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The construction and combined operation for fuzzy consistent matrixes*

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Abstract: Fuzziness is one of the general characteristics of human thinking and objective things. Introducing fuzzy techniques into decision-making yields very good results. Fuzzy consistent matrix has many excellent characteristics, especially center-division transitivity conforming to the reality of the human thinking process in decision-making. This paper presents a new approach for creating fuzzy consistent matrix from mutual supplementary matrix in fuzzy decision-making. At the same time, based on the distance between individual fuzzy consistent matrix and average fuzzy consistent matrix, a kind of combined operation for several fuzzy consistent matrixes is presented which reflects most opinions of experienced experts. Finally, a practical example shows its flexibility and practicability further.

Key words: Fuzzy consistent matrix, Fuzzy consistent relation, Compound operation
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INTRODUCTION

Fuzziness is one of the general characteristics of human thinking and objective things. On the one hand, there are a lot of decision-making problems difficult to quantify in human activity. On the other hand, in human thought activity, people exchange information in fuzzy language, then carry out inference and integrate judgement, finally implement decision-making. Combining fuzzy intuition with strict and logistic reasoning is one of the characteristics of human thinking activity. Therefore, it is reasonable and necessary to introduce fuzzy techniques in decision-making. Fuzzy techniques are used for fuzzy decision-making (Wang and Li, 1996; Smolikova and Wachowiak, 2002; Niskanen, 2002).

Fan *et al*.(2002) proposed a new approach to solve the multiple attribute decision making (MADM) problem, where the decision maker gives his or her preference on alternatives in a fuzzy relation. To reflect the decision maker's preference information, an optimization model is constructed to assess the attribute weights and then to select the most optimal alternatives. Herrera *et al*.(2002) devised a decision process for resolving selection problem in conditions of uncertainty, supplying a linguistic decision model for evaluating the satisfaction of the objectives by the potential solutions. Williams and Steele (2002) introduced the notion of a fuzzy index to model sets of linguistic terms for which there is no formal measurement scale. Zhou *et al*.(2002) proposed a fuzzy set approach that integrates objective and subjective information for evaluating grades of journals, which provided a comprehensive method for dealing with incomplete and imprecise information to support the whole evaluation process.

In the opinion of set theory, decision-making is very closely associated with relation. The decision-making process deals with relations among affair discourse, countermeasure discourse and benefit

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discourse. We present a new kind of fuzzy relation for the first time, i.e. fuzzy consistent relation (Yao and Huang, 1998). The fuzzy consistent relation on finite discourse may be expressed by fuzzy consistent matrix. Fuzzy consistent relation has many special characteristics, especially center-division transitivity, that makes fuzzy consistent relation conform to the psychological characteristics of human decision-making. Therefore, fuzzy consistent relation may be used as theoretical foundation for solving certain decision-making problems, and thus find broad applications in soft science such as fuzzy similar selection, fuzzy judgement, analytical hierarchical process, weighting analysis, etc., so fuzzy consistent matrix attracts the interest of many scholars, and is cited by them (Xu, 1999; Zhang, 2000; Li, 2001).

FUZZY CONSISTENT MATRIX AND ITS PROPERTIES

Definition 1 Assume there is a discourse $U=\{u_i|i\in I\}$, $I=\{1, 2, \ldots\}$ is an index set, a fuzzy subset in $U\times U$

$$\boldsymbol{R}:\ U\times U\rightarrow[0,1] \tag{1}$$

or

$$\boldsymbol{R}=\sum_{(i,j)}\frac{\mu_R(u_i,u_j)}{(u_i,u_j)} \tag{2}$$

is called fuzzy relation in U.

Definition 2 Let $\boldsymbol{R}$ be a fuzzy relation in U; if for any $u_i\in U$, $u_j\in U$, for $\forall k\in I$, if there is

$$\mu_R(u_i, u_j)=\mu_R(u_i, u_k)-\mu_R(u_j, u_k)+0.5 \tag{3}$$

then $\boldsymbol{R}$ is called fuzzy consistent relation.

It is necessary to note that the fuzzy consistent relation conforms consistently to the thinking process of human decision-making.

Definition 3 Let $U=\{u_1, u_2, \ldots, u_m\}$, then the fuzzy consistent relation $\boldsymbol{R}$ may be denoted by a fuzzy consistent matrix, i.e.

$$\boldsymbol{R}=(r_{ij})_{m\times m}$$

here

$$r_{ij}=\mu_R(u_i, u_j) \tag{4}$$

Apparently, the elements in fuzzy consistent matrix $\boldsymbol{R}$ satisfy

$$r_{ij}=r_{ik}-r_{jk}+0.5 \quad \forall k \tag{5}$$

Besides the general properties of fuzzy relation, fuzzy consistent relation has many special characteristics, especially center-division transitivity as described in the following theorem.

Theorem 1 Fuzzy consistent matrix $\boldsymbol{R}=(r_{ij})_{m\times m}$ has the following properties,

(1) $r_{ii}=0.5$;

(2) The sum of elements in column i and row i equals to m;

(3) $\boldsymbol{R}^{\mathrm{T}}=\boldsymbol{R}^{\mathrm{C}}$, $\boldsymbol{R}^{\mathrm{T}}$ and $\boldsymbol{R}^{\mathrm{C}}$ are transpose and complementary matrix of $\boldsymbol{R}$, respectively. They are fuzzy consistent matrixes, too;

(4) The sub-matrix obtained by deleting any row and corresponding column from $\boldsymbol{R}$, is still fuzzy consistent matrix;

(5) $\boldsymbol{R}$ satisfies center-division transitivity,

(a) When $\lambda\geq0.5$, if $r_{ij}\geq\lambda$, $r_{jk}\geq\lambda$ then $r_{ik}\geq\lambda$;

(b) When $\lambda\leq0.5$, if $r_{ij}\leq\lambda$, $r_{jk}\leq\lambda$ then $r_{ik}\leq\lambda$.

It is easy to prove the above theorem by means of Definition 2 and Definition 3. Readers may also find the proof of Theorem 1 in Yao and Huang (1998). It is necessary to note that the center-division transitivity conforms is consistent with the thinking process of human decision making, namely,

(a) Let $\lambda>0.5$, if u_i is more important than u_j ($r_{ij}\geq\lambda$), u_j is more important than u_k ($r_{jk}\geq\lambda$), then u_i is more important than u_k ($r_{ik}\geq\lambda$) certainly;

(b) Let $\lambda<0.5$, if u_i is less important than u_j ($r_{ij}\leq\lambda$), u_j is less important than u_k ($r_{jk}\leq\lambda$), then u_i is less important than u_k ($r_{ik}\leq\lambda$) certainly.

CREATION OF FCR

Definition 4 Let fuzzy matrix $\boldsymbol{F}=(f_{ij})_{m\times m}$; if

$$f_{ij}+f_{ji}=1 \tag{6}$$

then matrix $\boldsymbol{F}$ is fuzzy mutual supplementary matrix.

Theorem 2 Fuzzy consistent matrix must be fuzzy mutual supplementary matrix.

Proof Based on Definition 2, $r_{ij}=r_{ik}-r_{jk}+0.5$, then

$$r_{ij}+r_{ji}=(r_{ik}-r_{jk}+0.5)+(r_{jk}-r_{ik}+0.5)=0.5+0.5=1$$

Generally speaking, in the process of fuzzy decision-making, the estimation matrix constructed by decision-maker is commonly a fuzzy mutual supplementary matrix $\boldsymbol{F}=(f_{ij})_{m\times m}$, not a fuzzy consistent matrix. For example, assume that the object set to be estimated is $U=\{u_1, u_2, \ldots, u_m\}$. The estimation matrix $\boldsymbol{F}=(f_{ij})_{m\times m}$ is constructed by binary antitheses, here

$$f_{ij}=\begin{cases}0 & \text{if } u_j \text{ outbalances } u_i\\ 0.5 & \text{if } u_i \text{ is the same as } u_j\\ 1.0 & \text{if } u_i \text{ outbalances } u_j\end{cases}$$

then the matrix $\boldsymbol{F}$ is a fuzzy mutual supplementary matrix.

The remaining work is how to rebuild fuzzy consistent matrix from mutual supplementary matrix? Yao and Huang (1998) presented an approach for this task. Here, based on the characteristics of fuzzy mutual supplementary matrix and fuzzy consistent matrix, we propose another approach for constructing fuzzy consistent matrix, as described in the following theorem.

Theorem 3 If summing by column in fuzzy mutual supplementary matrix $\boldsymbol{F}=(f_{ij})_{m\times m}$, written as

$$r_j=\sum_{k=1}^{m} f_{kj} \qquad j=1, \ldots, m \tag{7}$$

and performing the following mathematical transform

$$r_{ij}=0.5-\frac{r_i-r_j}{2m} \tag{8}$$

Then the so-produced matrix $\boldsymbol{R}=(r_{ij})_{m\times m}$ is fuzzy consistent.

Proof (1) $r_{ij}+r_{ji}=0.5-\frac{r_i-r_j}{2m}+0.5-\frac{r_j-r_i}{2m}=1$

Thus $\boldsymbol{R}$ is fuzzy mutual supplementary;

(2) $$r_{ij}=0.5-\frac{r_i-r_j}{2m}=0.5-\frac{r_i-r_k-(r_j-r_k)}{2m}$$
$$=0.5-\frac{r_i-r_k}{2m}-(0.5-\frac{r_j-r_k}{2m})+0.5$$
$$=r_{ik}-r_{jk}+0.5$$

Thus $\boldsymbol{R}$ is fuzzy consistent.

The significance of Theorem 3 is that because the creation of fuzzy mutual supplementary matrix is simpler, the estimation matrixes built by decision-maker in the actual decision-making process are commonly fuzzy mutual supplementary matrix. At this time, fuzzy consistent matrix can be rebuilt from fuzzy mutual supplementary matrix by means of Theorem 3.

It is necessary to note that in order to adapt daedal decision-making situations, you can present more reasonable and more effective rebuilding algorithms according to the actual problems to be solved.

COMBINED OPERATION

Definition 5 Assume $\boldsymbol{R}^l=(r_{ij}^l)_{m\times m}$, $l=1,\cdots,n$ be n fuzzy relations in U, if

$$r_{ij}=\sum_{l=1}^{n} w_l r_{ij}^l \tag{9}$$

$$\sum_{l=1}^{n} w_l=1 \tag{10}$$

then fuzzy relation $\boldsymbol{R}=(r_{ij})_{m\times m}$ is called combined relation of $\boldsymbol{R}^l$ (l=1,…,n), written as $\boldsymbol{R}=\boldsymbol{R}^1\oplus\boldsymbol{R}^2\oplus\ldots\oplus\boldsymbol{R}^n$.

Theorem 4 Assume $\boldsymbol{R}^l$ (l=1, …, n) to be fuzzy consistent relations in U, fuzzy relation $\boldsymbol{R}$ is called combined relation of $\boldsymbol{R}^l$ (l=1, …, n), i.e.

$$\boldsymbol{R}=\boldsymbol{R}^1\oplus\boldsymbol{R}^2\oplus\ldots\oplus\boldsymbol{R}^n \tag{11}$$

Then $\boldsymbol{R}$ is also fuzzy consistent.

The proof of Theorem 4 is easy, here is omitted. It is necessary to note that the significance of combined operation of fuzzy consistent relation is that it may synthesize many decision-making results (i.e. many fuzzy consistent relations) effectively, and thus forms the whole fuzzy consistent relation.

Well then, how to determine weight coefficient in combined operation? If the degrees of significance for n individuals are equal, then $w_l=1/n$ is appropriate, the element of the whole fuzzy consistent matrix is

$$r_{ij}=\frac{1}{n}\sum_{l=1}^{n} r_{ij}^l \tag{12}$$

In fact, the weight coefficient w_l (l=1, …, n) in combined operation is not only relevant with personnel weights, but also relevant with the degree of coherence among decision-making results determined by n individuals. The former lies on the decision-making individuals' social status and experience. For example, if some decision-making individual is company leader or principal engineer, then his or her personnel weight should be larger than that of the others. The latter embodies most decision-making opinions adequately so as to prevent the leading function being determined by individual prejudice, especially that of decision-maker with larger personnel weight. If some individual decision-making result differs much from the results of others, then the weight of his or her decision-making opinion in the whole decision-making should be smaller than that of others. The quantitative analysis for coherence degree in decision-making is as follows.

Definition 6 Assume $\boldsymbol{R}^l$ (l=1, …, n) to be n fuzzy consistent matrixes in U, let

$$\overline{r}_{ij}=\frac{1}{n}\sum_{l=1}^{n}r_{ij}^{l} \qquad (13)$$

then fuzzy matrix $\overline{\boldsymbol{R}}=(\overline{r}_{ij})_{m\times m}$ is called average fuzzy matrix.

Theorem 5 Average fuzzy matrix defined by Definition 6 is fuzzy consistent matrix.

Proof

$$\overline{r}_{ij}=\frac{1}{n}\sum_{l=1}^{n}r_{ij}^{l}=\frac{1}{n}\sum_{l=1}^{n}(r_{ik}^{l}-r_{jk}^{l}+0.5)$$

$$=\frac{1}{n}\sum_{l=1}^{n}r_{ik}^{l}-\frac{1}{n}\sum_{l=1}^{n}r_{jk}^{l}+\frac{1}{n}\sum_{l=1}^{n}0.5$$

$$=\overline{r}_{ik}-\overline{r}_{jk}+0.5$$

Definition 7 Assume $\boldsymbol{R}^l$ (l=1, …, n) to be n fuzzy consistent matrixes in U, $\overline{\boldsymbol{R}}$ average fuzzy matrix, let

$$D_l=\left\|\boldsymbol{R}_l-\overline{\boldsymbol{R}}\right\| \qquad (14)$$

be the distance between fuzzy consistent matrix $\boldsymbol{R}^l$ (l=1, …, n) and $\overline{\boldsymbol{R}}$.

Definition 8 If D_l (l=1, …, n) be the distance between fuzzy consistent matrix $\boldsymbol{R}^l$ (l=1, …, n) and their average matrix, then

$$B_l=1-D_l\Big/\sum_{k=1}^{n}D_k \qquad (15)$$

is the coherence degree between fuzzy consistent matrix $\boldsymbol{R}^l$ and $\boldsymbol{R}^k$ (l=1, …, n, $k\neq l$).

Having the coherence degree of fuzzy consistent matrixes, the formula for computing weight coefficients may be given as follows.

Definition 9 Let A_l (l=1, …, n) be the personnel weight in creating n fuzzy consistent matrix $\boldsymbol{R}^l$ (l=1, …, n) in U, then the weight coefficients in the combined operation of fuzzy consistent matrixes $\boldsymbol{R}^l$ is

$$w_l=\alpha A_l+\beta B_l \qquad l=1,\ldots,n \qquad (16)$$

here α and β are appropriate constants satisfying $\alpha+\beta=1$. The above formula can reflect most opinions and embody the status of experienced experts.

Example Let five experts evaluate four schemes S^m(m=1, …, 4) of one task, and the five fuzzy mutual supplementary matrixes $\boldsymbol{F}^n$ (n=1, …, 5) are obtained as follows,

$$\boldsymbol{F}^{1-2}=\begin{bmatrix}0.5&1&1&1\\0&0.5&1&1\\0&0&0.5&1\\0&0&0&0.5\end{bmatrix},\boldsymbol{F}^{3}=\begin{bmatrix}0.5&0.5&0&0\\0.5&0.5&0&0\\1&1&0.5&0.5\\1&1&0.5&0.5\end{bmatrix}$$

$$\boldsymbol{F}^{4}=\begin{bmatrix}0.5&0&0&0\\1&0.5&0&0\\1&1&0.5&0\\1&1&1&0.5\end{bmatrix},\boldsymbol{F}^{5}=\begin{bmatrix}0.5&1&1&0\\0&0.5&0&0\\0&1&0.5&0\\1&1&1&0.5\end{bmatrix}$$

Then they can be transformed into five fuzzy consistent matrixes $\boldsymbol{R}^l$ (l=1, …, 5) by Theorem 3

$$\boldsymbol{R}^{1-2}=\begin{bmatrix}0.5&0.375&0.25&0.125\\0.625&0.5&0.375&0.25\\0.75&0.625&0.5&0.375\\0.875&0.75&0.625&0.5\end{bmatrix}$$

$$\boldsymbol{R}^{3}=\begin{bmatrix}0.5&0.5&0.75&0.75\\0.5&0.5&0.75&0.75\\0.25&0.25&0.5&0.5\\0.25&0.25&0.5&0.5\end{bmatrix}$$

$$\boldsymbol{R}^{4}=\begin{bmatrix}0.5&0.625&0.75&0.875\\0.375&0.5&0.625&0.75\\0.25&0.375&0.5&0.625\\0.125&0.25&0.375&0.5\end{bmatrix}$$

$$\boldsymbol{R}^5=\begin{bmatrix}0.5 & 0.25 & 0.375 & 0.625\\ 0.75 & 0.5 & 0.625 & 0.875\\ 0.625 & 0.375 & 0.5 & 0.75\\ 0.375 & 0.125 & 0.25 & 0.5\end{bmatrix}$$

Their average fuzzy matrix $\overline{\boldsymbol{R}}$ is

$$\overline{\boldsymbol{R}}=\begin{bmatrix}0.5 & 0.425 & 0.475 & 0.5\\ 0.575 & 0.5 & 0.55 & 0.575\\ 0.525 & 0.45 & 0.5 & 0.525\\ 0.5 & 0.425 & 0.475 & 0.5\end{bmatrix}$$

and the coherence degrees between fuzzy consistent matrixes are computed by Eq.(15)

B_1=0.776 B_2=0.776 B_3=0.842
B_4=0.776 B_5=0.829

Let $A_1=A_2=A_3=A_4=A_5=0.2$, $\beta=\alpha=0.5$, then the normalized weight coefficients are computed by Eq.(16)

w_1=0.195 w_2=0.195 w_3=0.209
w_4=0.195 w_5=0.206

By Definition 5, the combined fuzzy consistent matrix is

$$\boldsymbol{R}=\boldsymbol{R}^1\oplus\boldsymbol{R}^2\oplus\boldsymbol{R}^3\oplus\boldsymbol{R}^4\oplus\boldsymbol{R}^5=\begin{bmatrix}0.5 & 0.424 & 0.478 & 0.505\\ 0.576 & 0.5 & 0.554 & 0.581\\ 0.522 & 0.446 & 0.5 & 0.527\\ 0.495 & 0.419 & 0.473 & 0.5\end{bmatrix}$$

Finally, the preferential values of all schemes may be obtained by means of root-squaring method,

v_1=0.4756 v_2=0.5518
v_3=0.4977 v_4=0.4706

Obviously, the order of four schemes from good to bad is $S^4<S^1<S^3<S^2$.

CONCLUSION

From the previous discussion and analysis, we have the following conclusions:

1. Because the creation of fuzzy mutual supplementary matrix is simpler, fuzzy consistent matrix can be rebuilt from fuzzy mutual supplementary matrix by means of certain algorithm.

2. The significance of combined operation of fuzzy consistent relation is that it may synthesize many decision-making results effectively, and thus forms the whole fuzzy consistent relation. The formula for computing weight coefficients can reflect most opinions and embody the status of experienced experts.

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