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# Exact thickness-shear resonance frequency of electroded piezoelectric crystal plates*

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**Abstract:** The determination of the precise thickness-shear frequency of electroded crystal plates has practical importance in quartz crystal resonator design and fabrication, especially when the high fundamental thickness-shear frequency has reduced the crystal plate thickness to such a degree that proper consideration of the effect of electrodes is very important. The electrodes effect as mass loading in the estimation of the resonance frequency has to be modified to consider the stiffness of electrodes, as the relative strength is increasingly noticeable. By following a known procedure in the determination of the thickness-shear frequency of an infinite AT-cut crystal plate, frequency equations of crystal plate without and with piezoelectric effect are obtained in terms of elastic constants and the electrode material density. After solving these equations for the usual design parameters of crystal resonators, the design process can be optimized to pinpoint the precise configuration to avoid time-consuming trial and reduction steps. Since these equations and solutions are presented for widely used materials and parameters, they can be easily integrated into the existing crystal resonator design and manufacturing processes.

**Key words:** Thickness shear, Crystal plate, Piezoelectric
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## INTRODUCTION

In quartz crystal resonator design, the mass effect of the electrodes must be considered to achieve accurate prediction of the fundamental thickness-shear frequency, as analyzed and demonstrated by Bleustein and Tiersten (1968), which is considered to be the exact frequency solutions in making necessary comparisons with solutions from two-dimensional plate equations and deriving the correspondent correction factors (Mindlin, 1972). When the crystal plate is relatively thick or the fundamental thickness-shear frequency is relatively low, it is adequate to consider the effect of the electrodes as additional mass on crystal surface. As we know, earlier efforts in research on and production of piezoelectric resonators were concentrated on frequencies much lower than 100 MHz. In last few years, rapidly emerging applications and market driven efforts in reducing the size of crystal resonators and continuous push in reaching higher frequencies have led to thinner crystal plate blanks and relatively larger mass ratios of electrodes are increasingly important in determining the fundamental thickness-shear frequency. Apparently, in addition to the mass effect, the relative stiffness of the electrodes, which is proportional to the mass ratio, is also a factor to be considered. By assuming constant deformation in the electrodes, Mindlin (1963) introduced the stiffness ratios in terms of elastic constants and thicknesses of crystal plates and electrodes in the high frequency vibration equations of plated crystal plates. To improve the frequency prediction of crystal resonators with finite element method, Wang *et al.*(1999) treated the deformation of electrodes as separated variables in the finite element formulation and later solved the expanded equations numerically.

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Although much more accurate and realistic, the extra variables and their solutions actually make the extraction of useful information more difficult, thus losing their appeal in practical applications.

In this work, we are concerned about the fundamental thickness-shear vibration frequency of crystal resonators with relatively larger mass ratio

$$R = 2\bar{\rho}\bar{h} / \rho h \tag{1}$$

where $R$, $\bar{\rho}$, $\bar{h}$, $\rho$, and $h$ are mass ratio, electrode density, electrode thickness, crystal density, and crystal thickness, respectively. Following Bleustein and Tiersten (1968), we obtained the frequency equations of fundamental thickness-shear vibrations based on an infinite crystal plate with symmetric and full electrodes on both faces. The solution of these equations is the approximate resonance thickness-shear frequency of a resonator structure with typically complicated electrodes and support structures, because the vibrations are dominated by the driving voltage applied to the electroded area, which is usually in the center of the crystal blank. As before, these solutions can be used as references for crystal resonator design and fabrication, because fast and precise determination of the blank thickness is always of great importance, if the final target of precise thickness of the crystal blank in the process is known in advance. In addition, these solutions can be used to derive correction factors of the two-dimensional plate equations to aid further analytical efforts in obtaining other important parameters like capacitance ratio and thickness-shear displacement distribution with various methods including the straight-crested wave solutions that are popular for strip crystal resonator analysis. All these steps are closely related to resonator design and fabrication processes, have great potential to improve the efficiency in product development and production.

## THICKNESS-SHEAR VIBRATIONS OF ELECTRODED CRYSTAL PLATES

For an infinite crystal plate with symmetric electrodes in the upper and lower faces, as shown in Fig.1, we assume the thicknesses of the crystal plate and electrodes in $x_2$ direction are

$$h = 2b,\ \bar{h} = 2\bar{b} \tag{2}$$

respectively. The thickness-shear displacements satisfying the continuity boundary conditions on the interfaces are

$$\begin{aligned} &u_1 = A\sin(\eta x_2), \quad -b \le x_2 \le b \\ &\bar{u}_1 = \pm A\sin(\eta b)\cos[\bar{\eta}(x_2 \mp b)] + \bar{B}\sin[\bar{\eta}(x_2 \mp b)] \\ &\qquad b \le |x_2| \le b + 2\bar{b} \end{aligned} \tag{3}$$

for the crystal plate and electrodes, respectively, with wavenumbers in the crystal plate and electrodes being $\eta$ and $\bar{\eta}$. The displacement amplitudes $A$ and $\bar{B}$ in Eq.(3) are to be determined later with boundary conditions. As a result, the stress components corresponding to these displacements are

$$\begin{aligned} &T_6 = c_{66} A\eta\cos(\eta x_2) \\ &\bar{T}_6 = \bar{c}_{66}\bar{\eta}\{\mp A\sin(\eta b)\sin[\bar{\eta}(x_2 \mp b)] \\ &\qquad + \bar{B}\cos[\bar{\eta}(x_2 \mp b)]\} \end{aligned} \tag{4}$$

for the crystal plate and electrodes and $c_{66}$ and $\bar{c}_{66}$ are elastic constants in the crystal and electrodes, respectively.

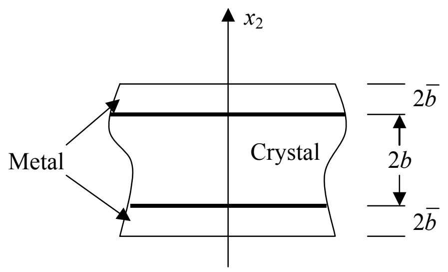


**Fig.1 Electroded crystal plate of a typical crystal resonator**

For infinite plate, the stress equations of motion are related to the displacements in Eq.(3) and stress components in Eq.(4) for the crystal blank and the electrodes, with simple stress state in Eq.(4) they can be simplified to

$$c_{66}\eta^2 - \omega^2\rho = 0, \quad \bar{c}_{66}\bar{\eta}^2 - \omega^2\bar{\rho} = 0 \tag{5}$$

where $\omega$ is the circular frequency, $\rho$ and $\bar{\rho}$ are the densities of crystal and electrodes, respectively. From traction-free boundary conditions

$$\overline{T}_6=0,\ x_2=\pm(2\overline{b}+b);\quad T_6=\overline{T}_6,\ x_2=\pm b \tag{6}$$

with stress components in Eq.(4), we have

$$\begin{aligned}&-\sin(\eta b)\sin(2\overline{\eta}\overline{b}A)+\cos(2\overline{\eta}\overline{b}\overline{B})=0\\&c_{66}\eta\cos(\eta bA)-\overline{c}_{66}\overline{\eta}\overline{B}=0\end{aligned} \tag{7}$$

For resonance to occur, from Eq.(7) we must have

$$\begin{vmatrix}-\sin(\eta b)\sin(2\overline{\eta}\overline{b}) & \cos(2\overline{\eta}\overline{b})\\ c_{66}\eta\cos(\eta b) & -\overline{c}_{66}\overline{\eta}\end{vmatrix}=0 \tag{8}$$

or

$$c_{66}\eta\cos(\eta b)\cos(2\overline{\eta}\overline{b})=\overline{c}_{66}\overline{\eta}\sin(\eta b)\sin(2\overline{\eta}\overline{b}) \tag{9}$$

To simplify Eq.(9), we define

$$\begin{aligned}&\eta=k\overline{\eta},\ k^2=\overline{v}_2^2/v_2^2,\ v_2^2=c_{66}/\rho,\ \overline{v}_2^2=\overline{c}_{66}/\overline{\rho}\\&B=\overline{b}/b,\ C_{66}=\overline{c}_{66}/c_{66},\ \xi=\eta b,\ S_{66}=2BC_{66}\end{aligned} \tag{10}$$

and Eq.(9) can be rewritten as

$$\tan\xi\tan\left(\frac{2B}{k}\xi\right)=\frac{k}{C_{66}} \tag{11}$$

With the solution $\xi$ for known parameters appearing in Eq.(11), we have the frequency solution as

$$\omega=\frac{\pi}{2b}\sqrt{\frac{c_{66}}{\rho}}\frac{2\xi}{\pi}=\omega_0X,\ f=\frac{1}{4b}\sqrt{\frac{c_{66}}{\rho}}\frac{2\xi}{\pi}=f_0X \tag{12}$$

where we define the normalized frequency solution as

$$X=2\xi/\pi \tag{13}$$

Finally, we can write the frequency equation in Eq.(11) as

$$\tan\left(\frac{\pi}{2}X\right)\tan\left(\frac{\pi B}{k}X\right)=\frac{k}{C_{66}} \tag{14}$$

As we can see from Eq.(14), both the thickness ratio $B$ and elastic constant ratio $C_{66}$ are presented, thus effectively taking into consideration the stiffness and mass effects of the electrodes on the resonator structure. In the earlier study by Bluestein and Tiersten (1968), the stiffness term was neglected, limiting the results to be applicable only to relatively thin electrodes with small mass ratio. The result presented here in Eq.(14), as indicated, should be accurate for a much large size range of electrodes with different materials and configurations.

## PIEZOELECTRIC CONSIDERATIONS

As given by Bluestein and Tiersten (1968), for the piezoelectric crystal plate we have the electric potential inside the piezoelectric plate as

$$\varphi=\frac{e_{26}}{\varepsilon_{22}}u_1+C_1x_2+C_0 \tag{15}$$

where $\varphi$, $e_{26}$, $\varepsilon_{22}$, $u_1$, $C_1$, and $C_0$ are electric potential, piezoelectric constant, dielectric constant, thickness-shear displacement, and two integration constants, respectively.

With alternating driving voltage $\varphi_0 e^{i\omega t}$ on the electroded faces, we have the electrical boundary conditions as

$$\varphi(\pm b)=\pm\varphi_0 \tag{16}$$

which simplifies Eq.(15) to

$$\varphi=\frac{e_{26}}{\varepsilon_{22}}A\left(\sin(\eta x_2)-\frac{x_2}{b}\sin(\eta b)\right)+\frac{x_2}{b}\varphi_0 \tag{17}$$

Consequently, the stress components in crystal plate with electric potential term will be

$$T_6=c_{66}\left(1+\frac{e_{26}^2}{c_{66}\varepsilon_{22}}\right)\eta A\cos(\eta x_2)+\frac{e_{26}}{b}\left(\varphi_0-\frac{e_{26}}{\varepsilon_{22}}A\sin(\eta b)\right) \tag{18}$$

Again, we apply the traction-free boundary conditions given in Eq.(6) to Eq.(4.2) and Eq.(18) for the undetermined $A$ and $\overline{B}$. For the resonance to occur, we must have

$$\begin{vmatrix} c_{66}\left[\left(1+k_{26}^2\right)\eta\cos(\eta b)-k_{26}^2\dfrac{\sin(\eta b)}{b}\right] & -\bar{c}_{66}\bar{\eta} \\ -\sin(\eta b)\sin(2\bar{\eta}\bar{b}) & \cos(2\bar{\eta}\bar{b}) \end{vmatrix}=0 \quad (19)$$

where

$$k_{26}^2=\frac{e_{26}^2}{c_{66}\varepsilon_{22}} \quad (20)$$

is the piezoelectric coupling constant. By further defining

$$K^2=\frac{1}{1+k_{26}^2}k^2 \quad (21)$$

we can rewrite Eq.(19) as

$$\xi\tan\xi\tan\left(\frac{2B}{K}\xi\right)=\frac{K}{C_{66}}\left[\left(1+k_{26}^2\right)\xi-k_{26}^2\tan\xi\right] \quad (22)$$

The frequency will be the same as in Eq.(11), and the equation for normalized frequency is

$$X\tan\left(\frac{\pi}{2}X\right)\tan\left(\frac{\pi B}{K}X\right)=\frac{K}{C_{66}}\left[\left(1+k_{26}^2\right)X-\frac{2k_{26}^2}{\pi}\tan\left(\frac{\pi}{2}X\right)\right] \quad (23)$$

In comparison to Eq.(14), we emphasize the new parameter $K$ as given in Eq.(21), which differs from the one in Eq.(10) with the introduction of piezoelectric coupling constant. The consideration of both the stiffness of the electrodes and piezoelectric effect of crystal plate will certainly make the frequency solution more accurate when the electrode presence cannot be neglected, which is true nowadays because the crystal blank has been shrunken a lot in achieving higher fundamental thickness-shear frequency.

## APPLICATIONS IN RESONATOR DESIGN

In crystal resonator design, how to quickly determine the parameters appearing in the above equations is very important in selecting the best initial configuration. We can certainly employ an iterative procedure based on the equations above, or we can use the known parameters, like the required frequency and electrodes based on the practical manufacturing capability to decide the thickness of the crystal plate so the iterations of reducing the thickness of crystal blank can be kept to minimum. With $f$ as the given frequency of the crystal resonator, we have thicknesses of crystal and metal electrodes in pure thickness-shear vibration mode as

$$b_0=\frac{1}{4f}\sqrt{\frac{c_{66}}{\rho}}, \quad (24.1)$$

$$\bar{b}_0=\frac{1}{4f}\sqrt{\frac{\bar{c}_{66}}{\bar{\rho}}} \quad (24.2)$$

Consequently, the equation for the crystal blank thickness can be deduced from Eqs.(14) and (23) and are

$$\tan\left(\frac{\pi}{2}\frac{b}{b_0}\right)\tan\left(\pi\frac{\bar{b}}{\bar{b}_0}\right)=\frac{k}{C_{66}} \quad (25)$$

and

$$\frac{b}{b_0}\tan\left(\frac{\pi}{2}\frac{b}{b_0}\right)\tan\left(\frac{\pi}{\sqrt{1+k_{26}^2}}\frac{\bar{b}}{\bar{b}_0}\right)=\frac{k}{C_{66}\sqrt{1+k_{26}^2}}\left[\left(1+k_{26}^2\right)\frac{b}{b_0}-\frac{2k_{26}^2}{\pi}\tan\left(\frac{\pi}{2}\frac{b}{b_0}\right)\right] \quad (26)$$

respectively. Since all the parameters except crystal thickness $b$ are known, we can use them for the determination of the thickness of crystal blanks. These equations are very similar to the frequency equations given in Eqs.(14) and (23), so the solution procedure will also be similar. In addition, we have noticed that Eq.(24.1) has been known to design engineers as the primary formula for the selection of crystal blank thickness in the initial design stage with given frequency.

## NUMERICAL EXAMPLES

With given frequency Eqs.(14) and (23), we can use known parameters like crystal cut, crystal blank thickness, electrode material, and electrode thickness to find the accurate resonance frequency of the resonator. To evaluate the effect of electrodes at larger mass ratio, we consider an AT-cut quartz crystal and copper electrodes with the following constants

$$c_{66}=29.01\times10^{9}\ \text{N/m}^2,\ k_{26}^2=7.8126\times10^{-3},$$
$$\rho=2649\ \text{kg/m}^3;$$
$$\bar{c}_{66}=4.37\times10^{10}\ \text{N/m}^2,\ \bar{\rho}=10500\ \text{kg/m}^3$$

for frequency solution in Eq.(23). The results in Fig.2, in comparison with that of Bluestein and Tiersten (1968), are very close, although there are small differences in the equations. Bluestein and Tiersten (1968) stated clearly that their approximate result is for mass ratios in the range of $0.005<R<0.05$, but we found the results were also good up to larger numbers, say around 0.3, or 30%. Since the results are for one electrode material only, we can say at least for copper, the effect is dominated by the mass loading.

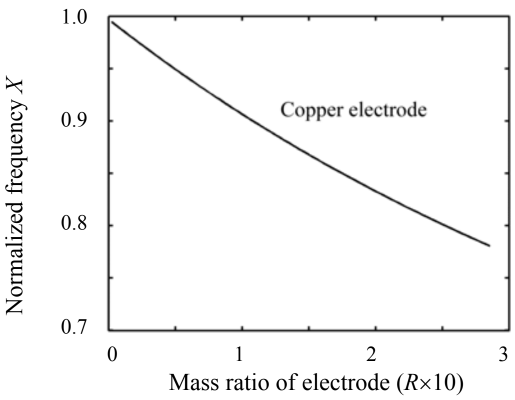


**Fig.2 Normalized frequency vs mass ratio of copper electrode**

It can be observed that indeed the effect of electrodes on the resonance frequency can be well predicted with the mass loading consideration when the thickness of the electrode is relatively small, or the mass ratio $R$ is in the small range specified by Bluestein and Tiersten (1968). As the thickness ratio $B$ or the mass ratio $R$ increases, the frequency will decrease, almost linearly manner.

In crystal resonator design and production, the precise determination of the crystal blanks with electrodes present will be important for many reasons like the reduction of etching process and related tuning and adjustments. Since the electrodes are generally known in the design process, we can use Eqs.(25) or (26) to calculate the precise crystal blank thickness in terms of the ratio with crystal plate without electrodes. This result is shown in Fig.3 with given frequency and electrodes as ratios defined in the equation. These results can be directly applied in the design process for given frequencies and electrodes.

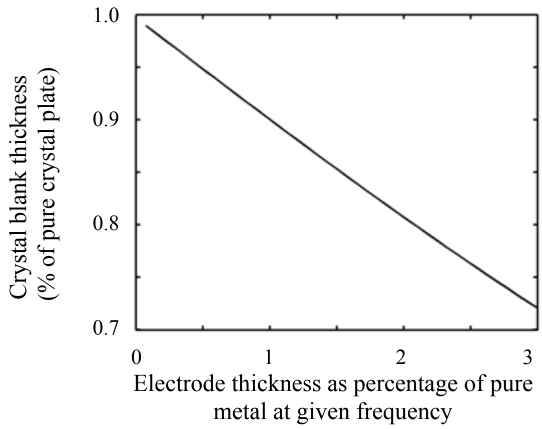


**Fig.3 Crystal blank thickness vs electrode thickness ratio for copper electrode**

## CONCLUSION

With a rigorous derivation of thickness-shear resonance frequency of electroded crystal plates, we have obtained the frequency equation in ratios of thicknesses and densities of crystal plates and electrodes. By evaluating the equation for solutions of an infinite AT-cut plate, we found that for larger mass or thickness ratios, the consideration of stiffness effect can improve the approximate frequency predictions. It is obvious that in applications like crystal resonator design today, the higher fundamental thickness-shear frequency has pushed down the crystal blank thickness, and the relative ratios of thickness and mass have been increasing to a level requiring further attention in making necessary revisions to the design theory and tools. The results presented in this paper are our initial response to this matter based on our extensive work on the computational tool development. We are considering incorporating these results into the basic thickness-shear vibration analysis through the correction factors based on the accurate frequency solutions. We believe that the new correction factors and proper consideration of electrode stiffness in the plate equations will make the analytical effort more suitable for practical applications.

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