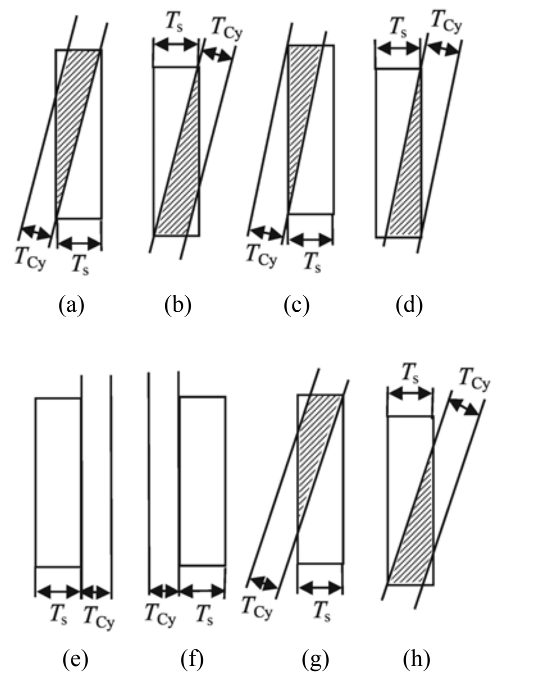

Fig.5 Some examples of size and cylindricity tolerance zone

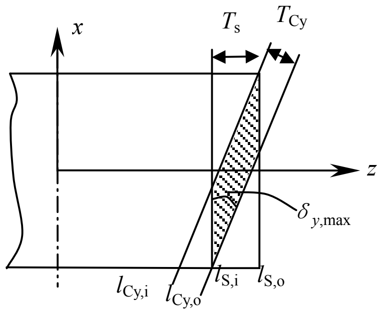

Fig. 6 Maximum orientation angle of cylindricity tolerance zone

When d_ϕ has the maximum value, point ($l/2$, $r+es/2$) is on line $l_{\mathrm{Cy,i}}(x,z)$, and the point ($-l/2$, $r+ei/2$) is on line $l_{\mathrm{Cy,o}}(x,z)$. From Eqs.(12) and (13):

$$r + es/2 = r + l \cdot d_{\phi,\max}/2 + d_z \tag{20}$$

$$r + ei/2 = r + (-l/2) \cdot d_{\phi,\max} + d_z + \frac{T_{\mathrm{Cy}}}{\cos d_{\phi,\max}} \tag{21}$$

Assuming that the variance of $d_{\phi,\max}$ is small, then $\cos d_{\phi,\max} \approx 1$. From the above equations,

$$d_{\phi,\max} = [(es - ei) - 2T_{\mathrm{Cy}}]/2l = (T_{\mathrm{s}} - 2T_{\mathrm{Cy}})/2l$$

The final variation zone for other cases can be obtained in a similar way. Therefore, the variation zone of the cylindricity tolerance zone as specified in the tolerance specification is as follows:

$$\begin{cases} -(T_{\mathrm{s}} - 2T_{\mathrm{Cy}})/2l \le d_\phi \le (T_{\mathrm{s}} - 2T_{\mathrm{Cy}})/2l \\ -(T_{\mathrm{s}} - 2T_{\mathrm{Cy}})/2l \le d_\psi \le (T_{\mathrm{s}} - 2T_{\mathrm{Cy}})/2l \\ ei - T_{\mathrm{Cy}} \le d_y \le es \\ ei - T_{\mathrm{Cy}} \le d_z \le es \end{cases} \tag{22}$$

INSTANCE

Fig.7 shows the connecting rod of air compressor brake for automobile. The hole, whose diameter is 12 mm, must accommodate the piston pin to be inserted into it and the cylindricity error should not exceed 0.003 mm. If inappropriate size tolerance is selected to control the form error, it will

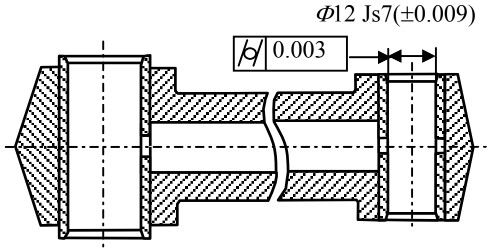

Fig.7 Connecting rod of air compressor brake for automobile

make manufacture more difficult. The automobile is a large size product with relatively big size tolerance Js7(±0.009), so smaller cylindricity tolerance (0.003 mm) should be given according to real production experience and assembly in groups to meet the function request. Next we are going to give the mathematical model of the hole tolerances. From the inequality Eq.(15), each degree variables' movement range of cylindricity tolerance can be derived as follows:

$$-0.0015 \le \sqrt{(y+\delta_y+x\cdot\delta_\psi)^2+(z+\delta_z+x\cdot\delta_\phi)^2}-6 \le 0.0015 \tag{23}$$

where: $-0.0015 \le \delta_y \le 0.0015, -0.0015 \le \delta_z \le 0.0015$, $-0.00005 \le \delta_\phi \le 0.00005$, $-0.00005 \le \delta_\psi \le 0.00005$.

The inequality in Eq.(23) shows points in the cylindricity tolerance zone and determines the size and form of the cylindricity tolerance zone. From the inequality in Eq.(22), each degree variables' movement range of the cylindricity tolerance zone's coordinates system are as follows:

$$\begin{cases} -0.0002 \le d_\phi \le 0.0002 \\ -0.0002 \le d_\psi \le 0.0002 \\ -0.012 \le d_y \le 0.009 \\ -0.012 \le d_z \le 0.009 \end{cases} \tag{24}$$

Eq.(24) determines the position and direction of the cylindricity tolerance zone. Based on the mathematical model above, we can derive the simulation cylinder of the hole.

CONCLUSION

Tolerance is essential for integration of CAD and CAM. How to interpret its semantics is becoming a focus of relevant study (Srinivasan, 1993). In this paper, based on the mathematical definition of form tolerance in ANSI Y14.5.1M-1994, the mathematical model of form tolerance for cylindrical feature was established. This model can interpret the semantics of form tolerance exactly and completely.

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