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Two kinds of B-basis of the algebraic hyperbolic space*

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Abstract: In this paper, two new kinds of B-basis functions called algebraic hyperbolic (AH) Bézier basis and AH B-Spline basis are presented in the space Γ_k=span$\{1,t,\ldots,t^{k-3},\sinh t,\cosh t\}$, in which K is an arbitrary integer larger than or equal to 3. They share most optimal properties as those of the Bézier basis and B-Spline basis respectively and can represent exactly some remarkable curves and surfaces such as the hyperbola, catenary, hyperbolic spiral and the hyperbolic paraboloid. The generation of tensor product surfaces of the AH B-Spline basis have two forms: AH B-Spline surface and AH T-Spline surface.

Key words: Algebraic hyperbolic Bézier basis, Algebraic hyperbolic B-Spline basis, Algebraic hyperbolic Bézier curve, Algebraic hyperbolic B-Spline curve
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INTRODUCTION

Bézier basis and B-Spline basis are two important bases of the polynomial space spanned by $\{1,t,\ldots,t^{n-2}, t^{n-1},t^n\}$, in which n is an arbitrary positive integer. However, as shown by Mainar *et al.* (2001), there still exist several limitations of Bézier model and B-Spline model. Chen and Wang (2003) and Wang *et al.*(2004) constructed C-Bézier basis and the non uniform algebraic trigonometric (NUAT) B-Spline basis of the space spanned by $\{1,t,\ldots,t^{n-2}, \sin t,\cos t\}$ which can represent exactly transcendental curves such as helix and cycloid. Koch and Lyche (1991) presented a kind of exponential spline. In this paper, we will give two bases for the algebraic hyperbolic space Γ_k spanned by $\{1,t,\ldots,t^{k-3}, \sinh t,\cosh t\}$, which can represent exactly some remarkable curves such as the hyperbola and the catenary.

ALGEBRAIC HYPERBOLIC (AH) BÉZIER BASIS

Definition of the AH Bézier basis

We first present the AH Bézier basis functions. The construction is recursive, starting with the two initial functions:

$$B_{0,1}(t)=\frac{\sinh(\alpha-t)}{\sinh\alpha},\quad B_{1,1}(t)=\frac{\sinh t}{\sinh\alpha}$$
$$t\in[0,\alpha],\qquad \alpha\in(0,+\infty) \tag{1}$$

For $n>1$, the AH Bézier basis functions $\{B_{0,n},B_{1,n},\ldots,B_{n,n}\}$ of the space Γ_{n+1} spanned by $\{1,t,\cdots,t^{n-2},\sinh t,\cosh t\}$ are defined recursively by:

$$B_{0,n}(t)=1-\int_0^t \delta_{0,n-1}B_{0,n-1}(s)\mathrm{d}s$$
$$B_{i,n}(t)=\int_0^t (\delta_{i-1,n-1}B_{i-1,n-1}(s)-\delta_{i,n-1}B_{i,n-1}(s))\mathrm{d}s$$
$$B_{n,n}(t)=\int_0^t \delta_{n-1,n-1}B_{n-1,n-1}(s)\mathrm{d}s \tag{2}$$

where $\delta_{i,n}=1\Big/\int_0^\alpha B_{i,n}(t)\mathrm{d}t$, $0<i<n$. Then we get the definition of the AH Bézier basis:

Definition 1 $\{B_{0,n},B_{1,n},\ldots,B_{n,n}\}$ is called the AH

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Bézier basis for the space Γ_{n+1} spanned by $\{1, t, \cdots, t^{n-2}, \sinh t, \cosh t\}$. See Fig.1.

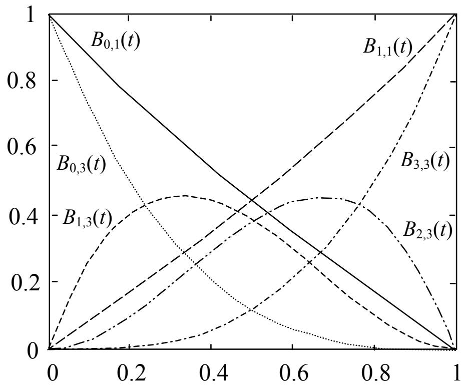

Fig.1 AH Bézier basis functions of order 2 and 4

The linear independence of the $\{B_{0,n}, B_{1,n}, \ldots, B_{n,n}\}$ will be proved in Section 2. In particular, if we use $B_{0,1}(t)=1-t$, $B_{1,1}(t)=t$ as the two initial functions, we get the Bézier basis for the polynomial space from Eq.(2); if we use "sin" to replace "sinh" in the initial functions, we will get the C-Bézier basis for the algebraic trigonometric space from Eq.(2).

Properties of the AH Bézier basis

(1) Partition of unity

By the definition of the AH Bézier basis, it is easy to prove $\sum_{i=0}^{n} B_{i,n}(t)=1$, $t\in[0,\alpha]$, $\alpha\in(0,+\infty)$.

(2) Symmetry

$B_{i,n}(t)=B_{n-i,n}(\alpha-t)$ for $t\in[0,\alpha]$, $i=0,1,\ldots,n$.

Proof Consider the situation when $n=1$, the symmetry property is obvious. Suppose it holds for $n-1$;

$$B_{i,n-1}(t)=B_{n-1-i,n-1}(\alpha-t)$$

We now prove the property by induction on n:

$$\int_0^{\alpha-t} B_{n-1-i,n-1}(s)\mathrm{d}s = -\int_0^{t} B_{n-1-i,n-1}(\alpha-s)\mathrm{d}s$$
$$= \int_t^{\alpha} B_{i,n-1}(s)\mathrm{d}s = \delta_{i,n-1}^{-1} - \int_0^{t} B_{i,n-1}(s)\mathrm{d}s$$

Letting $t=0$, we obtain $\delta_{i,n-1}=\delta_{n-1-i,n-1}$. For $1<i<n$, we have:

$$B_{n-i,n}(\alpha-t) = \delta_{n-1-i,n-1}\int_0^{\alpha-t} B_{n-1-i,n-1}(s)\mathrm{d}s - \delta_{n-i,n-1}\int_0^{\alpha-t} B_{n-i,n-1}(s)\mathrm{d}s$$
$$= \left(1-\delta_{i,n-1}\int_0^{t} B_{i,n-1}(s)\mathrm{d}s\right) - \left(1-\delta_{i-1,n-1}\int_0^{t} B_{i-1,n-1}(s)\mathrm{d}s\right)$$
$$= B_{i,n}(t)$$

The proof for the situation when $i=1$ and $i=n$ is similar. That is, the symmetry property is proved.

(3) Properties of the endpoints

At the endpoints, the AH Bézier basis has the same properties as the Bézier basis and C-Bézier basis. That is, for $n\geq 2$

(a) $B_{0,n}(0)=B_{n,n}(\alpha)=1$ (3)

(b) $B_{i,n}^{j}(0)=B_{i,n}^{j}(\alpha)=0$,

$j=0,1,\ldots,i-1$, $k=0,1,\ldots,n-i-1$ (4)

(c) $B_{i,n}^{(i)}(0)=\delta_{i-1,n-1}\delta_{i-2,n-2}\ldots\delta_{0,n-i}$, $i=1,\ldots,n$ (5)

(d) $B_{i,n}^{(n-i)}(\alpha)=\delta_{i,n-1}\delta_{i,n-2}\ldots\delta_{i,i}$, $i=0,\ldots,n-1$ (6)

Eq.(5) and Eq.(6) can be proved by induction on n.

(4) Linear independence

Let $\sum_{i=0}^{n} a_i B_{i,n}(t)=0$, $t\in[0,\alpha]$. By taking $t=0$, we get from Eq.(4) that $\alpha_0=0$. Differentiating the above formula k times, we deduce again from Eq.(4) that $\alpha_k=0$ for $k=1,\ldots,n$. Therefore $B_{0,n}, B_{1,n}, \ldots, B_{n,n}$ are linear independent and $\{B_{0,n}, B_{1,n}, \ldots, B_{n,n}\}$ is a basis of Γ_{n+1}.

(5) Degree elevation:

Differentiating $B_{i,n}(t)=\sum_{j=0}^{n+1} a_{i,j}B_{j,n+1}(t)$ k times, we have from property (3) of the AH Bézier basis that $\alpha_{i,k}=0$ for $k=0,1,\ldots,i-1,i+2,\ldots,n+1$. Thus:

$$B_{i,n}(t)=a_{i,i}B_{i,n+1}(t)+a_{i,i+1}B_{i+1,n+1}(t) \tag{7}$$

Using L'Hospital's Rule, we have

$$a_{i,i}=\frac{B_{i,n}(t)-a_{i,i+1}B_{i+1,n+1}(t)}{B_{i,n+1}(t)}$$
$$=\lim_{t\to 0+}\frac{B_{i,n}(t)-a_{i,i+1}B_{i+1,n+1}(t)}{B_{i,n+1}(t)}$$

$$=\lim_{t\to 0+}\frac{B'_{i,n}(t)-a_{i,i+1}B'_{i+1,n+1}(t)}{B'_{i,n+1}(t)}$$

$$=\frac{B_{i,n}^{(i)}(0)}{B_{i,n+1}^{(i)}(0)}=\frac{\delta_{i-1,n-1}\delta_{i-2,n-2}\cdots\delta_{0,n-i}}{\delta_{i-1,n}\delta_{i-2,n}\cdots\delta_{0,n+1-i}}$$

and $$a_{i,i+1}=\frac{B_{i,n}^{(n-i)}(\alpha)}{B_{i+1,n+1}^{(n-i)}(\alpha)}=\frac{\delta_{i,n-1}\delta_{i,n-2}\cdots\delta_{i,i}}{\delta_{i+1,n}\delta_{i+1,n-1}\cdots\delta_{i+1,i+1}}$$

From property (1) of the AH Bézier basis, we know

$$\sum_{i=0}^{n}B_{i,n}(t)=\sum_{i=0}^{n}[a_{i,i}B_{i,n+1}(t)+a_{i,i+1}B_{i+1,n+1}(t)]$$
$$=1=\sum_{i=0}^{n+1}B_{i,n+1}(t)$$

By the linear independence of the AH Bézier basis, we have

$$a_{i,i+1}=\begin{cases}1-a_{i+1,i+1}, & i=0,1,\cdots,n-1\\ 1, & i=n\end{cases}$$

In fact, $\alpha_{0,0}=\alpha_{n,n+1}=1$. Thus we have the degree elevation formula:

$$B_{i,n}(t)=\frac{B_{i,n}^{(i)}(0)}{B_{i,n+1}^{(i)}(0)}B_{i,n+1}(t)+\left(1-\frac{B_{i+1,n}^{(i+1)}(0)}{B_{i+1,n+1}^{(i)}(0)}\right)B_{i+1,n+1}(t)$$

Positivity $B_{i,n}(t)>0$ for $t\in(0,\alpha)$, so AH Bézier basis is a blending system.

Proof Consider an arbitrary AH Bézier basic function $B_{i,n}(t)$, $n\geq 2$, $0\leq i\leq n$. By the Rolle's Theorem and property (4) of the AH Bézier basis, we deduce that $B_{i,n}(t)$ has and only has n zeros at $[0,\alpha]$ including the i-fold zero at 0 and the $(n-i)$-fold zero at α, so $B_{i,n}(t)$ is either positive or negative on the interval $(0, \alpha)$. From property (5) of the AH Bézier basis, we get that the AH Bézier basis is positive for $t\in(0,\alpha)$.

AH Bézier curves

An order $n+1$ AH Bézier curve $p(t)$ with control points $\boldsymbol{p}_i$ is defined as:

$$p(t)=\sum_{i=0}^{n}B_{i,n}(t)\boldsymbol{p}_i,\ \ t\in[0,\alpha],\ \alpha\in(0,+\infty) \tag{8}$$

where $\{B_{i,n}(t)\}_{i=0}^{n}$ is the AH Bézier basis for the space Γ_{n+1}, and α is a global shape parameter. AH Bézier curve has many properties the same as those of the Bézier curve:

(1) Endpoints interpolation

$$p(0)=\boldsymbol{P}_0,\ \ p(\alpha)=\boldsymbol{P}_n.$$

(2) Convex hull property

The entire AH Bézier curve Eq.(8) must lie inside its control polygon spanned by $\boldsymbol{p}_0,\ldots,\boldsymbol{p}_n$ (Fig.2).

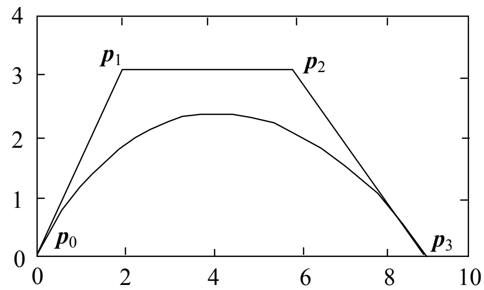

Fig.2 AH Bézier curve and control polygon

(3) Derivative

The derivative $p'(t)$ of an AH Bézier curve Eq.(8) is clearly an order n curve:

$$p'(t)=\sum_{i=0}^{n-1}B_{i,n-1}(t)\boldsymbol{q}_i,\ t\in[0,\alpha]$$

where $\boldsymbol{q}_i=\delta_{i,n-1}(\boldsymbol{p}_{i+1}-\boldsymbol{p}_i)$. In particular, we have

$$p^{(k)}(0)=\sum_{i=0}^{k}B_{i,n}^{(k)}(0)\boldsymbol{p}_i$$

(4) Degree elevation

By the elevation of the AH Bézier basis functions, we give the degree elevation of the AH Bézier curve easily as:

$$p(t)=\sum_{i=0}^{n}B_{i,n}(t)\boldsymbol{p}_i=\sum_{i=0}^{n+1}B_{i,n+1}(t)\boldsymbol{q}_i \tag{9}$$

here

$\boldsymbol{q}_0=\boldsymbol{p}_0$,

$$\boldsymbol{q}_i=\left(1-\frac{B_{i,n}^{(i)}(0)}{B_{i,n+1}^{(i)}(0)}\right)\boldsymbol{p}_{i-1}+\frac{B_{i,n}^{(i)}(0)}{B_{i,n+1}^{(i)}(0)}\boldsymbol{p}_i,\ i=0,1,\cdots,n,$$

$\boldsymbol{q}_{n+1}=\boldsymbol{p}_n$

In fact, a degree elevation procedure is a corner cutting procedure just as those of the Bézier curve. It can be verified that the sequence of control polygons we get recursively from Eq.(9) converges to the AH Bézier curve.

(5) Variation diminishing (V. D.) property

No plane intersects an AH Bézier curve more often than it intersects the corresponding control polygon. We will prove it in Section 3.

(6) Convexity preserving property

If the control polygon is convex, then the corresponding AH Bézier curve is also convex.

(7) The limit of the AH Bézier curves

As $\alpha\to 0$, the limit of an AH Bézier curve in the space Γ_{n+1} approaches a Bézier curve in the space spanned by $\{1,t,t^2,\ldots,t^n\}$. In Fig.3, Fig.3b is obtained by mapping the intervals $0\le t\le 1$ and $0\le t\le\alpha$ onto the interval $0\le t\le 1$.

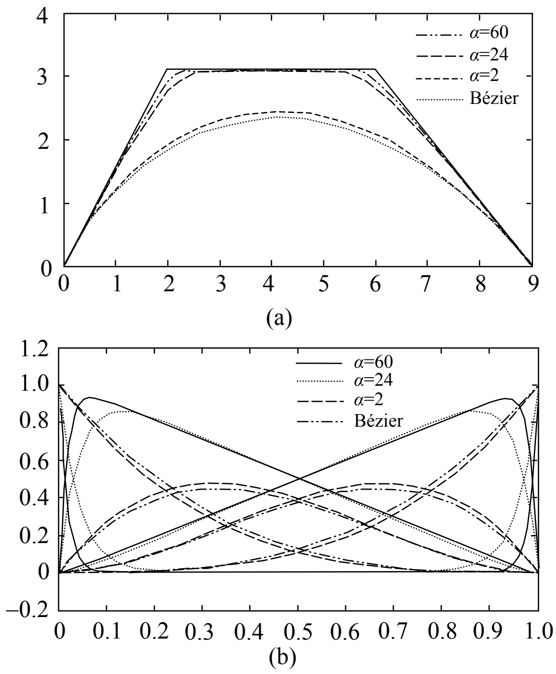

Fig.3 Bézier curve and AH Bézier curve (a) for α=60, 24, 2; (b) for α=60, 24, 2

Proof Reparametrizing by $\tau=t/\alpha$, it is easy to prove that the result holds for n=1. Suppose it holds for n and by inductive hypothesis:

$$B_{i,n-1}(t)=B_{i,n-1}(\alpha\tau),\quad \lim_{\alpha\to 0}B_{i,n-1}(\alpha\tau)=b_{i,n-1}(\tau)$$

here $b_{i,n-1}(\tau)$, $0\le\tau\le 1$ is a Bézier basis. We give the proof by induction by n:

$$B_{i,n}(t)=\int_0^t[\delta_{i-1,n-1}B_{i-1,n-1}(s)-\delta_{i,n-1}B_{i,n-1}(s)]\mathrm{d}s$$
$$=\frac{\int_0^t B_{i-1,n-1}(s)\mathrm{d}s}{\int_0^\alpha B_{i-1,n-1}(s)\mathrm{d}s}-\frac{\int_0^t B_{i,n-1}(s)\mathrm{d}s}{\int_0^\alpha B_{i,n-1}(s)\mathrm{d}s}$$

$$\lim_{\alpha\to 0}B'_{i,n}(\alpha\tau)=\lim_{\alpha\to 0}\frac{B_{i-1,n-1}(\alpha\tau)}{\int_0^1 B_{i-1,n-1}(\alpha\tau)\mathrm{d}\tau}-\frac{B_{i,n-1}(\alpha\tau)}{\int_0^1 B_{i,n-1}(\alpha\tau)\mathrm{d}\tau}$$
$$=\frac{b_{i-1,n-1}(\tau)}{\int_0^1 b_{i-1,n-1}(\tau)\mathrm{d}\tau}-\frac{b_{i,n-1}(\tau)}{\int_0^1 b_{i,n-1}(\tau)\mathrm{d}\tau}$$
$$=n(b_{i-1,n-1}(\tau)-b_{i,n-1}(\tau))=b'_{i,n}(\tau)$$

By the properties (3) and (4) of the AH Bézier basis, we have $B_{i,n}(0)=b_{i,n}(0)$, therefore:

$$\lim_{\alpha\to 0}B_{i,n}(\alpha\tau)=b_{i,n}(\tau),\ n\ge 1,\ i=0,1,\cdots,n$$

From the definitions of AH Bézier curve and Bézier curve, we get the property (8).

(8) The AH Bézier basis is B-basis

By the properties (1), (5) and (6), we have that AH Bézier basis is a totally positive basis. It is easy to get $\inf\{B_{i,n}(t)/B_{j,n}(t)|B_{j,n}(t)\ne 0\}=0$ by L'Hospital's Rule. From Proposition 3.12 in (Canicer and Peña, 1994), AH Bézier basis is B-basis, so it has optimal shape preserving properties [Chapter 4 of (Peña, 1999)] and optimal stability properties for the evaluation [Chapter 5 of (Peña, 1999)].

AH Bézier surface

An AH Bézier surface can be constructed using tensor product:

$$p(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}B_{i,m}(u)B_{j,n}(v)\boldsymbol{p}_{ij},$$
$$u\in[0,\alpha],\ v\in[0,\beta],\ \alpha,\beta\in(0,+\infty)$$

In which $B_{i,m}(u)$, $B_{j,n}(v)$ are the AH Bézier basis functions and $\boldsymbol{p}_{ij}$ are the control points. It should be noted that one can choose a different parameter β in the v direction. Its properties can be deduced by those of the AH Bézier curve.

ALGEBRAIC HYPERBOLIC B-SPLINE BASIS (AH B-SPLINE)

Definition of the AH B-Spline basis

Let T be a given knot sequence $\{t_i\}_{-\infty}^{+\infty}$ with $t_i \le t_{i+1}$ we first give a set of initial functions:

$$N_{i,2}(t)=\begin{cases}\sinh(t-t_i)/\sinh(t_{i+1}-t_i), & t_i<t\le t_{i+1}\\ \sinh(t_{i+2}-t)/\sinh(t_{i+2}-t_{i+1}), & t_{i+1}<t\le t_{i+2}\\ 0, & \text{otherwise}\end{cases} \quad (10)$$

Here we define that 0/0=0. Then the AH B-Spline basis functions of order k in the space Γ_{k-1}=span $\{1,t,\ldots,t^{k-3},\sinh t,\cosh t\}$ can be defined recursively as:

$$N_{i,k}(t)=\int_{-\infty}^{t}(\delta_{i,k-1}N_{i,k-1}(s)-\delta_{i+1,k-1}N_{i+1,k-1}(s))\mathrm{d}s \quad k\ge 3 \quad (11)$$

where $\delta_{i,k}=1\Big/\int_{-\infty}^{+\infty}N_{i,k}(t)\mathrm{d}t$. If $N_{i,k}(t)=0$, $\delta_{i,k}=\infty$ and $\delta_{i,k}N_{i,k}(t)=0$. We have from Eq.(11) the following:

$$\int_{-\infty}^{t}\delta_{i,k}N_{i,k}(s)\mathrm{d}s=\begin{cases}0, & t\le t_i\\ \ge 0, & t_i<t<t_{i+k}\\ 1, & t\ge t_{i+k}\end{cases}$$

so we get the definition of the AH B-Spline:

Definition 2 $\{N_{i-k+1,k},N_{i-k+2,k},\ldots,N_{i,k}\}$ is called the AH B-Spline basis of the space Γ_{k-1} spanned by $\{1,t,\ldots,t^{k-3},\sinh t,\cosh t\}$ for $t\in[t_i,t_{i+1}]$. The linear independence of the $\{N_{i-k+1,k},N_{i-k+2,k},\ldots,N_{i,k}\}$ will be proved in Section 3.

In particular, if we replace sinh$t$ by t in Eq.(10), we get the B-Spline basis for the polynomial space from Eq.(11); if we replace sinh$t$ by sin$t$ in Eq.(10), we will get the NUAT B-Spline basis for the algebraic trigonometric space from Eq.(11). If the knot sequence is uniform, we will get the uniform hyperbolic polynomial B-Spline basis described in (Lü *et al.*, 2002).

The sequence of $N_{i,k}(t)$, has the following properties just the same as those of the B-Spline.

Properties of the AH B-Spline basis

(1) Local support

$N_{i,k}(t)=0$, $t\notin[t_i, t_{i+k}]$.

(2) Partition of unity

$\sum_{i=-\infty}^{+\infty}N_{i,k}(t)=1$ for all $k\ge 3$ and all t.

(3) Derivative

$N'_{i,k}(t)=\delta_{i,k-1}N_{i,k-1}(t)-\delta_{i+1,k-1}N_{i+1,k-1}(t)$.

(4) Zero function

$N_{i,k}(t)\equiv 0$ if and only if $t_i=t_{i+1}=\ldots=t_{i+k}$.

(5) Positivity

$N_{i,k}(t)>0$ for $t\in(t_i,t_{i,k})$. Here $t_i<t_{i,k}$. $N_{i,k}(t)\ge 0$ for all t. This can be proved in the same way as that for AH Bézier basis.

(6) Differential

$N_{i,k}(t)$ is $(k-r_j-1)$ time continuously differential at the knot t_j with r_j the number of times t_j appears in the knot sequence $\{t_j\}_i^{i+n}$.

(7) Let $r=\max\{s|t_i=t_{i+s}\}$. If $r\ge k-2$, we then have:

$$N_{j,k}(t_i)=\begin{cases}1, & j=i-1\\ 0, & j\ne i-1\end{cases}$$

(8) Linear independence

$N_{i-k+1,k}(t),N_{i-k+2,k}(t),\ldots,N_{i,k}(t)$ are linearly independent on $[t_i,t_{i+1}]$ with $t_i<t_{i+1}$ for all i, $k\ge 2$. As a consequence of this, we get that $N_{i,k}(t)$, $i=0,\pm1,\ldots$ are linearly independent on $(-\infty,+\infty)$ if and only if the multiplicity of each knot of T is less than $k+1$. That is, there is no zero function in $N_{i,k}(t)$, $i=0,\pm1,\ldots$

(9) Relation with AH Bézier basis

In the case $t_{i-k+1}=t_{i-k+2}=\ldots=t_i<t_{i-k+1}=t_{i-k+2}=\ldots=t_{i+1}=t_{i+2}=\ldots=t_{i+k}$, $N_{i-k+1,k}(t),\ldots,N_{i,k}(t)$ are just the AH Bézier basis of order k on $[t_i,t_{i+1}]$. From the definitions of the two bases, the above property can be proved by induction on k.

Inserting a new knot

It is the same as the B-Spline and the NUAT B-Spline, we have the knot inserting theorem of the

AH B-Spline:

Theorem 1 Let $T=(t_i)_{-\infty}^{+\infty}$ be a given knot sequence with $0\le t_{i+1}-t_i<+\infty$, and let $T^1=(t_i^1)_{-\infty}^{+\infty}$ be a new knot sequence obtained by inserting a new knot u into T with $t_i\le u<t_{i+1}$, $N_{j,k}(t)$ and $N_{j,k}^1(t)$ are defined as in Eq.(11) for the knot sequence T and T^1 respectively. If r,s are multiplicities of the knots u,t_i in T respectively, we have for all $j,k\ge 2$,

$$N_{j,k}(t)=\alpha_{j,k}N_{j,k}^1(t)+\beta_{j+1,k}N_{j+1,k}^1(t) \tag{12}$$

where for $0\le r<k$:

$$\alpha_{j,k}=\begin{cases}1, & j\le i-k\\ \alpha_{j,k-1}\,\delta_{j,k-1}\big/\delta_{j,k-1}^1, & i-k<j<i-r+1\\ 0, & j\ge i-r+1\end{cases}$$

$$\beta_{j,k}=\begin{cases}0, & j\le i-k\\ \beta_{j+1,k-1}\,\delta_{j,k-1}\big/\delta_{j+1,k-1}^1, & i-k<j<i-r+1\\ 1, & j\ge i-r+1\end{cases}$$

and for $r\ge k$

$$\alpha_{j,k}=\begin{cases}1, & j\le i-k\\ 0, & j>i-k\end{cases}\qquad \beta_{j,k}=\begin{cases}0, & j\le i-k+1\\ 1, & j>i-k+1\end{cases}$$

and

$$\alpha_{j,2}=\begin{cases}1, & j\le i\\ \dfrac{\sinh(u-t_i)}{\sinh(t_{i+1}-t_i)}, & j=i\\ 0 & j\ge i+1\end{cases}$$
$$\beta_{j,2}=\begin{cases}0, & j\le i\\ \dfrac{\sinh(t_{i+1}-u)}{\sinh(t_{i+1}-t_i)}, & j=i\\ 1 & j\ge i+1\end{cases} \tag{13}$$

when $r=0$ and $s\ge k$, we have $\alpha_{i-k+1,k}=1$ (Fig.4).

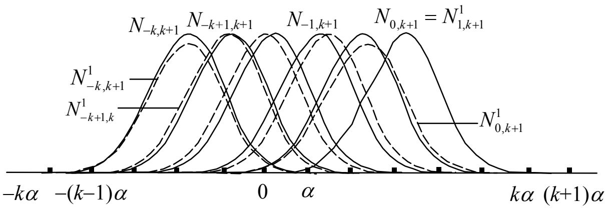

Fig.4 Inserting a knot at t=0

Proof Let k=2, we obtain Eq.(13) by direct calculation. Suppose the proposition holds for $k-1$:

$$N_{j,k-1}(t)=\alpha_{j,k-1}N_{j,k-1}^1(t)+\beta_{j+1,k-1}N_{j+1,k-1}^1(t)\ \text{ for all } j.$$

Now we will prove the case for k. By the definition we then have $N_{j,k}(t)=N_{j,k}^1(t)$ for $j\le i-k$ and $N_{j,k}(t)=N_{j+1,k}^1(t)$ for $j\ge r-k+1$ which implies $\alpha_{j,k}=1$, $\beta_{j+1,k}=0$ for $j\le i-k$ and $\alpha_{j,k}=0$, $\beta_{j+1,k}=1$ for $j\ge i-r+1$.

(1) $r\ge k$. It is easy to prove that $\alpha_{j,k}=0$, $\beta_{j+1,k}=1$ for $j\le i-k$ and $\alpha_{j,k}=0$, $\beta_{j+1,k}=1$ for $j\ge i-k+1$.

(2) $0<r\le k-1$. By assumption, we have

$$\begin{aligned}N_{j,k}(t)&=\int_{-\infty}^{t}(\delta_{j,k-1}N_{j,k-1}(t)-\delta_{j+1,k-1}N_{j+1,k-1}(t))\mathrm{d}t\\ &=\int_{-\infty}^{t}\Big[\delta_{j,k-1}\left(\alpha_{j,k-1}N_{j,k-1}^1(t)-\beta_{j+1,k-1}N_{j+1,k-1}^1(t)\right)\\ &\quad-\delta_{j+1,k-1}\left(\alpha_{j+1,k-1}N_{j+1,k-1}^1(t)-\beta_{j+2,k-1}N_{j+2,k-1}^1(t)\right)\Big]\mathrm{d}t\\ &=A_1(t)+A_2(t)+A_3(t)\ \text{ for } i-k+1\le j\le i-r\end{aligned}$$

where

$$\begin{aligned}A_1(t)&=\frac{\delta_{j,k-1}\alpha_{j,k-1}}{\delta_{j,k-1}^1}\\ &\quad\cdot\int_{-\infty}^{t}\left[\delta_{j,k-1}^1N_{j,k-1}^1(t)-\delta_{j+1,k-1}^1N_{j+1,k-1}^1(t)\right]\mathrm{d}t\\ &=\frac{\delta_{j,k-1}\alpha_{j,k-1}}{\delta_{j,k-1}^1}N_{j,k}^1(t)\end{aligned}$$

$$\begin{aligned}A_2(t)&=\frac{\delta_{j+1,k-1}\beta_{j+2,k-1}}{\delta_{j+2,k-1}^1}\\ &\quad\cdot\int_{-\infty}^{t}\left[(\delta_{j+1,k-1}^1N_{j+1,k-1}^1(t)-\delta_{j+2,k-1}^1N_{j+2,k-1}^1(t)\right]\mathrm{d}t\\ &=\frac{\delta_{j+1,k-1}\beta_{j+2,k-1}}{\delta_{j+2,k-1}^1}N_{j+1,k}^1(t)\end{aligned}$$

$A_3(t)=\lambda\int_{-\infty}^{t}\delta_{j+1,k-1}^1N_{j+1,k-1}^1(t)\mathrm{d}t$ with λ some real number.

Let $v\ge t_{i-r+k}=t_{i-r+k+1}^1$, we have $N_{j,k}(v)=A_1(v)=A_2(v)=0$ while $A_3(v)=\int_{-\infty}^{v}\delta_{j+1,k-1}^1N_{j+1,k-1}^1(t)\mathrm{d}t=1$ for $i-k+1\le j\le i-r$.

We get that $\lambda=0$, so we obtain

$$N_{j,k}(t)=\frac{\delta_{j,k-1}\alpha_{j,k-1}}{\delta_{j,k-1}^1}N_{j,k}^1(t)+\frac{\delta_{j+1,k-1}\beta_{j+2,k-1}}{\delta_{j+2,k-1}^1}N_{j+1,k}^1(t)$$

for $i-k+1\le j\le i-r$.

It means

$$\alpha_{j,k}=\frac{\delta_{j,k-1}\alpha_{j,k-1}}{\delta^1_{j,k-1}},\ \beta_{j+1,k}=\frac{\delta_{j+1,k-1}\beta_{j+2,k-1}}{\delta^1_{j+2,k-1}}$$

for $i-k+1\le j\le i-r$.

(3) $r=0$. When $s\le k-1$, the proof is similar to the case (2). If $s\ge k$, it is also similar to the case (2) for $i-k+1<j\le i$. For $j=i-k+1$, we have $N_{i-k+1,k-1}(t)=0$. We define:

$$\int_{-\infty}^{t}\delta_{i-k+1,k-1}N_{i-k+1,k-1}(t)\mathrm{d}t=\int_{-\infty}^{t}\delta^1_{i-k+1,k-1}N^1_{i-k+1,k-1}(t)\mathrm{d}t=\begin{cases}1, & t\ge t_i\\ 0, & t<t_i\end{cases}$$

so we have

$$N_{i-k+1,k}(t)=\int_{-\infty}^{t}\left(\delta_{i-k+1,k-1}N_{i-k+1,k-1}(t)-\delta_{i-k+2,k-1}N_{i-k+2,k-1}(t)\right)\mathrm{d}t$$
$$=\int_{-\infty}^{t}\Big[\delta^1_{i-k+1,k-1}N^1_{i-k+1,k-1}(t)-\delta_{i-k+2,k-1}\cdot(\alpha_{i-k+2,k-1}N^1_{i-k+2,k-1}+\beta_{i-k+3,k-1}N^1_{i-k+3,k-1})\Big]\mathrm{d}t$$

similar to the case (2), we have

$$N_{i-k+1,k}(t)=N^1_{i-k+1,k}(t)+\frac{\delta_{i-k+2,k-1}\beta_{i-k+3,k-1}}{\delta^1_{i-k+3,k-1}}N^1_{i-k-2,k}(t)$$

That is

$$\alpha_{i-k+1,k-1}=1,\ \beta_{i-k+2,k-1}=\frac{\delta_{i-k+2,k-1}\beta_{i-k+3,k-1}}{\delta^1_{i-k+3,k-1}}.$$

From the discussion above, we get that the proposition also holds for k. This proves Theorem 1 also gives a method to compute the coefficients α_j, β_{j+1} for all j. Furthermore, By the property of partition of unity and the linearly independence of $N^1_{i,k}(t),\ i=0,\pm1,\cdots$, we have the formula $\alpha_{j,k}+\beta_{j,k}=1$ for all i,k with $N^1_{i,k}(t)\ne 0$ for all i. Then Eq.(12) can be rewritten as

$$N_{i,k}(t)=\alpha_{i,k}N^1_{i,k}(t)+(1-\alpha_{i+1,k})N^1_{i+1,k}(t),\ 0\le\alpha_{i,k}\le 1 \tag{14}$$

Now, we give the proof of the non-negativity property of $N_{i,k}(t)$, $i=0,\pm1,\ldots$

Theorem 2 $N_{i,k}(t)\ge 0$ for all t.

Proof By the local support, we have that $N_{i,k}(t)$ can be non-zero only on $[t_i,t_{i+k}]$. If $t_i=t_{i+k}$, we have $N_{i,k}(t)\equiv 0$ from property (4). In the following, suppose $t_i<t_{i+k}$, we insert a series of new knots into knot sequence T such that the multiplicity of each knot t_j ($j=i,\ldots,i+k$) is k. We then obtain a new knot sequence denoted by T^1. Let $N^1_{i,k}$ be the new splines with the new knot sequence T^1, then $N_{i,k}(t)$ is a convex combination of $N^1_{i,k}(t)$. By property (9), we have that $N^1_{i,k}(t)$ determined by t_j ($j=i,\ldots,i+k$) is actually an AH Bézier basis on each interval $[t_j,t_{j+1}]$, $j=i,i+1,\ldots,i+k-1$, $t_j<t_{j+1}$. Thus, we have $N_{i,k}(t)\ge 0$ for all t.

AH B-Spline curves

AH B-Spline basis $N_{i,k}(t)$, $i=0,\pm1,\ldots$ has many good properties, so it can be used for geometric modelling. Because of the local support property, we can define a piece of AH B-Spline curve $p(t)$ with control points $\boldsymbol{p}_i$ in a finite interval such as $[t_{k-1},t_{m+1}]$ (Fig.5).

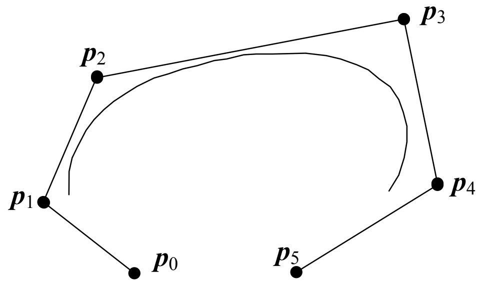

Fig.5 A piece of AH B-Spline curve

$$p(t)=\sum_{i=0}^{m}N_{i,k}(t)\boldsymbol{p}_i,\ t\in[t_{k-1},t_{m+1}],\ m\ge k-1 \tag{15}$$

where $\{N_{i,k}(t)\}_{i=0}^{m}$ is the AH B-Spline basis with the knot sequence $T=\{t_i\}_{i=0}^{m+k}$ for the space Γ_{k-1}. AH B-Spline curve has many properties the same as those of the B-Spline curve:

(1) Derivative

The derivative $p'(t)$ of an order k AH B-Spline curve $p(t)$ is clearly a degree k curve:

$$p'(t)=\sum_{i=1}^{m}N_{i,k-1}(t)\delta_{i,k-1}\Delta\boldsymbol{p}_i$$
$$=\sum_{i=1}^{m}N_{i,k-1}(t)\delta_{i,k-1}(\boldsymbol{p}_i-\boldsymbol{p}_{i-1}),\ t\in[t_{k-1},t_{m+1}]$$

(2) Local control property

Change of one control point will alter at most k segments of the original AH B-Spline curve of order k. Hence local adjustment can be made without disturbing the rest of the curve.

(3) Geometric invariance

The shape of the AH B-Spline curve is independent of the choice of the coordinate system because $p(t)$ is an affine combination of the control points.

(4) Convex hull property

The entire AH B-Spline curve must lie inside its control polygon (Fig.5). It follows from the non-negative and partition of unity of the AH B-Spline basis.

(5) Differential

$p(t)$ is $k-r-1$ continuously differential at a knot of multiplicity r.

(6) Subdivision of the curves

Substituting Eq.(14) into Eq.(15), we have

$$p(t)=\sum_{i=0}^{m}\left(\alpha_{i,k}N_{i,k}^{1}(t)+(1-\alpha_{i+1,k})N_{i+1,k}^{1}(t)\right)\boldsymbol{p}_i$$
$$=\sum_{i=0}^{m+1}N_{i,k}^{1}(t)\boldsymbol{p}_i^{1} \tag{16}$$

where

$$\boldsymbol{p}_i^{1}=(1-\alpha_{i,k})\boldsymbol{p}_{i-1}+\alpha_{i,k}\boldsymbol{p}_i,\quad \boldsymbol{p}_{-1}=\boldsymbol{p}_{m+1}=0,$$
$$0\le\alpha_{i,k}\le 1. \tag{17}$$

with $\alpha_{i,k}$ and $N_{i,k}^{1}(t)$ as defined in Section 3. From Eq.(17) we get that the new control points can be obtained from the old control points after subdivision. In fact, the process of inserting a new knot is actually a corner cutting process. If we insert the same knot u, $t_i<u<t_{i+1}$, iteratively, then we will get a series of new control points $\boldsymbol{p}_i^{l}$, $i=0,1,\dots,m+l$ from Eq.(17). Here l is the times of inserting knot.

In particular, when $l=k-1$, since

$$N_{j,k}^{l}(u)=\begin{cases}1, & j=i\\ 0, & j\ne i\end{cases}$$

Then

$$p(u)=\sum_{j=0}^{m+k-1}\boldsymbol{p}_j^{k-1}N_{j,k}^{k-1}(u)=\boldsymbol{p}_i^{k-1} \tag{18}$$

We use $p(t)$ to denote an AH B-Spline curve of order k with $\boldsymbol{p}=\Phi^0[\boldsymbol{p}]=[\boldsymbol{p}_0^0,\boldsymbol{p}_1^0,\cdots,\boldsymbol{p}_m^0]$, $T=T^0=(t_i^0)$, $i=0,1,\dots,m+k$ as the control polygon and knot sequence. Let $T^r=(t_i^r)$, $i=0,1,\dots,m+k+r$ be the new knot sequence obtained by inserting a new knot into T^{r-1} and $\Phi^r[\boldsymbol{p}]=\Phi[\Phi^{r-1}[\boldsymbol{p}]]=[\boldsymbol{p}_0^r,\boldsymbol{p}_1^r,\cdots,\boldsymbol{p}_{m+r}^r]$ be the control polygon where the control points $\boldsymbol{p}_i^r$, $i=0,1,\dots,m+r$ are given by Eq.(17). And let $\Delta^r=\max\left|t_{i+1}^r-t_i^r\right|$, $i=0,1,\dots,m+k+r-1$. We have the following theorem.

Theorem 3 When $\Delta^r\to 0$ as the number of subdivisions r increases, the sequence of control polygons $\Phi^r[\boldsymbol{p}]$ converges to the AH spline curve $p(t)$.

Proof Let $N_{i,k}^r(t)$ be the AH B-Spline basis functions with the knot sequence T^r. By the definition of $\Phi^r[\boldsymbol{p}]$, we have

$$p(t)=\sum_{i=0}^{m}N_{i,k}(t)\boldsymbol{p}_i=\cdots=\sum_{i=0}^{m+r}N_{i,k}^r(t)\boldsymbol{p}_i^r,\ t\in[t_{k-1},t_{m+1}]$$

From the property (1) of the curves, we have

$$p'(t)=\sum_{i=0}^{m}\delta_{i,k-1}N_{i,k-1}(t)\Delta\boldsymbol{p}_i=\cdots=\sum_{i=0}^{m+r}\delta_{i,k-1}^rN_{i,k}^r(t)\Delta\boldsymbol{p}_i^r,$$
$$t\in[t_{k-1},t_{m+1}]$$

where $\Delta\boldsymbol{p}_i^r=\boldsymbol{p}_i^r-\boldsymbol{p}_{i-1}^r$. It can be easily seen that $p'(t)$ is a piece of AH B-Spline curve of order $k-1$ and the control polygon $\delta_{i,k-1}^r\Delta\boldsymbol{p}_i^r$, $i=0,1,\dots,m+r$ can be obtained from $\delta_{i,k-1}\Delta\boldsymbol{p}_i$, $i=0,1,\dots,m$ after a series of subdivisions. Hence $\delta_{i,k-1}^r\Delta\boldsymbol{p}_i^r$, $i=0,1,\dots,m+r$ is bounded by the convex hull of $\delta_{i,k-1}\Delta\boldsymbol{p}_i$, $i=0,1,\dots,m$. However, note that

$$\delta_{i,k-1}^r=\left(\int_{-\infty}^{+\infty}N_{i,k-1}^r(t)\mathrm{d}t\right)^{-1}$$
$$=\left(\int_{t_i^r}^{t_{i+k-1}^r}N_{i,k-1}^r(t)\mathrm{d}t\right)^{-1}\ge(t_{i+k-1}^r-t_i^r)^{-1}\ge\frac{k-1}{\Delta^r}$$

We have $\lim_{\Delta^r\to 0}\delta^r_{i,k-1}=\infty,\ i=1,\cdots,m+r.$

Therefore

$$\lim_{\Delta^r\to 0}\left|\Delta \boldsymbol{p}_i^r\right| = \lim_{\Delta^r\to 0}\left|\boldsymbol{p}_i^r-\boldsymbol{p}_{i-1}^r\right|=0,\ i=1,\cdots,m+r \quad (19)$$

Note that $\delta^r_{i,k-1}N^r_{i,k-1}(t)=0$ when $N^r_{i,k-1}(t)=0.$ In this case, we have $\Delta\boldsymbol{p}_i^r=0.$ From the convex hull property, for any $u\in[t_{k-1},t_{m+1}]$ we know that $p(u)$ lies within the convex hull of $\boldsymbol{p}_i^r,\boldsymbol{p}_{i+1}^r,\cdots,\boldsymbol{p}_{i+k}^r$ for some i. Together with Eq.(19), we conclude the theorem.

Theorem 3 means that recursive subdivision of control polygon leads to its corresponding AH B-Spline curve, so it is not difficult to prove the V. D. property and the convexity preserving property below:

(7) V. D. property

No plane intersects an AH B-Spline curve more often than it intersects the corresponding control polygon. Because AH Bézier basis is special AH B-Spline basis, we easily get that AH Bézier curves have V. D. property too.

(8) Convexity preserving property

If the control polygon is convex, then the corresponding AH B-Spline curve is also convex.

By the knot inserting theorem, we can easily get the property (9) of the AH Spline:

(9) The AH B-Spline basis is B-basis:

Proof Insert a knot at $t=t_0$ and $t=t_l$ repeatedly until the obtained multiplicities of knot t_0 and t_l are n respectively, and the old basis can be expressed by the new basis. Let us denote the basis functions generated after the pth knot inserting on as $\boldsymbol{N}_k^p=(N^p_{-k,k+1}(t),$ $N^p_{-k+1,k+1}(t),...,N^p_{l-1,k+1}(t)),$ where $\boldsymbol{N}_k^0$ is $\boldsymbol{N}_k$, thus the new basis $\boldsymbol{N}_k^{2k}$ has endpoint interpolation property. Form [Section 2 of (Mainar and Peña, 1999) and Section 2 of (Mainar *et al.*, 2001)] we know that if a basis has endpoint interpolation property, V. D. property, partition of unity and convex hull properties, it is normalized totally positive, so the new basis $\boldsymbol{N}_k^{2k}$ is normalized totally positive.

According to the property of inserting a new knot (Theorem 1), the transformation matrix between $\boldsymbol{N}_k^p$ and $\boldsymbol{N}_k^{p+1}$ is one-banded matrix, so the transformation matrix between $\boldsymbol{N}_k^0$ and $\boldsymbol{N}_k^{2k}$ can be decomposed into a product of bidiagonal factors. Thus we deduce that the AH B-Spline basis $\boldsymbol{N}_k$ is normalized totally positive.

By the definition of the B-basis, we will show the following equation holds for every $i\neq j$:

$$\inf\left\{\frac{N_{i,k+1}(t)}{N_{j,k+1}(t)}\middle| t\in[t_0,t_l],N_{j,k+1}(t)\neq 0\right\}=0 \qquad (20)$$

If $i<j$ and $t_{i+k+1}<t_{j+k+1}$, let $\tilde{t}=(t_{j+k+1}+t_{i+k+1})/2,$ we have:

$$\inf\left\{\frac{N_{i,k+1}(\tilde{t})}{N_{j,k+1}(\tilde{t})}\middle| N_{j,k+1}(\tilde{t})\neq 0\right\}=0$$

If $t_{i+k+1}=t_{j+k+1}$, let $t\to t^-_{i+k+1}$, then the infinite minimum order of $N_{i,k+1}$ is $j-i$ order higher than infinite minimum order of $N_{j,k+1}$, so

$$\lim_{t\to t^-_{i+k+1}}\left[N_{i,k+1}(t)/N_{j,k+1}(t)\right]=0,$$

that is Eq.(20) holds.

When $i>j$ and $t_i>t_j$, let $\tilde{t}=(t_j+t_i)/2$, we have Eq.(20) holds. When $i>j$ and $t_i=t_j$, let $t\to t_i^+$, then the infinite minimum order of $N_{i,k+1}$ is $i-j$ order higher than infinite minimum order of $N_{j,k+1}$, so $\lim_{t\to t_i^+}\left[N_{i,k+1}(t)/N_{j,k+1}(t)\right]=0,$ that is Eq.(20) holds.

By Proposition 3.12 in (Carnicer and Peña, 1994) and Section 3, AH B-Spline basis is normalized B-basis. Therefore AH B-Spline basis has optimal shape preserving properties and optimal stability properties.

AH B-Spline surface

Exactly as in the construction of B-Spline tensor product surfaces from B-Spline curves, an AH B-Spline surface can be constructed using tensor product. Let $U=\{u_i\}_{i=0}^{m+k}$, $V=\{v_j\}_{j=0}^{n+k}$ be two knot sequences, then an AH B-Spline surface with control meshes $\boldsymbol{p}_{ij}$ can be defined as

$$p(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}N_{i,k}(u)N_{j,k}(v)\boldsymbol{p}_{ij},$$

$$u \in [u_{k-1}, u_{m+1}], v \in [v_{l-1}, v_{n+1}], m \geq k-1, n \geq l-1.$$

In which $N_{i,k}(u)$, $N_{j,k}(v)$ are the AH B-Spline basis functions with knot sequence U, V respectively. Its properties can be deduced from the properties of the AH B-Spline such as the convex hull property and the convexity preserving property. The subdivision formulae can be used in both directions, etc.

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