Journal of Zhejiang University SCIENCE
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

Riemann surface with almost positive definite metric*

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Received Nov. 8, 2004; revision accepted Feb. 22, 2005

Abstract: In this paper, we consider and resolve a geometric problem by using $\mu(z)$-homeomorphic theory, which is the generalization of quasiconformal mappings. A sufficient condition is given such that a C^1-two-real-dimensional connected orientable manifold with almost positive definite metric can be made into a Riemann surface by the method of isothermal coordinates. The result obtained here is actually a generalization of Chern's work in 1955.

Keywords: Quasiconformal mapping, $\mu(z)$-homeomorphisms, Beltrami equation, Isothermal coordinates
doi:10.1631/jzus.2005.A0747 **Document code:** A **CLC number:** O174.5; O186

Let M be a C^1-two-real-dimensional connected orientable manifold with maximal set of charts $\{U_\alpha, z_\alpha\}_{\alpha\in\Lambda}$ on M. The metric on M is defined locally by the line elements ds as

$$ds^2=g_{11}dx^2+2g_{12}dxdy+g_{22}dy^2 \tag{1}$$

By adopting the complex notation dz=dx+idy, d$\bar{z}$ = dx−idy, the above equality can be written as

$$ds=\nu|dz+\mu d\bar{z}| \tag{2}$$

where

$$\nu^2=\left(g_{11}+g_{22}+2\sqrt{g_{11}g_{22}-g_{12}^2}\right)/4,$$

$$\mu=\frac{g_{11}-g_{22}+2ig_{12}}{g_{11}+g_{22}+2\sqrt{g_{11}g_{22}-g_{12}^2}}. \tag{3}$$

It is a classical result that M is a Riemann surface if $g_{11}=g_{22}$ and $g_{12}=0$. In this case,

$$ds=\nu|dz|. \tag{4}$$

Local coordinates z of M with this property are called isothermal.

Let us consider another local parameter of M which defines the local coordinates $\tilde{z}$. If the coordinates z and $\tilde{z}$ are both isothermal ($ds=\tilde{\nu}|d\tilde{z}|$) and if the induced mapping $z\to\tilde{z}$ is defined in some non-empty set of the plane, then

$$\tilde{\nu}|d\tilde{z}|=\nu|dz| \tag{5}$$

This shows that $z\to\tilde{z}$ is conformal.

In order that isothermal parameters exist it is necessary to impose on the metric some regularity assumptions. In fact, it was observed by Hartman and Wintner (1953) that it is not sufficient to assume the functions g_{ij} to be only continuous. Chern (1955) proved that the isothermal parameters exist by requiring that functions g_{ij} satisfy a Hölder condition of the same order τ, $0<\tau\leq 1$. In this paper, we suppose that $g_{ij}\in C^\tau$, $0<\tau\leq 1$ and consider the 2-real dimensional manifold M with almost positive definite metric. More precisely we have the following theorem:

Theorem 1 Let M be a $C^{1+\tau}$-two-real-dimensional connected orientable manifold with maximal set of charts $\{U_\alpha, z_\alpha\}_{\alpha\in\Lambda}$ on M and with the line elements ds as in Eq.(1). Assume that

(i) $E=\{z\,|\,g_{11}g_{22}-g_{12}^2=0\}$ is composed of at

*Project (No. 10101023) supported by the National Natural Science Foundation of China

most countable curves;

(ii) $G(z)=(g_{11}+g_{22})/\sqrt{g_{11}g_{22}-g_{12}^2}$ is locally in L^p (p>1);

(iii) for any z_0 in the neighborhood U,

$$\int_{\varepsilon}^{r_0}\frac{\mathrm{d}r}{rG^*(z_0,r)}\to\infty \quad \text{as } \varepsilon\to 0$$

where

$$G^*(z_0,r)=\frac{1}{2\pi}\int_0^{2\pi}G(z_0+r\mathrm{e}^{i\theta})\mathrm{d}\theta .$$

Then M can be made into a Riemann surface.

Proof of the theorem Let z be the local parameter of M. For convenience, we identify $z(U)$ with U. In order that the isothermal coordinates exist, we consider the following Beltrami equation

$$w_{\bar{z}}(z)=\mu(z)w_z(z) \tag{6}$$

where $\mu(z)$ is represented in Eq.(3).

Denote the dilatation function of w by $D(z)=(1+|\mu(z)|)/(1-|\mu(z)|)$. Then, following from Eq.(3), we have

$$D(z)=\frac{g_{11}+g_{22}+\sqrt{(g_{11}-g_{22})^2+4g_{12}^2}}{2\sqrt{g_{11}g_{22}-g_{12}^2}}. \tag{7}$$

It is obvious that

$$G(z)/2\le D(z)\le G(z).$$

So, $G(z)$ can be replaced by $D(z)$ in Assumptions (ii) and (iii).

In Assumption (i), E is the exceptional set, where the dilatation function $D(z)$ takes value of infinity. By the continuity of functions g_{ij}, E is closed. In view of Theorem 1 in Chen (2003), Assumptions (ii) and (iii) allow Eq.(6) a $\mu(z)$-homeomorphic solution $w(z)$.

On the other hand, the assumptions $g_{ij}\in C^{\alpha}$, imply the locally Hölder continuity of $\mu(z)$. Because the exceptional set E is closed, according to Theorem 7.2 in Lehto and Virtanen (1973)'s, $\partial_z w$ is continuous except on E. By Eq.(6) and topologicity of w, $|\partial_z w|>0$, $z\in U-E$. Hence,

$$|\mathrm{d}w|=|\partial_z w\mathrm{d}z+\partial_{\bar{z}}w\mathrm{d}\bar{z}|=|\partial_z w||\mathrm{d}z+\mu\mathrm{d}\bar{z}|. \tag{8}$$

Substituting Eqs.(2) and (4) into Eq.(8), we have

$$|\mathrm{d}w|=\frac{|\partial_z w|}{v}\mathrm{d}s.$$

That is,

$$\mathrm{d}s=\frac{v}{|\partial_z w|}|\mathrm{d}w|=\lambda|\mathrm{d}w| \tag{9}$$

The function $\lambda(z)$ is positive and continuous except on $U-E$.

Let $p\in M$ and U_p its neighbour domain belonging to $U_1\cap U_2$. Let z_1 and z_2 be parameters of U_1 and U_2 respectively. Let w_1 and w_2 be the corresponding $\mu(z)$-homeomorphic solution of Eq.(6). By Eq.(9), we have

$$\mathrm{d}s=\lambda_1|\mathrm{d}w_1|=\lambda_2|\mathrm{d}w_2| \tag{10}$$

where λ_1 and λ_2 are positive and continuous respectively on U_1-E_1 and U_2-E_2.

The isothermal coordinates from Eq.(10) show that the map $w_1\to w_2$ is conformal from $w_1(U_1)-w_1(E_1)$ onto $w_2(U_2)-w_2(E_2)$. Since E_i is composed of at most countable curves, so is $w_i(E_i)$ (i=1,2). Thus the homeomorphism $w_1\to w_2$ is conformal almost everywhere and therefore is a conformal mapping. Therefore the charts $\bigcup_{\alpha\in\Lambda}\{U_\alpha, w_\alpha\circ z_\alpha\}$ make a conformal structure for the manifold M. The theorem is proved.

Remark This short paper is an application of $\mu(z)$-homeomorphisms (or μ-homeomorphisms, mappings of finite distortion) on Riemann surfaces. A topological mapping f of a region Ω is called a $\mu(z)$-homeomorphism if f is ACL and satisfies the Beltrami Eq.(6), where the Beltrami coefficient $\mu(z)$ is a measurable function defined a.e. in Ω. Recently, the degeneration of μ in terms of $\mu(z)$ has been extensively studied. Many interesting results have been obtained. See David (1988), Brakalova and Jenkins (1998), Gutlyanskii *et al.*(2005), Iwaniec and Martin (2001), Koskela and Rajara (2003), Koskela and Malý (2003), Koskela and Onninen (2003), Martio and Miklyukov (2004), Rajara (2004), Ryazanov *et al.*(2001a; 2001b; in preprint) and the references therein. The theory of $\mu(z)$-homeomorphisms also has important applications on dynamic systems. See Petersen and Zakeri (2004).

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