Journal of Zhejiang University SCIENCE
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

Differentiability of the Pritchard-Salamon systems with admissible state-feedback*

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Received Feb. 2, 2004; revision accepted July 28, 2004

Abstract: Many papers on a wide range of control problems for Pritchard-Salamon systems have appeared and many of its important mathematical and system theoretical properties have been revealed. This paper deals with the differentiability of the Pritchard-Salamon system with admissible state-feedback. Spectrum analysis showed that under definite condition, the unbounded perturbation semigroup of the Pritchard-Salamon system is eventually differentiable.

Key words: C_0-semigroup, Differentiability, Admissible operator, Pritchard-Salamon systems, Perturbation
doi:10.1631/jzus.2005.A0311 **Document code:** A **CLC number:** O231.4

INTRODUCTION

The Pritchard-Salamon class of infinite dimensional systems was first introduced by Pritchard and Salamon (1985; 1987) to provide a general abstract framework for linear quadratic control problems. Now, many papers on a wide range of control problems for Pritchard-Salamon systems have appeared and many of its important mathematical and system theoretical properties have been revealed. We give the following definition.

Definition 1 Let W, V, U, Y be Hilbert spaces. Suppose that $W\subset V$ and that the canonical injection $W\to V$: $x\to x$ is continuous and dense. $S(t)$ ($t\geq 0$) is a C_0-semigroup on V which is restricted to a C_0-semigroup on W.

(i) An operator $B\in L(U, V)$ is called an admissible input operator (with respect to (W, V) for $S(t)$) if there exist $t_1>0$ and $\alpha>0$ such that for all $u\in L^2(0,t_1;U)$,

$$\int_0^{t_1} S(t_1-s)Bu(s)\mathrm{d}s\in W,$$

$$\left\|\int_0^{t_1} S(t_1-s)Bu(s)\mathrm{d}s\right\|_W\leq\alpha\|u\|_{L^2(0,t_1;U)}.$$

(ii) An operator $C\in L(W, Y)$ is called an admissible output operator (with respect to (W, V) for $S(t)$) if there exist $t_2>0$ and $\beta>0$ such that for all $x\in W$,

$$\|CS(\cdot)x\|_{L^2(0,t_2;Y)}\leq\beta\|x\|_V.$$

(iii) Let $B\in L(U, V)$ and $C\in L(W, Y)$ be the admissible input and output operators (with respect to (W, V) for $S(t)$), respectively, and suppose that $D\in L(U, Y)$. The system given by

$$x(t)=S(t)x_0+\int_0^t S(t-s)Bu(s)\mathrm{d}s,$$
$$y(t)=Cx(t)+Du(t)$$

where $x_0\in V$, $t\geq 0$ and $u\in L^2_{loc}(0,\infty;U)$, is called a Pritchard-Salamon system (with respect to (W, V) for $S(t)$) and is denoted by $\Sigma(S(\cdot),B,C,D)$.

Let $X=W$ or V. We use superscript A^X to denote the infinitesimal generator of a C_0-semigroup $S^X(t)$ on X. We also denote the growth bound of $S^X(t)$ by

*Project (No. 10271111) supported partially by the National Natural Science Foundation of China

$\omega(S^X(t))$. Curtain *et al.*(1997) established the theory of perturbation induced by admissible state-feedback for Pritchard-Salamon system, and Guo *et al.*(2003) improved this result. This is their theorem.

Theorem 1 Let $\Sigma(S(\cdot),B,C,D)$ be a Pritchard-Salamon system, $F\in L(W, U)$ be an admissible output operator for $S(t)$. The following are true.

(a) There exists a unique C_0-semigroup $S_{BF}(t)$ on W such that for all $x\in W$,

$$S_{BF}(t)x = S(t)x + \int_0^t S(t-s)BFS_{BF}(s)x\mathrm{d}s.$$

Moreover, $S_{BF}(t)$ extends to a C_0-semigroup on V, $\Sigma(S(\cdot),B,C,D)$ is a Pritchard-Salamon system and for all $x\in V$,

$$\begin{aligned} S_{BF}(t)x &= S(t)x + \int_0^t S(t-s)B\overline{FS_{BF}(s)}x\mathrm{d}s \\ &= S(t)x + \int_0^t S_{BF}(t-s)B\overline{FS(s)}x\mathrm{d}s. \end{aligned} \quad (1)$$

In addition, B, F are the admissible input and output operators for $S_{BF}(t)$ respectively.

(b) For $x\in D(A^V)\cap W$,

$$\overline{F(\lambda - A^W)^{-1}}(\lambda - A^V)x = Fx.$$

(c) The generators of $S_{BF}(t)$ on W and V satisfy

$$D(A_{BF}^V) = D(A^V),$$
$$D(A_{BF}^W) = \{x\in D(A_{BF}^V)\cap W \mid A^V x + BFx \in W\},$$
$$A_{BF}^V x = \begin{cases} A^V x + BFx, & F\in L(V,U) \text{ or } x\in D(A^V)\cap W, \\ A^V x + B\overline{F(\lambda - A^W)^{-1}}(\lambda - A^V)x, & \text{else} \end{cases}$$
$$\text{for } x\in D(A_{BF}^V)$$

and

$$A_{BF}^W x = A^V x + BFx \text{ for } x\in D(A_{BF}^W),$$

where λ is any number with real part larger than the growth bounds $\omega(S^W(t))$, $\omega(S^V(t))$, $\omega(S_{BF}^W(t))$ and $\omega(S_{BF}^V(t))$.

Gu (2004) treated the regularity of the unbounded perturbation semigroup $S_{BF}(t)$ on W and V. But to the best of our knowledge, there are no results available in the literature on the differentiability of $S_{BF}(t)$. Our main goal here is to study the differentiability of $S_{BF}(t)$. We show that under definite condition, the eventual differentiability is a stable property under the perturbation induced by admissible state-feedback for Pritchard-Salamon system.

MAIN RESULTS AND THEIR PROOFS

We need the following Lemma.

Lemma 1 Suppose that $\Sigma(S(\cdot),B,C,D)$ is a Pritchard-Salamon system and that $F\in L(W, U)$ is an admissible output operator for $S(t)$. Then the following hold:

(i) $BF\in L(W, V)$ is an admissible input operator with respect to (W, V) for $S(t)$.

(ii) $BF\in L(W, V)$ is an admissible output operator with respect to (W, V) for $S(t)$.

Proof See Gu (2004).

Now, we give the first main result of this paper.

Theorem 2 Suppose that $\Sigma(S(\cdot),B,C,D)$ is a Pritchard-Salamon system and that $F\in L(W, U)$ is an admissible output operator for $S(t)$. If for some $\mu\in\mathbb{R}$,

$$\limsup_{|\tau|\to\infty} \log|\tau| \left\| R(\mu + i\tau, A^W)\right\|_{L(W,W)} = C < \infty, \quad (2)$$

then $S_{BF}^W(t)$ is differentiable for $t>3C$. If $C=0$, $S_{BF}^W(t)$ is a differentiable semigroup.

Proof Let $1>\varepsilon>0$ and μ_0 be any real number satisfying

$$\mu_0 > \max\left\{\omega(S^W(t)),\ \omega(S^V(t)),\ \omega(S_{BF}^W(t)),\ \omega(S_{BF}^V(t))\right\}.$$

By Lemma 1, we see that $BF\in L(W, V)$ is an admissible input operator with respect to (W, V) for $S(t)$. Thus it follows from Theorem 1 that BF is also an admissible input operator with respect to (W, V) for $S_{BF}(t)$. According to Lemma 2.12 in Curtain *et al.*(1997), there exists $C_{\mu_0}>0$ such that

$$(\lambda - A_{BF}^V)^{-1}BF \in L(W,W)$$

and for $\mathrm{Re}\lambda>\mu_0$,

$$\left\|(\lambda - A_{BF}^{V})^{-1} BF\right\|_{L(W,W)} \leq \frac{C_{\mu_0}}{\sqrt{\mathrm{Re}\,\lambda - \mu_0}}.$$

Thus we can find a $\gamma > \mu_0$ such that

$$\left\|(\gamma + i\tau - A_{BF}^{V})^{-1} BF\right\|_{L(W,W)} \leq \varepsilon,\ \ \tau \in \mathbb{R}. \tag{3}$$

By Eq.(2), there exists $N>0$ such that

$$|\mu - \gamma| \left\|(\mu + i\tau - A^{W})^{-1}\right\|_{L(W,W)} \leq \varepsilon \ \text{ for } |\tau| > N.$$

Noting that

$$\begin{aligned}&(\gamma + i\tau - A^{W})^{-1}\\ &\ = (\mu + i\tau - A^{W})^{-1}(1 - (\mu - \gamma)(\mu + i\tau - A^{W})^{-1})^{-1},\end{aligned}$$

we have that

$$\limsup_{|\tau|\to\infty} \log|\tau| \left\|R(\gamma + i\tau, A^{W})\right\|_{L(W,W)} \leq \frac{C}{1-\varepsilon}. \tag{4}$$

By Eq.(1), the following holds for $x \in W$

$$S_{BF}(t)x = S(t)x + \int_0^t S_{BF}(t-s) BFS(s)x\mathrm{d}s. \tag{5}$$

It follows from Eq.(5) via Laplace transformation that

$$\begin{aligned}(\lambda - A_{BF}^{W})^{-1} &= (\lambda - A^{W})^{-1}\\ &\quad + (\lambda - A_{BF}^{V})^{-1} BF(\lambda - A^{W})^{-1},\ \ \mathrm{Re}\,\lambda > \mu_0.\end{aligned} \tag{6}$$

Combining Eqs.(3) and (6), we obtain

$$\begin{aligned}&\left\|(\gamma + i\tau - A_{BF}^{W})^{-1}\right\|_{L(W,W)}\\ &\leq (1+\varepsilon)\left\|(\gamma + i\tau - A^{W})^{-1}\right\|_{L(W,W)},\end{aligned}$$

which implies that

$$\limsup_{|\tau|\to\infty} \log|\tau| \left\|R(\gamma + i\tau, A_{BF}^{W})\right\|_{L(W,W)} \leq \frac{1+\varepsilon}{1-\varepsilon} C.$$

Since ε is an arbitrary number, the differentiability of $S_{BF}^{W}(t)$ for $t>3C$ follows from Corollary 2.2.10 in Pazy (1983).

The following is the second result of this paper.

Theorem 3 Suppose that $\Sigma(S(\cdot),B,C,D)$ is a Pritchard-Salamon system and that $F \in L(W, U)$ is an admissible output operator for $S(t)$. If for some $\mu \in \mathbb{R}$,

$$\limsup_{|\tau|\to\infty} \log|\tau| \left\|R(\mu + i\tau, A^{V})\right\|_{L(V,V)} = C < \infty, \tag{7}$$

then $S_{BF}^{V}(t)$ is differentiable for $t>3C$. If $C=0$, $S_{BF}^{V}(t)$ is a differentiable semigroup.

Proof Let $1>\varepsilon>0$ and μ_0 be any real number satisfying

$$\mu_0 > \max\left\{\omega(S^{W}(t)),\ \omega(S^{V}(t)),\ \omega(S_{BF}^{W}(t)),\ \omega(S_{BF}^{V}(t))\right\}.$$

By Lemma 1, we see that $BF \in L(W, V)$ is an admissible output operator with respect to (W, V) for $S(t)$. According to Lemma 2.12 in Curtain *et al.*(1997), $BF(\lambda - A^{W})^{-1}$ can be extended to $\overline{BF(\lambda - A^{W})^{-1}} \in L(V,V)$ and there exists $C_{\mu_0}>0$ such that

$$\left\|\overline{BF(\lambda - A^{W})^{-1}}\right\|_{L(V,V)} \leq \frac{C_{\mu_0}}{\sqrt{\mathrm{Re}\,\lambda - \mu_0}},\ \mathrm{Re}\,\lambda > \mu_0.$$

Like the discussions in proof of Theorem 2, we can find a $\gamma > \mu_0$ such that

$$\left\|\overline{BF(\gamma + i\tau - A^{W})^{-1}}\right\|_{L(V,V)} \leq \varepsilon,\ \ \ \tau \in \mathbb{R}. \tag{8}$$

and

$$\limsup_{|\tau|\to\infty} \log|\tau| \left\|R(\gamma + i\tau, A^{V})\right\|_{L(V,V)} \leq \frac{C}{1-\varepsilon}. \tag{9}$$

It is easy to see from (b) and (c) of Theorem 1 that

$$\begin{aligned}&(\gamma + i\tau - A_{BF}^{V})^{-1}\\ &\ = (\gamma + i\tau - A^{V})^{-1}(1 - \overline{BF(\gamma + i\tau - A^{W})^{-1}})^{-1}.\end{aligned} \tag{10}$$

Combining Eqs.(8)–(10), we have

$$\limsup_{|\tau|\to\infty} \log|\tau| \left\|R(\gamma+i\tau, A_{BF}^{V})\right\|_{L(V,V)}$$
$$\le \frac{1}{1-\varepsilon} \limsup_{|\tau|\to\infty} \log|\tau| \left\|R(\gamma+i\tau, A^{V})\right\|_{L(V,V)}$$
$$\le \frac{C}{(1-\varepsilon)^2}.$$

Since ε is an arbitrary number, the differentiability of $S_{BF}^{V}(t)$ for t>3C follows from Corollary 2.2.10 in Pazy (1983).

Remark 1 It is well known that under the condition which is the type of Eqs.(2) or (7), the eventual differentiability is a stable property under the bounded perturbation for a C_0-semigroup (Pazy, 1968). Here we show that under the condition of Eqs.(2) or (7), the unbounded perturbation semigroup for Pritchard-Salamon system is also eventually differentiable. Moreover, it is easy to see 3C is the optimal number such that the unbounded perturbation semigroup is differentiable for t>3C.

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