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Exceptional family of elements and the solvability of complementarity problems in uniformly smooth and uniformly convex Banach spaces

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Abstract: The notion of "exceptional family of elements (EFE)" plays a very important role in solving complementarity problems. It has been applied in finite dimensional spaces and Hilbert spaces by many authors. In this paper, by using the generalized projection defined by Alber, we extend this notion from Hilbert spaces to uniformly smooth and uniformly convex Banach spaces, and apply this extension to the study of nonlinear complementarity problems in Banach spaces.

Key words: Exceptional family of elements (EFE), Banach spaces and complementarity

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INTRODUCTION

New topological method in complementarity theory is now developing. This method is based on the notion of "exceptional family of elements (EFE)" initially defined in 1991–1993 by G. Isac as "radial family of elements". The name of EFE was given in (Isac *et al*., 1997), after the authors found a connection between this notion and the topological degree in R^n.

We note that after 1997, this notion was extended to completely continuous fields and to other classes of mappings in general Hilbert spaces. Now it is known (Isac, 2001) that this notion is related to the Leray-Schauder Alternative, which is one of the most important theorems in Nonlinear Analysis. The notion of EFE for a function in arbitrary Hilbert spaces, has recently been used in several papers for example, (Bulavski *et al*., 1998; Isac, 1998; 1999a; 1999b; 2000a; 2000b; 2000c; 2001; Isac *et al*., 1997; Isac and Carbone, 1999; Isac and Li, 2001; Isac and Obuchowska, 1998; Isac and Zhao, 2000; Kalashnikov, 1995; Kalashnikov and Isac, 2002; Zhao, 1997; 1998; Zhao and Isac, 2000a; 2000b), among others. In this paper we generalize the notion of EFE to uniformly smooth and uniformly convex Banach spaces, by using the "generalized projection operator" defined by Alber (1996) and the Leray-Schauder Alternative for completely continuous mappings. We use this generalization to give some new existence theorems of solutions for complementarity problems in Banach spaces.

PRELIMINARY

Let $(E, \|.\|)$ be a Banach space. We denote by E^* the topological dual of E, and by $\langle E, E^*\rangle$ the natural duality pairing between E^* and E, i.e., $\langle\varphi, x\rangle=\varphi(x)$, where $\varphi\in E^*$ and $x\in E$. We denote by Ω a nonempty, closed and convex subset in E. The set Ω may be a closed convex cone K in E, i.e., K is a closed subset of E satisfying:

(k_1) $\lambda K\subseteq K$, for all $\lambda\geq 0$;

(k_2) $K+K\subseteq K$.

The dual cone K^* of the cone K is by definition:

$$K^*=\{y\in E^*:\langle y, x\rangle\geq 0, \text{ for all } x\in K\}.$$

We need to recall several known definitions. A Banach space $(E, \|.\|)$ is called strictly convex if for two elements $x, y\in E$ which are linearly independent, we have that $\|x+y\|<\|x\|+\|y\|$.

Also, a Banach space $(E, \|.\|)$ is called uniformly convex if and only if, for every sequences $\{x_n\}_{x\in N}$, $\{y_n\}_{y\in N}\subset E$, with $\|x_n\|=\|y_n\|=1$, for any $n\in N$ and such that

$$\|x_n+y_n\|\to 2 \text{ implies } \|x_n-y_n\|\to 0.$$

Equivalently $(E, \|.\|)$ is uniformly convex if and only if, for every $\varepsilon>0$, there exists $\delta>0$ such that for any $x, y\in E$ with $\|x\|=\|y\|=1$ and $\|x-y\|\geq\varepsilon$ we have that $\|x+y\|\leq 2(1-\delta)$.

It is known (Cioranescu, 1990; Takahashi, 2000) that any uniformly Banach space is strictly convex.

A Banach space $(E, \|.\|)$ is smooth if and only if its norm $\|.\|$ is G-differentiable on $E\backslash\{0\}$ (Takahashi, 2000). A Hilbert space is smooth but there exist smooth Banach spaces that are not Hilbert spaces.

Finally, we say that a Banach space $(E, \|.\|)$ is uniformly smooth, if and only if the norm $f(x)=\|x\|$ is F-differentiable and

$$\lim_{t\to 0}\sup_{\|x\|=\|y\|=1}\left|\frac{\|x+ty\|-\|x\|}{t}-<f'(x),y>\right|=0.$$

Also, we recall that a Banach space $(E, \|.\|)$ is uniformly convex if and only if its dual E^* is uniformly smooth, and conversely E is uniformly smooth if and only if E^* is uniformly convex (Cioranescu, 1990; Takahashi, 2000). The Banach space $(E, \|.\|)$ is uniformly convex if and only if the function $f(x)=(1/2)\|x\|^2$ is strictly convex. It is also important, for the results of this paper to recall the notion of "duality mapping" associated with a Banach space.

Let $(E, \|.\|)$ be an arbitrary Banach space. The normalized duality mapping between E and E^* is by definition:

$$J(x)=\{x^*\in E^*: \langle x^*, x\rangle=\|x^*\|\|x\|,\ \|x\|=\|x^*\|\},$$

that is, for any $x^*\in J(x)$, we have

$$<x^*, x>=\|x^*\|_*\|x\|=\|x\|^2,$$

for any $x\in E$, where $\|x^*\|_*$ is the norm in the dual space E^* and $\|x\|$ is the E-norm. Obviously, we have $J{:}E\to 2^{E^*}$.

J is a monotone, coercive, homogeneous, odd and bounded operator in arbitrary Banach spaces. J is also uniformly continuous on each bounded set in uniformly smooth Banach spaces and is the identical operator in Hilbert spaces. It is known that if $(E, \|.\|)$ is uniformly smooth Banach space, then J is a single valued mapping and is continuous from norm topology to norm topology and moreover

$$J(x)=\text{grad}(\|x\|^2/2).$$

Regarding the classes of Banach spaces presented above and the duality mapping, the reader is referred to (Cioranescu, 1990; Takahashi, 2000). We note, as cited in (Alber, 1996), the Banach spaces l^p, L^p and W_m^p, $p\in(0, \infty)$ are uniformly convex and uniformly smooth.

COMPLEMENTARITY PROBLEMS AND VARIATIONAL INEQUALITIES

Let $(E, \|.\|)$ be a Banach space, $\Omega\subset E$ a nonempty, closed and convex subset, $K\subset E$ a closed convex cone and $f{:}E\to E^*$ a mapping, the nonlinear complementarity problem defined by f and K is by definition:

$$NCP(f,K){:}\begin{cases}\text{find } x_*\in K \text{ such that}\\ f(x_*)\in K^* \text{ and } \langle x_*, f(x_*)\rangle=0.\end{cases}$$

The $NCP(f, K)$ is the mathematical model of many problems considered in optimization, game theory, economics, engineering and mechanics, among others. Generally, the $NCP(f, K)$ is related to equilibrium problems (Cottle *et al.*, 1992; Isac, 1992; 2000a; Kalashnikov, 1995). The (classical) variational inequality defined by the mapping f and the set Ω is:

$$VI(f,\Omega){:}\begin{cases}\text{find } x_*\in\Omega \text{ such that}\\ \langle f(x_*), x-x_*\rangle\geq 0, \text{ for every } x\in\Omega.\end{cases}$$

The variational inequalities have many applications in economics, physics and technology. The Variational Inequalities Theory is a popular domain in applied mathematics.

It is known (Cottle *et al*., 1992; Isac, 1992; 2000a) that when Ω is a closed convex cone K, then the problems $NCP(f, K)$ and $VI(f, K)$ are equivalent.

A GENERALIZED PROJECTION OPERATOR

Let $(E, \|.\|)$ be a uniformly convex and uniformly smooth Banach space. Let E^* be the topological dual of E and $\Omega \subset E$ a closed and convex set. We consider the functional $V: E^* \times E \to R$ that is defined in Alber (1996):

$$V(y_*, x)=\|y_*\|_*^2-2\langle y_*, x\rangle+\|x\|^2,$$

where $y_* \in E^*$ and $x \in E$ and $\|.\|_*$ is the norm in E^*.

It is known (Alber, 1996) that the minimization problem

$$\begin{cases} \text{given } y_* \in E^*, \text{ find } x(y_*) \in \Omega \text{ such that} \\ V(y_*, x(y_*))=\inf\limits_{y\in\Omega} V(y_*, y) \end{cases}$$

has a unique solution. The operator $\pi_\Omega: E^* \to \Omega$ defined by $\pi_K(y_*)=x(y_*)$ is called the generalized projection operator.

The generalized projection operator was defined by Alber (1996) and has several interesting properties. For these properties, the reader is referred to (Alber, 1996).

We need to recall the following two results:

Lemma 1 (Alber, 1996) The functional $V(y_*, x)$ is convex with respect to y_* when x is fixed and with respect to x when y_* is fixed.

Proof A proof of this result is given in (Alber, 1996).

Lemma 2 (Bulavski *et al*., 1998) $\mathrm{grad}_x V(y_*, x)=2(J(x)-y_*)$.

Proof This formula is a consequence of the fact that on a uniformly smooth Banach space, we have that

$$J(x)=\mathrm{grad}(\|x\|^2/2).$$

Lemma 3 A differentiable functional $f: E \to \mathrm{R}$ is convex if and only if

$$f(x)-f(x_0) \geq \langle \mathrm{grad}\, f(x_0), x-x_0\rangle,$$

for any $x, x_0 \in E$.

Proof This is a classical result and well known in convex analysis.

The following two results are due to Alber (1996).

Theorem 1 (Alber, 1996) The point y_0 is the generalized projection of $y_* \in E^*$ on Ω, i.e., $y_0=\pi_\Omega(y_*)$, if and only if

$$\langle y_*-J(y_0), y_0-u\rangle \geq 0, \text{ for all } u \in \Omega. \quad (1)$$

Proof Because by definition $y_0=\pi_\Omega(y_*)$, we have

$$V(y_*, y_0) \leq V(y_*, y_0+t(u-y_0)),$$

where $u \in \Omega$, $t \in [0, 1]$ and $y_0+t(u-y_0) \in \Omega$ (using the convexity of Ω). Considering Lemmas 1, 2 and 3, we deduce the following inequalities,

$$\begin{aligned} 0 &\geq V(y_*, y_0)-V(y_*, y_0+t(u-y_0)) \\ &\geq 2\langle J(y_0+t(u-y_0))-y_*, y_0-y_0-t(u-y_0)\rangle, \end{aligned}$$

which implies

$$\langle J(y_0+t(u-y_0))-y_*, u-y_0\rangle \geq 0.$$

Letting $t \to 0$, we have

$$\langle J(y_0)-y_*, u-y_0\rangle \geq 0, \text{ for all } u \in \Omega,$$

so Eq.(1) is satisfied.

Conversely, if Eq.(1) is satisfied, then we have (using Lemma 2 again),

$$V(y_*, u)-V(y_*, y_0) \geq 2\langle J(y_0)-y_*, u-y_0\rangle \geq 0, \text{ for all } u \in \Omega,$$

which implies that

$$V(y_*, u) \geq V(y_*, y_0), \text{ for all } u \in \Omega,$$

that is, $y_0=\pi_\Omega(y_*)$ and the proof is completed.

Theorem 2 (Alber, 1996) Let f be a mapping from E into E^*, $\Omega \subset E$ a closed and convex set and α an arbitrarily fixed positive number. Then an element $x_* \in \Omega$ is a solution of the $VI(f, \Omega)$ if and only if x_* is a fixed point of the mapping

$$\Psi_\Omega(x)=\pi_\Omega(J(x)-\alpha f(x)), x \in E,$$
$$\text{i.e., } x_*=\pi_\Omega(J(x_*)-\alpha f(x_*)).$$

Proof Indeed, we remark that $VI(f, \Omega)$ has the equivalent representation

$$\langle J(x)-\alpha(f(x)-J(x)), x-u\rangle \geq 0, \text{ for all } u\in\Omega \quad (2)$$

Considering Eq.(2) and taking in Theorem 1, $y_*=J(x)-\alpha f(x_*)\in E^*$ and $x=y_0\in\Omega\subset E$, the conclusion of the theorem is achieved.

Corollary 1 Let f be a mapping from E into E^* and $K\subset E$ a closed convex cone. Then an element $x_*\in K$ is a solution of the $NCP(f, K)$ if and only if x_* is a fixed point of the mapping

$$x_*=\pi_K(J(x_*)-f(x_*)).$$

Remarks (i) For the main results of this paper, it is important to recall that in the case of uniformly convex and uniformly smooth Banach space, the generalized projection π_Ω is uniformly continuous on each bounded subset of the space.

(ii) The assumptions that the space E must be uniformly convex and uniformly smooth are essential for the definition of π_Ω and for its properties necessary for the results of this paper.

A NOTION OF EXCEPTIONAL FAMILY OF ELEMENTS IN BANACH SPACES

The notion of "exceptional family of elements (EFE)" as it was defined in (Isac *et al.*, 1997) has been systematically used in many papers for example, (Bulavski *et al.*, 1998; Isac, 1998; 1999a; 1999b; 2000a; 2000b; 2000c; 2001; Isac *et al.*, 1997; Isac and Carbone, 1999; Isac and Li, 2001; Isac and Obuchowska, 1998; Isac and Zhao, 2000; Kalashnikov, 1995; Zhao, 1997; Zhao and Isac, 2000a; 2000b), among others. In the cited papers, the notion of EFE has been used in the study of complemetarity problems and in the study of variational inequalities. Until now this notion has been considered only in Hilbert spaces.

Now, in this paper we introduce the notion of EFE in uniformly convex and uniformly smooth Banach spaces and we apply this notion to the study of complemetarity problems.

Let $(E, \|.\|)$ be a uniformly convex and uniformly smooth Banach space. Let $\Omega\subset E$ be a closed and convex set and let $f:E\to E^*$ be a mapping.

Definition 1 If $x\in\Omega$, then, the generalized normal cone of Ω at the point x is

$$\mathsf{N}_\Omega(x)=\{y_*\in E^*:\langle y_*, u-x\rangle\leq 0, \text{ for all } u\in\Omega\}.$$

Remark The generalized normal cone $\mathsf{N}_\Omega(x)$ is a subset of the dual space E^*. If E is a Hilbert space, $\Omega\subset E$ is a closed and convex set and $x\in\Omega$, then in this case $\mathsf{N}_\Omega(x)$ is the normal cone $N_\Omega(x)\subset E$ of the set Ω at point x.

The importance of this notion is given by the following results.

Proposition 1 An element $y_0\in\Omega$ has the property that $y_0=\pi_\Omega(y_*)$, where $y_*\in E^*$ if and only if $y_*\in J(y_0)+\mathsf{N}_K(y_0)$.

Proof Indeed, by Theorem 1 we have that $y_0=\pi_\Omega(y_*)$, if and only if, for any $u\in\Omega$, we have

$$\langle y_*-J(y_0), y_0-u\rangle\geq 0,$$

or

$$\langle y_*-J(y_0), u-y_0\rangle\leq 0, \text{ for all } u\in\Omega,$$

that is, $y_*-J(y_0)\in\mathsf{N}_K(y_0)$, i.e., $y_*\in J(y_0)+\mathsf{N}_K(y_0)$. This proposition is proved.

Now, we suppose that $\Omega=K\subset E$, where K is a closed convex cone. We recall that a mapping $T:E\to E^*$ is completely continuous if T is continuous and for any bounded set $D\subset E$, we have that $T(D)$ is relatively compact.

Definition 2 We say that a mapping $f:E\to E^*$ is a J-completely continuous field if f has a representation $f(x)=J(x)-T(x)$, for all $x\in E$, where $T:E\to E^*$ is a completely continuous mapping.

Now we can define a notion of EFE for J-completely continuous fields.

Definition 3 We say that a family of elements $\{x_r\}_{r>0}\subset K$ is an exceptional family of elements (EFE) for a J-completely continuous field $f(x)=J(x)-T(x)$ with respect to a closed convex cone $K\subset E$, if and only if, for every real number $r>0$ there exists a real number $\mu_r>1$ such that

(i) $\|x_r\|\to+\infty$ as $r\to+\infty$

(ii) $T(x_r)-J(\mu_r x_r)\in\mathsf{N}_K(\mu_r x_r)$.

Remark In the case of a Hilbert space, the notion of EFE defined by Definition 3 is the notion of EFE used in the papers (Bulavski *et al.*, 1998; Isac, 1998; 1999a; 1999b; 2000b; 2000c; 2001; Isac *et al.*, 1997; Isac and Carbone, 1999; Isac and Li, 2001; Isac and Obuchowska, 1998; Isac and Zhao, 2000; Kalashnikov, 1995).

The importance of this notion of EFE is given in the next section.

APPLICATIONS TO THE STUDY OF COMPLEMENTARITY PROBLEMS

We present in this section some existence theorems for nonlinear Complementarity Problems in uniformly convex and uniformly smooth Banach spaces. Our results are based on the notion of EFE and on an alternative theorem deduced from the Leray-Schauder Alternative. We note that the Leray-Schauder Alternative is one of the most important theorems in Nonlinear Analysis.

First, we recall the Leray-Schauder Alternative. Let $E(\tau)$ be a locally convex space, $D\subset E$ a subset and $f:E\to E^*$ a mapping. We say that f is compact if $f(D)$ is relatively compact.

Theorem 3 (Leray-Schauder Alternative) Let $(E, \|.\|)$ be a Banach space, $C\subset E$ a convex set and $V\subset C$ a subset open with respect to C and such that $0\in V$. Then each continuous compact mapping $f:clV\to C$ has at least one of the following two properties: f has a fixed point, there exists $x_*\in\partial V$ and there is a real number $\lambda_*\in(0, 1)$ such that $x_*\in\lambda_* f(x_*)$.

Proof The reference about the proof of this result is in (Isac, 2001).

We have the following result.

Theorem 4 Let $(E, \|.\|)$ be a uniformly convex and uniformly smooth Banach space, $K\subset E$ a closed convex cone and $f:E\to E^*$ a J-completely continuous field with the representation $f(x)=J(x)-T(x)$. Then there exists either a solution to the problem $NCP(f, K)$ or f has an exceptional family of elements (EFE) with respect to K.

Proof From Theorem 2, the problem $NCP(f, K)$ has a solution if and only if the following mapping

$$\Psi_K(x)=\pi_K(J(x)-f(x))=\pi_K(T(x)),\ \text{ for all } x\in E,$$

has a fixed point (which is obviously in K).

If $\Psi_K(x)$ has a fixed point, the proof is completed.

Assume that the problem $NCP(f, K)$ has no solution. Obviously the mapping Ψ_K is fixed point free. We observe that Ψ_K satisfies the assumptions of Theorem 3 with respect to each set $B_r=\{x\in E:\|x\|\le r\}$ with $r>0$ (Because T is completely continuous and π_Ω is uniformly continuous on each bounded subset of the space). Then applying Theorem 3 to each set B_r we obtain for each $r>0$, that there exists $x_r\in\partial B_r$ and there is a real number $\lambda_r\in[0, 1]$ such that $x_r=\lambda_r\pi_K(T(x_r))$. We have that $x_r\in K$ for each $r>0$. From Proposition 1 we obtain that $T(x_r)\in J(x_r/\lambda_r)+N_K(x_r/\lambda_r)$. Let $\mu_r=1/\lambda_r$, for all $r>0$, then we obtain

(d_1) $\|x_r\|=r$, and $\mu_r>1$, for all $r>0$,

(d_2) $\|x_r\|\to+\infty$ as $r\to+\infty$,

(d_3) $T(x_r)-J(\mu_r x_r)\in N_K(\mu_r x_r)$,

and the conclusion of the theorem is achieved.

A consequence of Theorem 4 is the fact that if we know that the J-completely continuous field $f:E\to E^*$ is without EFE, then the $NCP(f, K)$ has a solution. Therefore, it is interesting to have some conditions that imply the nonexistence of EFE for a given mapping. We give some results in this sense.

The first author of this paper introduced "condition θ" in (Isac, 1999a; Kalashnikov and Isac, 2002) and used this condition in (Isac, 1998; 1999b; 2000c; 2001; Isac and Li, 2001).

Now we will show that this condition also applies to Banach spaces.

Definition 4 Let $(E, \|.\|)$ be a uniformly convex and uniformly smooth Banach space. We say that a mapping $f:E\to E^*$ satisfies the condition (Θ) with respect to a closed convex cone $K\subset E$ if there exists a real number $\rho>0$ such that for each $x\in K$ with $\|x\|>\rho$, there exists $y\in K$ such that $\|y\|<\|x\|$ and $\langle f(x), x-y\rangle\ge0$.

Theorem 5 If $f:E\to E^*$ is a J-completely continuous field satisfying the condition (Θ) with respect to a closed convex cone $K\subseteq E$, then f is without exceptional family of elements with respect to K and the $NCP(f, K)$ has a solution.

Proof Suppose, by contradiction, that f has an exceptional family of elements $\{x_r\}$ with respect to K. Then for all $r>0$, we have $\|x_r\|=r$, $\mu_r x_r\in K$ such that $\mu_r>1$, and $J(x_r)-f(x_r)-J(\mu_r x_r)\in N_K(\mu_r x_r)$, that is,

$$\langle J(x_r)-f(x_r)-J(\mu_r x_r),\ y-\mu_r x_r\rangle\le0, \text{ for all } y\in K. \quad (3)$$

Because f satisfies the condition (Θ), with respect to K, we have that for any r sufficiently big, there exists $y_r\in K$ such that $\|x_r\|>\rho$, $\|y_r\|<\|x_r\|$ and $\langle f(x_r), x_r-y_r\rangle\ge0$.

Considering Eq.(3) and using the fact that the operator J is homogenous, we have

$$\begin{aligned}0&\leq\langle f(x_r), x_r-y_r\rangle\\ &=\langle -J(x_r)+f(x_r)+J(\mu_r x_r)+J(x_r)-J(\mu_r x_r), x_r-y_r\rangle\\ &=\langle -J(x_r)+f(x_r)+J(\mu_r x_r), x_r-y_r\rangle+\langle J(x_r)-J(\mu_r x_r), x_r-y_r\rangle\\ &\leq\langle J(x_r)-J(\mu_r x_r), x_r-y_r\rangle\end{aligned}$$

We used the following relation in the last inequality:

$$\begin{aligned}&\mu_r\langle -J(x_r)+f(x_r)+J(\mu_r x_r), x_r-y_r\rangle\\ &\quad=\langle -J(x_r)+f(x_r)+J(\mu_r x_r), \mu_r x_r-\mu_r y_r\rangle\\ &\quad=\langle J(x_r)-f(x_r)-J(\mu_r x_r), \mu_r y_r-\mu_r x_r\rangle\leq 0,\end{aligned}$$

(because $\mu_r y_r\in K$), which implies

$$\langle -J(x_r)+f(x_r)+J(\mu_r x_r), x_r-y_r\rangle\leq 0.$$

Therefore we have

$$\begin{aligned}0&\leq\langle J(x_r)-J(\mu_r x_r), x_r-y_r\rangle\\ &=\leq\langle J(x_r)-\mu_r J(x_r), x_r-y_r\rangle\\ &=(1-\mu_r)\langle J(x_r), x_r-y_r\rangle\\ &=(1-\mu_r)(\|x_r\|^2-\langle J(x_r), y_r\rangle)\\ &=(1-\mu_r)(\|x_r\|^2+(\mu_r-1)\langle J(x_r), y_r\rangle)\\ &\leq(1-\mu_r)\|x_r\|^2+(\mu_r-1)\|J(x_r)\|\|y_r\|\\ &=(1-\mu_r)\|x_r\|^2+(\mu_r-1)\|x_r\|\|y_r\|\\ &<(1-\mu_r)\|x_r\|^2+(\mu_r-1)\|x_r\|^2\\ &=0,\end{aligned}$$

which is a contradiction.

Hence f is without EFE with respect to K. The conclusion of the theorem follows from Theorem 4 and the proof is completed.

The following condition is a generalization of the Ding-Tan (DT) condition considered in (Isac, 1999a).

Definition 5 We say that a mapping f:$E\to E^*$ satisfies the DT condition with respect to a closed convex cone $K\subset E$ if there exist two bounded subsets D_0 and D^* in K such that for each $x\in K\backslash D^*$ there is a $y\in conv(D_0\cap\{x\})$ such that $\langle f(x), x-y\rangle>0$.

We have the following result.

Proposition 2 If f:$E\to E^*$ satisfies the DT condition with respect to a closed convex cone $K\subset E$, then f satisfies the condition (Θ).

Proof Because D_0 and D^* are bounded subsets in K, there exists a real number $\rho>0$ such that D_0, $D^*\subset(clB_\rho)\cup K$. If $x\in K$ is such that $\|x\|>\rho$, then by the DT condition, there is an element $y\in conv(D_0\cap\{x\})$ such that $\langle f(x), x-y\rangle>0$. We have, $y=\lambda x_0+(1-\lambda)x$ with $\lambda\in[0, 1]$ and $x_0\in D_0$, which implies

$$\|y\|\leq\lambda\|x_0\|+(1-\lambda)\|x\|<\lambda\|x\|+(1-\lambda)\|x\|=\|x\|.$$

Therefore, f satisfies condition (Θ) with respect to K. This proposition is proved.

The following notion is related to a similar notion defined in (Isac *et al*., 1997).

Definition 6 Let f, g:$E\to E^*$ be two mappings. We say that mapping f is asymptotically g-pseudomonotone with respect to a closed convex cone $K\subset E$, if there exists a real number $\rho>0$ such that for all x, $y\in K$ with $\max\{\|y\|, \rho\}<\|x\|$, we have that

$$\langle g(y), x-y\rangle\geq 0 \text{ implies } \langle f(x), x-y\rangle\geq 0.$$

This notion implies the following result.

Theorem 6 Let $(E, \|.\|)$ be a uniformly convex and uniformly smooth Banach space, $K\subset E$ an arbitrary closed convex cone and f, g:$E\to E^*$ be two mappings such that f is a J-completely continuous field. If f is asymptotically g-pseudomonotone with respect to K and the problem $NCP(g, K)$ has a solution, then f is without EFE with respect to K and the problem $NCP(f, K)$ has a solution.

Proof Let x_* be a solution to the problem $NCP(g, K)$. Then for all $y\in K$, we have $\langle g(x_*), y-x_*\rangle\geq 0$. Since f is an asymptotically g-pseudomonotone with respect to K, there exists a real number $\rho>0$ such that for all x, $y\in K$ with $\max\{\|y\|, \rho\}<\|x\|$, we have that $\langle g(y), x-y\rangle\geq 0$ implies $\langle f(x), x-y\rangle\geq 0$.

Take $\rho_0=\max\{\|x_*\|+1, \rho+1\}$. Then for any $x\in K$ with $\|x\|>\rho_0$, we may take $x_*\in K$. Because $\|x_*\|<\|x\|$ and $\langle g(x_*), x-x_*\rangle\geq 0$, we have that $\langle f(x), x-x_*\rangle\geq 0$, that is, f satisfies the condition (Θ) with respect to K. Now, applying Theorem 5, the conclusion of this theorem is achieved.

Remark If in Definition 6 we take $g=f$, we say that f is asymptotically pseudomonotone with respect to K. Obviously, if f is monotonic, it is asymptotically pseudomonotone, but the converse is not true.

The following result follows from Theorem 6 immediately.

Corollary 2 Let $(E, \|.\|)$ be a uniformly convex and uniformly smooth Banach space, $K \subset E$ a closed convex cone and $f:E \to E^*$ a J-completely continuous field. If f is asymptotically pseudomonotone with respect to K, then the problem $NCP(f, K)$ has a solution if and only if f is without EFE with respect to K.

COMMENTS

We introduced in this paper the notion of EFE for a J-completely continuous field $f:E \to E^*$, where E is a uniformly convex and uniformly smooth Banach space. By using this notion, we obtained an existence theorem for the general nonlinear complementarity problem. Other results obtained in (Bulavski *et al.*, 1998; Isac, 1998; 1999a; 2000a; 2000b), in Hilbert spaces may be extended to this class of Banach spaces.

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