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Bézier curves with shape parameter*

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Abstract: In this paper, Bézier basis with shape parameter is constructed by an integral approach. Based on this basis, we define the Bézier curves with shape parameter. The Bézier basis curves with shape parameter have most properties of Bernstein basis and the Bézier curves. Moreover the shape parameter can adjust the curves' shape with the same control polygon. As the increase of the shape parameter, the Bézier curves with shape parameter approximate to the control polygon. In the last, the Bézier surface with shape parameter is also constructed and it has most properties of Bézier surface.

Keywords: Bézier curve, Bézier basis with shape parameter, Bernstein basis
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INTRODUCTION

The Bézier curves and surfaces form a basic tool for constructing free form curves and surfaces. Many basis-like Bézier basis are presented. Said (1989) and Goodman and Said (1991) constructed the Ball basis. Mainar *et al*.(2001) found some bases for the spaces $\{1, t, \cos t, \sin t, \cos 2t, \sin 2t\}$, $\{1, t, t^2, \cos t, \sin t\}$, and $\{1, t, \cos t, \sin t, t\cos t, t\sin t\}$. Chen and Wang (2003) gave the C-Bézier basis in the space $\{1, t, t^2, \ldots, t^{n-2}, \sin t, \cos t\}$. Wang and Wang (2004) put forward Uniform B Spline with shape parameter which is extended from Uniform B Spline and possesses many excellent properties. In this paper, we present Bézier basis with shape parameter by an integral approach. The shape parameter can adjust the curve's position and the Bézier curves with shape parameter have most properties of the Bézier curves.

The rest of this paper is organized as follows. Section 2 gives an algorithm for constructing the basis. Some properties of the Bézier basis with shape parameter are discussed in Section 3. Using this basis, we give the definition of a Bézier curve with shape parameter which has most properties of the Bézier curve in Section 4. Some figure examples are shown in Section 4.

CONSTRUCTION OF THE BÉZIER BASIS WITH SHAPE PARAMETER

We first give two initial functions (Fig.1)

$$N_{0,1}(t)=\frac{3}{2}\lambda(1-t)^2+(1-\lambda)(1-t),$$
$$N_{1,1}(t)=\frac{3}{2}\lambda t^2+(1-\lambda)t, \tag{1}$$

where $t\in[0, 1]$.

For $n\geq 2$ we define the Bézier basis with shape parameter $\{N_{0,n}(t), N_{1,n}(t), \ldots, N_{n,n}(t)\}$ recursively by

$$N_{0,n}(t)=1-\int_0^t \delta_{0,n-1}N_{0,n-1}(x)\mathrm{d}x,$$
$$N_{i,n}(t)=\int_0^t (\delta_{i-1,n-1}N_{i-1,n-1}(x)-\delta_{i,n-1}N_{i,n-1}(x))\mathrm{d}x,$$
$$N_{n,n}(t)=\int_0^t \delta_{n-1,n-1}N_{n-1,n-1}(x)\mathrm{d}x. \tag{2}$$

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In these formulae,

$$\delta_{i,n}=\int_0^1 N_{i,n}(t)\mathrm{d}t,\quad i=0,1,...,n.$$

$\lambda\in(-\infty,-2)\cup(-2,1]$ for $n\geq3$;

when n=2, $\lambda\in(-2,1]$.

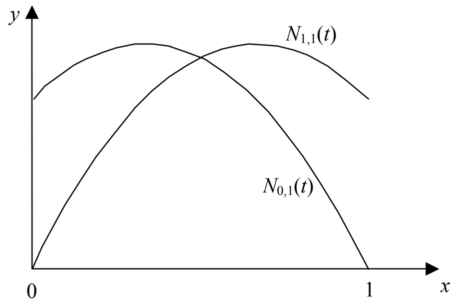


Fig.1 The two initial functions (λ=−1)

If λ=−2, then $N_{0,1}(t)$=$N_{1,1}(t)$, $N_{1,2}(t)$=0 and the quadratic Bézier curve with shape parameter is line, so we make $\lambda\neq-2$. We choose the two symmetric initial functions in Eq.(1) in order that $\int_0^1(N_{0,1}(t)+N_{0,1}(t))\mathrm{d}t=1$ and $N_{0,1}(t)$, $N_{1,1}(t)$ are quadratic function of t with shape parameter λ. Defining the basis functions in Eq.(2) recursively ensures possession of many properties that will be discussed in Section 3. The parameter λ can change the shape of the curve constructed by this basis and in particular, when λ=0, we get the Bézier basis for the polynomial space from Eqs.(1) and (2), so we name it Bézier basis with shape parameter. Fig.2 shows the quadratic Bézier basis with shape parameter

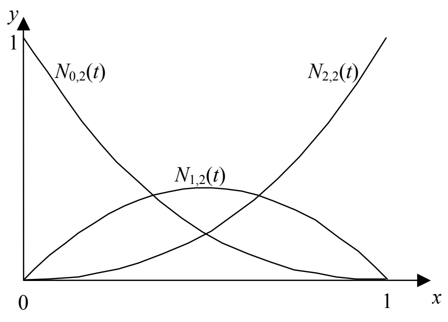


Fig.2 The quadratic Bézier basis with shape parameter (λ=0.5)

PROPERTIES OF THE BASIS

Properties at the endpoints

Lemma 1 At the endpoints, the Bézier basis with shape parameter has the same properties as the Bézier basis. That is, for $n\geq2$,

(a) $N_{0,n}(0)=N_{n,n}(1)=1$, (3)

(b) $N_{n,n}^{(n-1)}(0)=N_{0,n}^{(n-1)}(1)=0$,

$N_{i,n}^{(j)}(0)=N_{i,n}^{(k)}(1)=0$,

j=0,1,…,i−1; k=0,1,…,n−i−1; i=1,2,…,n−1. (4)

(c) $N_{i,n}^{(i)}(0)=\delta_{0,n-i}\delta_{1,n-i+1}\cdots\delta_{i-2,n-2}\delta_{i-1,n-1}$,

i=1,2,…,n.

By Eqs.(1) and (2), it is easy to prove Lemma 1 by induction on n.

Linear independence

In order to check the independence of $\{N_{0,n}, N_{1,n}, …, N_{n,n}\}$, we consider a trivial linear combination $\sum_{i=0}^n \alpha_i N_{i,n}(t)=0$, $t\in[0,1]$. By taking t=0, we get from Eq.(4) that α_0=0.

Differentiating the linear combination k times we deduce again from Eq.(4) that α_k=0 for k=1, …, n. That is, $N_{i,n}(t)$ (i=0, 1, …, n) are linearly independent. Therefore $\{N_{0,n}, N_{1,n}, …, N_{n,n}\}$ is linearly independent.

Positivity

Lemma 2 $N_{i,n}(t)$ (i=0, 1, …, n) has no zero on $t\in(0, 1)$.

Proof Using Eq.(2) recursively, we get $N_{0,n}^{(n-1)}(t)=(-1)^{n-1}a_0N_{0,1}(t)$, $N_{n,n}^{(n-1)}(t)=b_nN_{1,1}(t)$, $a_0,b_n>0$, $N_{i,n}^{(n-1)}(t)=a_iN_{0,1}(t)-b_iN_{1,1}(t)$, $a_ib_i>0$, $(i=1,\dots,n-1)$, a_i,b_i are constants that are independent of t and λ. Obviously $N_{0,n}^{(n-1)},N_{n,n}^{(n-1)}$ has only one zero on $t\in[0,1]$. Because $N_{i,n}^{(n-1)}(0)N_{i,n}^{(n-1)}(1)=-\left(\frac{2+\lambda}{2}\right)^2a_ib_i<0$ and $N_{i,n}^{(n-1)}$ is a quadratic function of t, $N_{i,n}^{(n-1)}$ $(i=1,\cdots,n-1)$ has also only one zero on $t\in[0,1]$. By Rolle's Theorem, we have that $N_{i,n}$ (i=0, …, n) has at most n zeros on $t\in[0, 1]$. We see in Eq.(4) that $N_{i,n}$ has n zeros on $t\in[0, 1]$, including the i-fold zero at 0 and the (n−i)-fold zero at 1. So $N_{i,n}(t)$ (i=0, 1,

…, n) has no zero on $t\in(0, 1)$.

Lemma 3 The Bézier basis with shape parameter are positive on [0, 1].

Proof Consider an arbitrary Bézier basis function with shape parameter $N_{i,n}(t)$, $n\geq 2$, $0\leq i\leq n$.

From Lemma 2, we conclude that $N_{i,n}(t)$ has no zero on $t\in(0, 1)$. In other words, $N_{i,n}(t)$ is either positive or negative on the interval. We conclude from Eq.(4) that $N_{i,n}(t)$ is positive on (0, 1).

Since $N_{i,n}(t)$ is arbitrary, we know that the Bézier basis with shape parameter is positive on [0, 1].

Lemma 4 The Bézier basis with shape parameter is normalized, that is,

$$\sum_{i=0}^{n} N_{i,n}(t)=1 .$$

We summarize Lemmas 3 and 4 in Proposition 1.

Proposition 1 The Bézier basis with shape parameter is a blending system.

Symmetry

Proposition 2 $N_{i,n}(t)=N_{n-i,n}(1-t)$ for $t\in[0, 1]$, (i=0, 1, …, n).

Proof We prove this proposition by induction. When n=1, the proposition obviously holds by the definition of the Bézier basis with shape parameter. Assume that the property holds for $n=k$, that is $N_{i,k}(t)=N_{k-i,k}(1-t)$. Hence we have,

$$\int_0^{1-t} N_{k-i,k}(x)\mathrm{d}x=-\int_1^{t} N_{k-i,k}(1-x)\mathrm{d}x$$
$$=\int_t^{1} N_{i,k}(x)\mathrm{d}x=\delta_{i,k}^{-1}-\int_0^{t} N_{i,k}(x)\mathrm{d}x$$

By letting t=0, we obtain $\delta_{k-i,k}=\delta_{i,k}$. Therefore, for $1<i<k+1$, we have

$$\begin{aligned}&N_{k+1-i,k+1}(1-t)\\&=\delta_{k-i,k}\int_0^{1-t} N_{k-i,k}(x)\mathrm{d}x-\delta_{k+1-i,k}\int_0^{1-t} N_{k+1-i,k}(x)\mathrm{d}x\\&=\left(1-\delta_{i,k}\int_0^{t} N_{i,k}(x)\mathrm{d}x\right)-\left(1-\delta_{i-1,k}\int_0^{t} N_{i-1,k}(x)\mathrm{d}x\right)\\&=N_{i,k+1}(t) .\end{aligned}$$

The proof for the case when i=1 and i=k+1 is similar. So, the proposition holds by induction on n.

GEOMETRIC PROPERTIES OF THE BÉZIER CURVE WITH SHAPE PARAMETER

A Bézier curve with shape parameter $p(t)$ with control points p_i is defined by

$$p(t)=\sum_{i=0}^{n} p_i N_{i,n}(t), t\in[0,1] \tag{5}$$

where $\{N_{i,n}(t)\}$ is Bézier basis with shape parameter.

Geometric properties at the endpoints

The geometric properties at the endpoints of the Bézier curves with shape parameter can be easily deduced from those of the Bézier basis with shape parameter.

(a) $p(0)=p_0$, $p(1)=p_n$, (6)

(b) $p^{(k)}(0)=\sum_{i=0}^{k} p_i N_{i,n}^{(k)}(0)$. (7)

Convex hull property

The entire Bézier curve with shape parameter Eq.(5) must lie inside its control polygon spanned by p_0, p_1, …, p_n. This property is a consequence of Proposition 1. Fig.3 shows convex hull property.

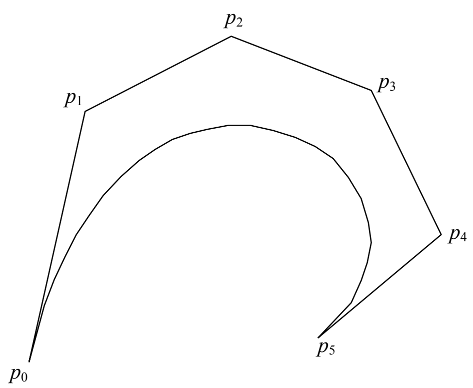


Fig.3 Convex hull property

Differentiation

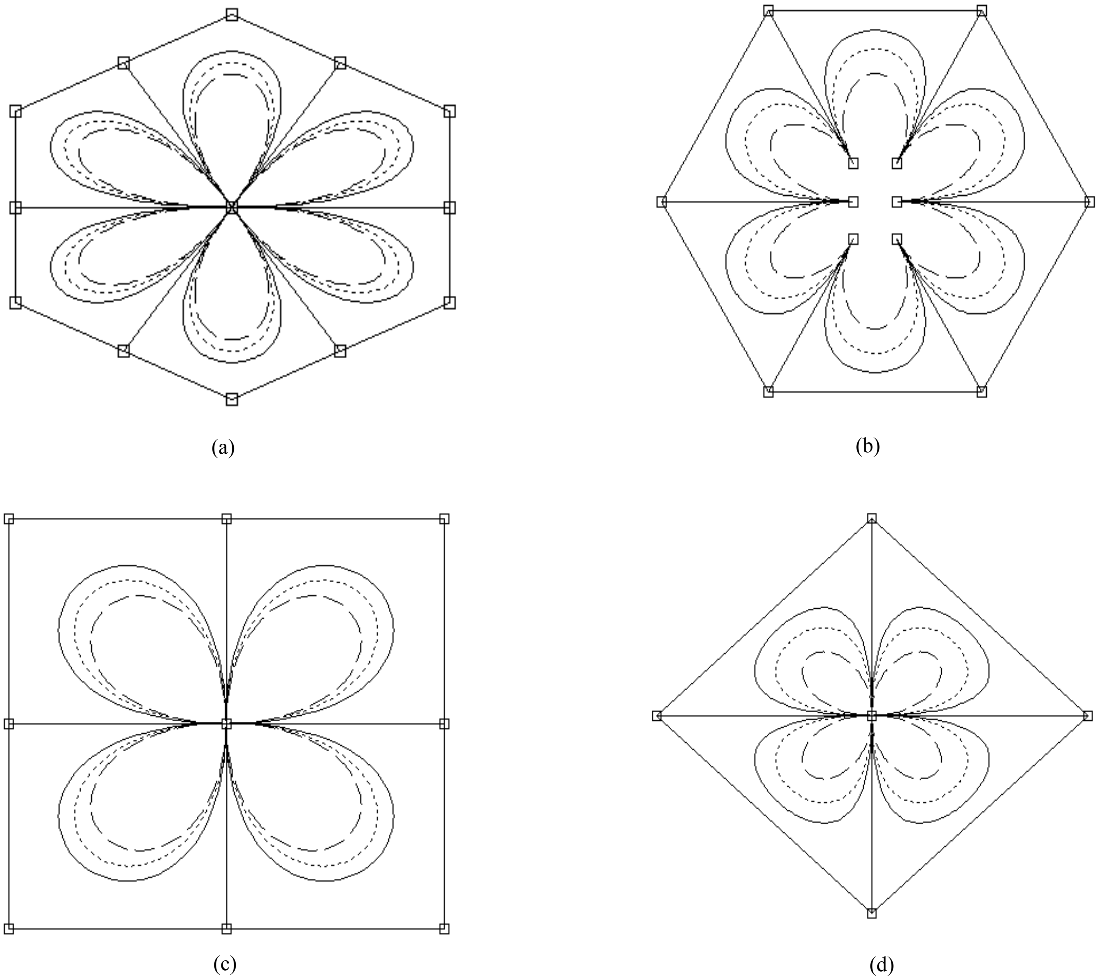
The derivative $p'(t)$ of degree-(n+1) Bézier curves with shape parameter $p(t)$ is clearly a degree-n curve. Such a curve can be written in Bézier curves with shape parameter-like form as

$$p'(t)=\sum_{i=0}^{n-1}p_iN_{i,n-1}(t),t\in[0,1] \tag{8}$$

where p_i (i=0, 1, …, n−1) are the control points of $p'(t)$. Differentiating the functions in Eq.(2) and after some algebraic manipulations, we find that the control points of $p'(t)$ in the above form are given by $\delta_{k,n-1}(P_{i+1}-P_i)$.

Some examples

Figs.4a and 4b respectively show the hexagon consisting of six symmetric control polygons and the flowers consisting of six symmetric quartic and cubic Bézier curves with shape parameter for λ=1, 0, −3 (solid, dotted and dashed lines). Fig.4c and 4d respectively show the square consisting of four symmetric control polygons and the flowers consisting of four symmetric quartic and cubic Bézier curves with shape parameter for λ=1, 0, −3 (solid, dotted and dashed lines). The symbol "□" is the control point in all figures. Fig.5 shows degree-6 Bézier curves with shape parameter for λ=1, 0, −2.01, −15 (solid, dotted, dashed and dashed-dot lines). The figures show that the Bézier curves with shape parameter approximate to the control polygon as the increase of the shape parameter.



(a) (b) (c) (d)

Fig.4 The flowers consisting of symmetric Bézier curves with shape parameter
(a) In six symmetric closed control polygons; (b) In six symmetric open control polygons; (c) In four symmetric closed control polygons; (d) In four symmetric open control polygons

BÉZIER SURFACE WITH SHAPE PARAMETER

Using the tensor product, we can construct Bézier surface with shape parameter

$$p(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}b_{i,j}N_{i,m}(u)N_{j,n}(v),\quad 1\le u,\ v\le 1,$$

in which $N_{i,m}(u)$, $N_{j,n}(v)$ are the Bézier basis functions with shape parameter, and $b_{i,j}$ is the control point. The tensor product of Bézier surface with shape parameter has properties similar to those of the tensor product of Bézier surface.

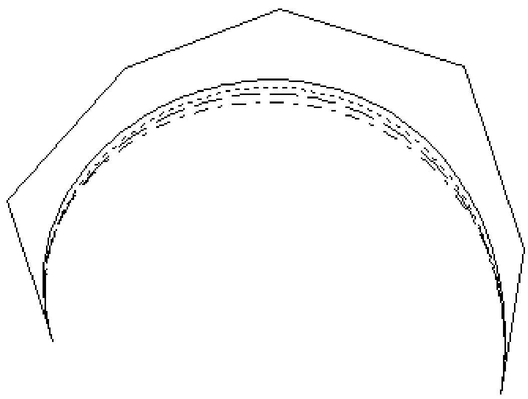


Fig.5 Degree-6 Bézier curves with shape parameter

CONCLUSION

Different curves lying on the Bézier curve of degree-k nearby can be created by this way in the paper. The Bézier curves with shape parameter approximate to the control polygon as the increase of the shape parameter λ. We can design Bézier curves by choosing different shape parameter in $\lambda\in(-\infty,-2)\cup(-2,1]$. Since Bézier curves with shape parameter have many of the same of their properties and structure as those of ordinary Bézier curves and preserve some practical geometry properties, they can more conveniently be used as such. However there are some deficiencies in Bézier basis with shape parameter, such as how to control the shape parameter and what is the geometric meaning of the shape parameter, etc. In the future we will research those problems.

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