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A rigidity theorem for submanifolds in S^{n+p} with constant scalar curvature*

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Abstract: Let M^n be a closed submanifold isometrically immersed in a unit sphere S^{n+p}. Denote by R, H and S, the normalized scalar curvature, the mean curvature, and the square of the length of the second fundamental form of M^n, respectively. Suppose R is constant and ≥1. We study the pinching problem on S and prove a rigidity theorem for M^n immersed in S^{n+p} with parallel normalized mean curvature vector field. When n≥8 or, n=7 and p≤2, the pinching constant is best.

Key words: Scalar curvature, Mean curvature vector, The second fundamental form
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INTRODUCTION

Let M^n be a connected and oriented submanifold isometrically immersed in a space form $M_c^{n+p}(c\geq 0)$. We say M^n is closed if it is compact and without boundary. Denote by R, H and S, the normalized scalar curvature, mean curvature, and square of the length of the second fundamental form of M^n, respectively.

Application of the approach of Simons (1968) yields many rigidity results for minimal submanifolds and submanifolds with parallel mean curvature vector field immersed into a unit sphere (Alencar and de Carmo, 1994; Chern *et al.*, 1970; Hou, 1997).

Cheng and Yau (1977) first studied the rigidity problem for hypersurface M^n immersed into a space form $M_c^{n+1}(c\geq 0)$ with constant scalar curvature by introducing a self-adjoint second order differential operator. They proved that, for a hypersurface M^n immersed into space form M_c^{n+1}, if R is constant and ≥c, then $|\nabla\sigma|^2\geq n^2|\nabla H|^2$ where σ denotes the second fundamental form of M^n. By using Cheng-Yau's technique, Li (1994) studied the pinching problem on S for hypersurfaces with constant scalar curvature in M_c^{n+1} where c≥0 and Hou (1998) studied the pinching problem on H for submanifold M^n with constant scalar curvature in M_c^{n+p}. They also proved some rigidity theorems by adding an assumption that the normalized mean curvature vector field of M^n is parallel.

In this paper, we study the pinching problem on S for submanifold M^n in a unit sphere S^{n+p} with constant scalar curvature and parallel normalized mean curvature vector field, and prove the following theorem.

Main theorem Let M^n be a closed submanifold with parallel normalized mean curvature vector field immersed into a unit sphere S^{n+p}. Suppose that R is constant and ≥1. Then

(i) If n≥8 or, n≥3 and p≤2, $S\leq 2\sqrt{n-1}$, then either

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1) $S=n(R-1)$ and M^n is a sphere $S^n(1/\sqrt{R})$ in S^{n+p}; or

2) $n\geq 8$ or, $n=7$ and $p\leq 2$, $S=2\sqrt{n-1}\neq n(R-1)$, and M^n is product of the form $S^{n-1}(r_1)\times S^1(r_2)$, and lies in a totally geodesic submanifold S^{n+1} of S^{n+p}, where $r_1^2=\sqrt{n-1}/(1+\sqrt{n-1}), r_2^2=1/(1+\sqrt{n-1})$;

(ii) If $3\leq n\leq 7$, $S\leq 2n/3$, then $S=n(R-1)$ and M^n is a sphere $S^n(1/\sqrt{R})$ in S^{n+p};

(iii) If $n=2$, $S\leq n(5R-1)$, then either

1) $S=n(R-1)$ and M^2 is a sphere $S^2(1/\sqrt{R})$ in S^{2+p}; or

2) $S=n(5R-1)$, $p\geq 3$ and M^2 is a Veronese surface in $S^4(1/\sqrt{3R})$, which is a totally geodesic submanifold of $S^{2+(p-1)}(1/\sqrt{3R})$, and $S^{2+(p-1)}(1/\sqrt{3R})$ is a totally umbilical hypersurface of S^{2+p}.

When $p=1$, we have the following corollary.

Corollary Let M^n be a closed hypersurface in a unit sphere S^{n+1} with constant scalar curvature $R\geq 1$. Then

(i) If $n\geq 3$ and $S<2\sqrt{n-1}$, then $S=n(R-1)$ and M^n is a sphere $S^n(1/\sqrt{R})$ in S^{n+1};

(ii) If $n\geq 3$ and $S=2\sqrt{n-1}$, then M^n is either a sphere $S^n(1/\sqrt{R})$ or $n\geq 7$ and M^n is a product of the form $S^{n-1}(r_1)\times S^1(r_2)$, where r_1 and r_2 are taken as before;

(iii) If $n=2$, $S\leq n(5R-1)$, then $S=n(R-1)$ and M^2 is a sphere $S^2(1/\sqrt{R})$ in S^3.

In Section 4, we shall give an example which is an n-dimension submanifold of S^{n+p} with $S=2\sqrt{n-1}$. So, when $n\geq 8$ or, $n=7$ and $p\leq 2$, $2\sqrt{n-1}$ is the best pinching constant.

PRELIMINARIES

Let M^n be a connected and oriented submanifold isometrically immersed in a unit sphere S^{n+p}. We choose a local field of orthonormal frames $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+p}\}$ such that, restricted to M^n, the $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ are tangent to M^n. We shall make use of the following convention on the range of induces

$$1\leq A,B,C,\cdots\leq n+p;\ 1\leq i,j,k,\cdots\leq n;$$

$$n+1\leq \alpha,\beta,\gamma,\cdots\leq n+p.$$

Let $\{\omega_1, \ldots, \omega_{n+p}\}$ be the field of dual frames. Then the second fundamental form of M^n can be expressed as

$$\boldsymbol{\sigma}=\sum_{i,j,\alpha} h_{ij}^{\alpha}\omega_i\omega_j\boldsymbol{e}_\alpha .$$

Denote $\boldsymbol{L}_\alpha=(h_{ij}^{\alpha})_{n\times n}$ and $H_\alpha=1/n\sum_i h_{ii}^{\alpha}$ for $\alpha=n+1, \ldots, n+p$. Then the mean curvature vector field ξ, the mean curvature H and the square of the second fundamental form are expressed as

$$\xi=\sum_\alpha H_\alpha\boldsymbol{e}_\alpha;\ H=|\xi|;\ S=|\boldsymbol{\sigma}|^2=\sum_{i,j,\alpha}(h_{ij}^{\alpha})^2.$$

Moreover, the Riemannian curvature tensor $\{R_{ijkl}\}$, the normal curvature tensor $\{R_{\alpha\beta kl}\}$, the Ricci curvature tensor $\{R_{ik}\}$ and the normalized scalar curvature R are expressed as

$$R_{ijkl}=(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum_\alpha(h_{ik}^{\alpha}h_{jl}^{\alpha}-h_{il}^{\alpha}h_{jk}^{\alpha}),$$
$$R_{\alpha\beta kl}=\sum_i(h_{ik}^{\alpha}h_{il}^{\beta}-h_{il}^{\alpha}h_{ik}^{\beta}),$$
$$R_{ik}=(n-1)\delta_{ik}+\sum_\alpha(nH_\alpha)h_{ik}^{\alpha}-\sum_{j,\alpha}h_{ij}^{\alpha}h_{jk}^{\alpha},$$
$$n(n-1)R=\sum_i R_{ii}=n(n-1)+n^2H^2-S. \qquad (1)$$

Define the first and the second covariant derivatives of $\{h_{ij}^{\alpha}\}$, say $\{h_{ijk}^{\alpha}\}$ and $\{h_{ijkl}^{\alpha}\}$ by

$$\sum h_{ijk}^{\alpha}\omega_k=dh_{ij}^{\alpha}-\sum h_{ik}^{\alpha}\omega_{kj}-\sum h_{jk}^{\alpha}\omega_{ki}-\sum h_{ij}^{\beta}\omega_{\beta\alpha} \qquad (2)$$

$$\sum h_{ijkl}^{\alpha}\omega_l=dh_{ijk}^{\alpha}-\sum h_{ljk}^{\alpha}\omega_{li}-\sum h_{ilk}^{\alpha}\omega_{lj}-\sum h_{ijl}^{\alpha}\omega_{lk}-\sum h_{ijk}^{\beta}\omega_{\beta\alpha} \qquad (3)$$

It follows form Ricci's identity that

$$h_{ijk}^{\alpha}-h_{ikj}^{\alpha}=K_{ikj}^{\alpha}, \qquad (4)$$

$$h_{ijkl}^{\alpha}-h_{ijlk}^{\alpha}=\sum R_{\beta\alpha kl}h_{ij}^{\beta}+\sum R_{mikl}h_{mj}^{\alpha}+\sum R_{mjkl}h_{mi}^{\alpha} \qquad (5)$$

The Laplacian of h_{ij}^{α} is defined by $\Delta h_{ij}^{\alpha}=\sum_k h_{ijkk}^{\alpha}$.

From Eqs.(1)–(5), we have

$$\sum_{i,j} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} = n\sum_{i,j} H_{\alpha,ij}h_{ij}^{\alpha} - n^2H^2 + n\sum_{\beta} H_{\beta}Tr(\boldsymbol{L}_{\alpha}^2\boldsymbol{L}_{\beta}) - \sum_{\beta} S_{\alpha\beta}^2 - \sum_{\beta} N(\boldsymbol{L}_{\alpha}\boldsymbol{L}_{\beta} - \boldsymbol{L}_{\beta}\boldsymbol{L}_{\alpha}) + nS_{\alpha} \quad (6)$$

where $S_{\alpha\beta} = \sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta}$, $S_{\alpha} = S_{\alpha\alpha} = \sum_{i,j}(h_{ij}^{\alpha})^2$ for all α, β. And for any real matrix $\boldsymbol{A}=(a_{ij})_{n\times n}$, define $N(\boldsymbol{A}) = \sum_{i,j}(a_{ij})^2$.

Suppose $H>0$ on M^n and choose $\boldsymbol{e}_{n+1}=\boldsymbol{\xi}/H$. Then it follows that

$$H_{n+1}=H;\ H_{\alpha}=0 \text{ for any } \alpha>n+1. \quad (7)$$

From Eqs.(3) and (7) we can see that

$$H_{n+1,k}\omega_k=dH \quad (8)$$

$$H_{\alpha,k}\omega_k=H\omega_{n+1,\alpha} \text{ for any } \alpha>n+1. \quad (9)$$

From Eqs.(3), (7), (8) and (9) we have

$$H_{n+1,kl} = H_{kl} - \frac{1}{H}\sum_{\beta>n+1} H_{\beta,k}H_{\beta,l}, \quad (10)$$

where $\nabla H_k = \sum_l H_{kl}\omega_l = dH_k + \sum_i H_i\omega_{ik}$ and $dH = \sum_i H_i\omega_i$ for all k.

Using Eqs.(6) and (10), we have

$$\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} = n\sum_{i,j} H_{ij}h_{ij}^{n+1} + nS_{n+1} - n^2H^2 - S_{n+1}^2 - \frac{n}{H}\sum_{i,j,\beta>n+1} H_{\beta,i}H_{\beta,j}h_{ij}^{n+1} + nHTr(\boldsymbol{L}_{n+1}^3) - \sum_{\beta>n+1} S_{(n+1)\beta}^2 - \sum_{\beta>n+1} N(\boldsymbol{L}_{n+1}\boldsymbol{L}_{\beta} - \boldsymbol{L}_{\beta}\boldsymbol{L}_{n+1}) \quad (11)$$

Let $T = \sum_{ij} T_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n and defined as

$$T_{ij} = h_{ij}^{n+1} - nH\delta_{ij}. \quad (12)$$

We introduce an operator □ associated to T acting on $f\in C^2(M^n)$ by

$$\Box f = \sum_{i,j} T_{ij}f_{ij} = \sum_{i,j} h_{ij}^{n+1}f_{ij} - nH\Delta f,$$

where Δ is the Laplacian. Since T_{ij} is divergence-free, it follows from a result in (Cheng and Yau, 1977) that the operator □ is self-adjoint relative to the L^2-inner product of M^n.

Choosing $f=H$ in above expression, we have

$$\sum_{i,j} h_{ij}^{n+1}H_{ij} = \Box H + nH\Delta H. \quad (13)$$

From Eqs.(11) and (13), we have

$$\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} = n\Box H + \frac{1}{2}n^2\Delta(H^2) - n^2|\nabla H|^2 - S_{n+1}^2 - \frac{n}{H}\sum_{i,j,\beta>n+1} H_{\beta,i}H_{\beta,j}h_{ij}^{n+1} + nHTr(\boldsymbol{L}_{n+1}^3) - n^2H^2 - \sum_{\beta>n+1} S_{(n+1)\beta}^2 + nS_{n+1} - \sum_{\beta>n+1} N(\boldsymbol{L}_{n+1}\boldsymbol{L}_{\beta} - \boldsymbol{L}_{\beta}\boldsymbol{L}_{n+1}) \quad (14)$$

Moreover, we need the following theorem:

Lemma 1 (Hou, 1998) Let M^n be a connected submanifold in M_c^{n+p} with nowhere zero mean curvature H. Suppose that R is constant and $\geq c$. Then

$$|\nabla\boldsymbol{\sigma}|^2 = \sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2 \geq n^2|\nabla H|^2, \quad (15)$$

and the symmetric tensor T denoted by Eq.(12) is negative semi-definite. Moreover,

(i) when $R-c>0$, if the equality in Eq.(15) holds on M^n, then H is constant and T is negative definite;

(ii) when $R-c=0$, if the equality in Eq.(15) holds on M^n, then either H is constant or M^n lies in a totally geodesic subspace of M_c^{n+p}. In this latter case, if H is not constant on M^n, then $r(\boldsymbol{L}_{n+1})=1$, where $r(\boldsymbol{L}_{n+1})$ denotes the rank of $\boldsymbol{L}_{n+1}$.

PROOF OF AIN THEOREM

We need the following algebraic lemma.

Lemma 2 (Okumura, 1974; Zhang, 1999) Let b_1, …, b_n be real numbers satisfying $\sum_i b_i = 0$. Then

$$\left|\sum_i b_i^3\right| \le \frac{n-2}{\sqrt{n(n-1)}}(\sum_i b_i^2)^{3/2};$$

$$\sum_i b_i^4 \le \frac{n^2-3n+3}{n(n-1)}(\sum_i b_i^2)^2,$$

where the equality holds in each inequality if and only if at least $(n-1)$ numbers of the b_i's are same with each other.

Lemma 3 Let $a_1, \ldots, a_n$; $b_1, \ldots, b_n$ ($n\ge 2$) be real numbers satisfying $\sum_i b_i = 0$. Then

$$\sum_{i,j} a_i a_j (b_i - b_j)^2 \ge -\frac{n}{\sqrt{n-1}}\left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right).$$

Proof Using Lemma 2 it is easy to get

$$\sum_{i<j}(b_i-b_j)^4 = \frac{1}{2}\sum_{i,j}(b_i-b_j)^4 \le \frac{n^2}{n-1}\left(\sum_i b_i^2\right)^2.$$

If $a_i\ge 0$ for all i, the inequality holds obviously. Without loss of generality, we may assume that $a_1\ge a_2\ge\ldots\ge a_m\ge 0> a_{m+1}\ge\ldots\ge a_n$. Then

$$\begin{aligned}\sum_{i,j} a_i a_j (b_i-b_j)^2 &\ge -2\sum_{1\le i\le m, m+1\le j\le n} a_i a_j (b_i-b_j)^2\\ &\ge -2\left(\sum_{1\le i\le m, m+1\le j\le n}(a_i a_j)^2\right)^{\frac{1}{2}}\left(\sum_{1\le i\le m, m+1\le j\le n}(b_i-b_j)^4\right)^{\frac{1}{2}}\\ &\ge -\left(\sum_{1\le i\le m,} a_i^2 + \sum_{m+1\le j\le n} a_j^2\right)\left(\sum_{i<j}(b_i-b_j)^4\right)^{\frac{1}{2}}\\ &\ge -\frac{n}{\sqrt{n-1}}\left(\sum_i a_i^2\right)\left(\sum_j b_j^2\right).\end{aligned}$$

Lemma 4 (Li and Li, 1992) Let $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_p$ be symmetric $n\times n$-matrices. Set $S_{\alpha\beta} = Tr(\boldsymbol{A}_\alpha^{\mathrm{T}}\boldsymbol{A}_\beta)$, $S_\alpha = S_{\alpha\alpha} = N(\boldsymbol{A}_\alpha)$, $S=S_1+\ldots+S_p$. Then

$$\sum_{\alpha,\beta} N(\boldsymbol{A}_\alpha\boldsymbol{A}_\beta - \boldsymbol{A}_\beta\boldsymbol{A}_\alpha) + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \le \left(1+\frac{1}{2}\mathrm{sgn}(p-1)\right)S^2$$

where $\mathrm{sgn}(\cdot)$ is the standard sign function. Moreover, the equality holds if and only if at most two matrices are not zero.

Proof of main theorem Since ξ/H is parallel, we choose $\boldsymbol{e}_{n+1}=\xi/H$. Then $\omega_{(n+1)\alpha}=0$ for all α. It follows from Eqs.(3) and (9) that

$$H_{\alpha,k}=0,\quad H_{\alpha,kl}=0, \tag{16}$$

for all $\alpha>n+1$ and $k, l=1, \ldots, n$. Moreover, we have $\boldsymbol{L}_\alpha\boldsymbol{L}_{n+1} = \boldsymbol{L}_{n+1}\boldsymbol{L}_\alpha$ for all α.

We define $\overline{\boldsymbol{L}}_{n+1}, \overline{S}_{n+1}$ and $\overline{S}$ by

$$\overline{\boldsymbol{L}}_{n+1} = \boldsymbol{L}_{n+1} - H\boldsymbol{I}_n,\quad \overline{S}_{n+1} = S_{n+1} - nH^2,\quad \overline{S} = S - nH^2, \tag{17}$$

where $\boldsymbol{I}_n$ denotes the identity matrix of degree n. Moreover, we denote $S_I = \sum_{\beta>n+1} S_\beta$.

We now consider the cases $n\ge 3$ and $n=2$ separately.

Case $n\ge 3$. It follows from Eq.(7) that

$$\sum_{\beta>n+1} S_{n+1\beta}^2 = \sum_{\beta>n+1}\left\{\sum_{ij}(h_{ij}^{n+1} - H\delta_{ij})h_{ij}^\beta\right\} \le \overline{S}_{n+1}S_I. \tag{18}$$

On the other hand, it is easy to check that $Tr(\overline{\boldsymbol{L}}_{n+1})=0$. By using Lemma 2, we obtain

$$\begin{aligned}&nHTr(\boldsymbol{L}_{n+1}^3) = nHTr(\overline{\boldsymbol{L}}_{n+1}^3) + 3nH^2\overline{S}_{n+1} + n^2H^4\\ &\ge -\frac{n(n-2)}{\sqrt{n(n-1)}}H(\overline{S}_{n+1})^{\frac{3}{2}} + 3nH^2\overline{S}_{n+1} + n^2H^4\\ &\ge -\frac{n-2}{2\sqrt{n-1}}\overline{S}_{n+1}(anH^2 + \frac{1}{a}\overline{S}_{n+1}) + 3nH^2\overline{S}_{n+1} + n^2H^4,\end{aligned} \tag{19}$$

where a is any positive number.

Letting $\alpha=(n-2\sqrt{n-1})/(n-2)$, and substituting Eqs.(16), (18) and (19) into Eq.(14), we get

$$\begin{aligned}\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} &= n\Box H + \frac{1}{2}n^2\Delta(H^2) - n^2|\nabla H|^2 - nS_{n+1}^2\\ &\quad + nHTr(\boldsymbol{L}_{n+1}^3) - n^2H^2 - S_{n+1}^2 - \sum_{\beta>n+1} S_{n+1\beta}^2\end{aligned}$$

$$\geq n\Box H+\frac{1}{2}n^2\Delta(H^2)-n^2\left|\nabla H\right|^2$$
$$+\bar{S}_{n+1}\left(n-\frac{n-2\sqrt{n-1}}{2\sqrt{n-1}}S_{n+1}-S\right)$$
$$\geq n\Box H+\frac{1}{2}n^2\Delta(H^2)-n^2\left|\nabla H\right|^2$$
$$+\bar{S}_{n+1}\left(n-\frac{n}{2\sqrt{n-1}}S\right) \tag{20}$$

Taking the sum with respect to $\alpha>n+1$ on both side of Eq.(6), we have

$$\sum_{\substack{i,j,\\ \alpha>n+1}} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}=nS_I-\sum_{\substack{\alpha>n+1\\ \beta>n+1}}\left\{S_{\alpha\beta}^2+N(\boldsymbol{L}_\alpha\boldsymbol{L}_\beta-\boldsymbol{L}_\beta\boldsymbol{L}_\alpha)\right\}$$
$$+nH\sum_{\alpha>n+1}Tr(\boldsymbol{L}_\alpha^2\boldsymbol{L}_{n+1})-\sum_{\alpha>n+1}S_{\alpha(n+1)}^2. \tag{21}$$

For a given $\alpha>n+1$, we may choose $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ such that $h_{ij}^{\alpha}=\delta_{ij}h_{ii}^{\alpha}, h_{ij}^{n+1}=\delta_{ij}h_{ii}^{n+1}$. Then

$$nHTr(\boldsymbol{L}_{n+1}\boldsymbol{L}_\alpha^2)-S_{\alpha(n+1)}^2=\frac{1}{2}\sum_{i,j}h_{ii}^{n+1}h_{jj}^{n+1}(h_{ii}^{\alpha}-h_{jj}^{\alpha})^2,$$

since $\sum_i h_{ii}^{\alpha}=nH_\alpha=0,$ we can use Lemma 3 and get

$$nHTr(\boldsymbol{L}_{n+1}\boldsymbol{L}_\alpha^2)-S_{\alpha(n+1)}^2\geq-\frac{n}{2\sqrt{n-1}}S_\alpha S_{n+1}. \tag{22}$$

By using Lemma 4, we have

$$\sum_{\substack{\alpha>n+1,\\ \beta>n+1}}\left\{S_{\alpha\beta}^2+N(\boldsymbol{L}_\alpha\boldsymbol{L}_\beta-\boldsymbol{L}_\beta\boldsymbol{L}_\alpha)\right\}\leq\left(1+\frac{1}{2}\mathrm{sgn}(p-2)\right)S_I^2 \tag{23}$$

Substituting Eqs.(22) and (23) into Eq.(21), we have

$$\sum_{\substack{i,j,\\ \alpha>n+1}}h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}\geq S_I(n-\frac{n}{2\sqrt{n-1}}S_{n+1}-\left(1+\frac{1}{2}\mathrm{sgn}(p-2)\right)S_I) \tag{24}$$

If $n\geq 8$, or $n\geq 3$ and $p\leq 2$, then Eq.(24) becomes

$$\sum_{i,j,\alpha>n+1}h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}\geq S_I\left(n-\frac{n}{2\sqrt{n-1}}S\right). \tag{25}$$

From Eqs.(20) and (25), we have

$$\frac{1}{2}\Delta S=\sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2+\sum_{i,j}h_{ij}^{n+1}\Delta h_{ij}^{n+1}+\sum_{i,j,\alpha>n+1}h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}$$
$$\geq n\Box H+\frac{1}{2}n^2\Delta(H^2)-n^2\left|\nabla H\right|^2$$
$$+\sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2+\bar{S}\left(n-\frac{n}{2\sqrt{n-1}}S\right).$$

Note that $\Delta S=\Delta(n^2H^2)$. It follows that

$$0\geq n\Box H-n^2\left|\nabla H\right|^2+\sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2+\bar{S}\left(n-\frac{n}{2\sqrt{n-1}}S\right) \tag{26}$$

By Lemma 1, we have

$$\sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2=n^2\left|\nabla H\right|^2; \tag{27}$$

$$\bar{S}\left(n-\frac{n}{2\sqrt{n-1}}S\right)=0. \tag{28}$$

From Eqs.(1), (27), (28) and Lemma 1, it is easy to know that H is constant, and so is S. If $S\leq 2\sqrt{n-1}$, then either $\bar{S}=0$, then $S=nH^2=n(R-1)$, and M^n is a sphere $S^n(1/\sqrt{R})$ in S^{n+p}; or $S=2\sqrt{n-1}$, then the inequalities in Eqs.(18), (19), and Eqs.(22)–(25) become equal. Then we know $S_I=0$, and M^n lies in a totally geodesic submanifold S^{n+1} of S^{n+p}. The same arguments as those developed by Chern *et al.*(1970) show that M^n is locally a part of $S^{n-1}(r_1)\times S^1(r_2)$ in S^{n+1}, where

$$r_1^2=\sqrt{n-1}/(1+\sqrt{n-1}),\ \ r_2^2=1/(1+\sqrt{n-1}).$$

Moreover, since $n(n-1)(R-1)=n^2H^2-S\geq 0$ we need $n\geq 7$.

If $3\leq n\leq 7$ and $p\geq 3$, then Eq.(24) becomes

$$\sum_{i,j,\alpha>n+1} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \geq S_I(n-\frac{3}{2}S). \quad (29)$$

From Eqs.(20) and (29), we have

$$\frac{1}{2}\Delta S \geq n\square H + \frac{1}{2}n^2\Delta(H^2) - n^2|\nabla H|^2 + \sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2 + \bar{S}(n-\frac{3}{2}S).$$

Note that $\Delta S=\Delta(n^2H^2)$. It follows that

$$0 \geq n\square H - n^2|\nabla H|^2 + \sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2 + \bar{S}\left(n-\frac{3}{2}S\right) \quad (30)$$

By Lemma 1, we have

$$\sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2 = n^2|\nabla H|^2; \quad \bar{S}\left(n-\frac{3}{2}S\right)=0.$$

With the same arguments as above, we know H, S are constant. If $S\leq 2n/3$, then $S=n(R-1)$ and M^n is a sphere $S^n(1/\sqrt{R})$ in S^{n+p}; Since $H>0$ on M^n, combining Eq.(29), the case $S=2n/3$ cannot occur.

Case $n=2$. From Eq.(14), we get

$$\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} = n\square H + \frac{1}{2}n^2\Delta(H^2) - n^2|\nabla H|^2 + nS_{n+1} - S_{n+1}^2 - n^2H^2 - \sum_{\beta>n+1} S_{(n+1)\beta}^2 + nHTr(\boldsymbol{L}_{n+1}^3)$$

After a straight-forward computation, we get

$$\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} = n\square H + \frac{1}{2}n^2\Delta(H^2) - n^2|\nabla H|^2 + \bar{S}_{n+1}(2+2H^2-\bar{S})$$
$$\geq n\square H + \frac{1}{2}n^2\Delta(H^2) - n^2|\nabla H|^2 + \bar{S}_{n+1}(2+4H^2-S). \quad (31)$$

From Eq.(21) and Lemma 4, we get

$$\sum_{i,j,\alpha>n+1} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} = nS_I - \left(1+\frac{1}{2}\text{sgn}(p-2)\right)S_I^2 + nH\sum_{\alpha>n+1} Tr(\boldsymbol{L}_{\alpha}^2\boldsymbol{L}_{n+1}) - \sum_{\alpha>n+1} S_{\alpha(n+1)}^2. \quad (32)$$

If $p\geq 3$ then Eq.(32) becomes

$$\sum_{i,j,\alpha>n+1} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \geq nS_I - \frac{3}{2}SS_I + \frac{3}{2}S_{n+1}S_I + nH\sum_{\alpha>n+1} Tr(\boldsymbol{L}_{\alpha}^2\boldsymbol{L}_{n+1}) - \sum_{\alpha>n+1} S_{\alpha(n+1)}^2. \quad (33)$$

For a given $\alpha>n+1$, we may choose $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ such that $h_{ij}^{\alpha}=\delta_{ij}h_{ii}^{\alpha}, h_{ij}^{n+1}=\delta_{ij}h_{ii}^{n+1}$, for i, j=1, 2. Then $h_{11}^{\alpha}+h_{22}^{\alpha}=0$, we can get

$$nHTr(\boldsymbol{L}_{\alpha}^2\boldsymbol{L}_{n+1}) - S_{\alpha(n+1)}^2 + \frac{3}{2}S_{\alpha}S_{n+1}$$
$$=(h_{11}^{\alpha})^2(2(h_{11}^{n+1}+h_{22}^{n+1})^2 + (h_{11}^{n+1})^2 + (h_{22}^{n+1})^2)$$
$$=(h_{11}^{\alpha})^2(8H^2+S_{n+1}) \geq 5H^2S_{\alpha}. \quad (34)$$

Using Eq.(34), Eq.(33) becomes

$$\sum_{i,j,\alpha>n+1} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \geq S_I(n+5H^2-\frac{3}{2}S). \quad (35)$$

It is clear that $4H^2+2>2(5H^2+2)/3$. From Eqs.(31) and (35), we get

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2 + \sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} + \sum_{i,j,\alpha>n+1} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}$$
$$\geq n\square H + \frac{1}{2}n^2\Delta(H^2) - n^2|\nabla H|^2 + \sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2 + \bar{S}(n+5H^2-\frac{3}{2}S). \quad (36)$$

With the same argument as above, we have

$$\sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2 = n^2|\nabla H|^2; \; \bar{S}(n+5H^2-\frac{3}{2}S)=0.$$

And we again know that H, S are constant. If $S\leq n(5R-1)$, we know $S\leq 2(5H^2+2)/3$ by Eq.(1). Then either $\bar{S}=0$, and M^2 is a sphere $S^2(1/\sqrt{R})$ in S^{2+p}; or $S=n(5R-1)$, and all the inequality in Eqs.(31)–(36)

become equal. First, it is clear that $\overline{S}_{n+1}=0$, so M^2 is pseudo-umbilical. Combining the assumption $\boldsymbol{e}_{n+1}$ is parallel, M^2 lies in a totally umbilical submanifold $S^{2+(p-1)}(1/\sqrt{3R})$ of S^{2+p} (Yau, 1974). Moreover, M^2 is a minimal surface of a sphere $S^{2+(p-1)}(1/\sqrt{3R})$ with second fundamental form of constant length. By Lemma 4, we know there are two matrices of $\{\boldsymbol{L}_{2+1}, \ldots, \boldsymbol{L}_{2+p}\}$ which are not zero. Using Theorem 3 in (Chern *et al*., 1970), we know M^2 is a Veronese surface in $S^4(1/\sqrt{3R})$, then the conclusion holds.

If $p=2$, from Eq.(32) we can get

$$\begin{aligned}&\sum_{i,j} h_{ij}^{n+2}\Delta h_{ij}^{n+2}\\ &= nS_{n+2}+nHTr(\boldsymbol{L}_{n+1}\boldsymbol{L}_{n+2}^2)-S_{(n+2)(n+1)}^2-S_{n+2}^2\\ &= S_{n+2}(2+2H^2-S_{n+2})\geq S_{n+2}(2+4H^2-S). \quad (37)\end{aligned}$$

Using Eqs.(31) and (37), we can get at last

$$\sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2=n^2\left|\nabla H\right|^2;\ \overline{S}(n+4H^2-S)=0.$$

Since $n^2H^2-S=n(n-1)(R-1)\geq 0$, $S\leq n^2H^2<4H^2+2$, we have $\overline{S}=0$; $S=2H^2=2(R-1)$, M^2 is a sphere $S^2(1/\sqrt{R})$ in S^4.

We finish the proof of the main theorem.

RELATED EXAMPLE

Example Let $S^n(r)$ denote an n-dimensional sphere in R^{n+1} with radius r. Let

$$M_1=S^{n-1}((1+(n-1)^{-1/2})^{-1/2}),$$
$$M_2=S^1((1+(n-1)^{1/2})^{-1/2}).$$

Then the mapping from $M_1\times M_2$ to $S^{n+1}(1)$ is defined as follows. Let $(\boldsymbol{U}, \boldsymbol{V})$ be a point of $M_1\times M_2$, where $\boldsymbol{U}$ is a vector in R^n with length $(1+(n-1)^{-1/2})^{-1/2}$, and $\boldsymbol{V}$ is a vector in R^2 with length $(1+(n-1)^{-1/2})^{-1/2}$. Then $(\boldsymbol{U}, \boldsymbol{V})$ is a unit vector of $R^{n+2}=R^n\times R^2$. With the same argument as Chern *et al.*(1970), we can know $M_1\times M_2$ is a closed submanifold with parallel normalized mean curvature vector field immersed into a unit sphere $S^{n+1}(1)$ (In fact, the mean curvature vector field is parallel in this example). Moreover, we can know

$$R=\frac{n-2}{n}\left(1+\frac{1}{\sqrt{n-1}}\right), \quad S=2\sqrt{n-1}.$$

When $n\geq 7$, R is constant and ≥ 1.

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