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Hollow dimension of modules

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Abstract: In this paper, we are interested in the following general question: Given a module M which has finite hollow dimension and which has a finite collection of submodules K_i ($1\le i\le n$) such that $M=K_1+...+K_n$, can we find an expression for the hollow dimension of M in terms of hollow dimensions of modules built up in some way from $K_1,...,K_n$? We prove the following theorem: Let M be an amply supplemented module having finite hollow dimension and let K_i ($1\le i\le n$) be a finite collection of submodules of M such that $M=K_1+...+K_n$. Then the hollow dimension $h(M)$ of M is the sum of the hollow dimensions of K_i ($1\le i\le n$) if and only if K_i is a supplement of $K_1+...+K_{i-1}+K_{i+1}+...+K_n$ in M for each $1\le i\le n$.

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INTRODUCTION

Throughout this note R will denote an arbitrary associative ring with identity and all modules will be unital right R-modules. Let M be any R-module and S a submodule of M. S is called a small submodule of M (denoted as $S<<M$) if for every submodule T of M with $M=S+T$, then $M=T$. Let M be an R-module. Let N be a submodule of M. If any submodule K of M is minimal with the property that $M=N+K$, then the submodule K is called a supplement of N in M. It is not hard to see that K is a supplement of N in M if and only if $M=N+K$ and $N\cap K<<K$. A submodule K of M is called a supplement submodule of M if there exists a submodule N of M such that K is a supplement of N in M. Every submodule of any module M need not have a supplement in M. For example, consider the Z-module $M_Z=Z$. In this module, just the submodules 0 and M have supplements (in fact they are supplements of each other in M). Therefore any R-module M is called supplemented if every submodule of M has a supplement in M. On the other hand, let M be any R-module. M is called amply supplemented if for any two submodules A and B of M with $M=A+B$, A has a supplement in M that is contained in B. Since every artinian module is amply supplemented, the Z-module $M=Z/pZ\oplus Z/p^3Z$ is an amply supplemented module, where p is any prime integer. Clearly, every amply supplemented module is supplemented, but the converse is not true in general (Lemma A.5 in (Mohamed and Müller, 1990)). Any module M is called hollow, if every proper submodule of M is small in M.

In this note, we are interested in the following general question: Given a module M which has finite hollow dimension and a finite collection of submodules K_i ($1\le i\le n$) such that $M=K_1+...+K_n$, can we find an expression for the hollow dimension of M in terms of hollow dimensions of modules built up in some way from K_1, ..., K_n?

HOLLOW DIMENSION

Let M be an R-module. A non-empty family $\{N_i\}_I$ of proper submodules of M is called coindependent if for any $i\in I$ and any finite subset $F\subseteq I\setminus\{i\}$· $N_i+\bigcap_{k\in F} N_k=M$, with the convention that the intersection with an empty index set is set to be M.

Proposition 1 For a non-zero module M the following equivalent (3.1.2 in (Lomp, 1996); Grezeszcuk and Puczylowski, 1984):

(1) M does not contain an infinite coindependent family of submodules.

(2) M contains a finite coindependent family of submodules $\{N_1, ..., N_n\}$ such that $\bigcap_{i=1}^{n} N_i << M$ and M/N_i is a hollow module for every $1 \le i \le n$.

(3) $\mathrm{Sup}\{k | M$ contains a coindependent family of submodules of cardinality equal to $k\} = n$.

(4) For any descending chain $N_1 \supset N_2 \supset ...$ of submodules of M there exists j such that for all $k \ge j$, $N_j/N_k << M/N_k$.

(5) There exists an epimorphism from M to a finite direct sum of n hollow factor modules with the small kernel.

An R-module M is said to have finite hollow dimension if it satisfies one of the conditions in Proposition 1. We denote the hollow dimension of M with $h(M)$. For the module M as in Proposition 1, $h(M)=n$. Note that $M=0 \Leftrightarrow h(M)=0$ and M is hollow $\Leftrightarrow h(M)=1$.

Proposition 2 Let M be any R-module. Then the following hold:

(1) If $M=M_1 \oplus \ldots \oplus M_n$, then $h(M)=\sum_{i=1}^{n} h(M_i)$ (5.13 in (Miyashita, 1966); 3.1.10 (1) in (Lomp, 1996)).

(2) If M has finite hollow dimension, then for any submodule N of M, $N << M \Leftrightarrow h(M)=h(M/N)$ (Hanna and Shamsuddin, 1984; 3.1.10 (2) in (Lomp, 1996)).

(3) Assume the sequence $0 \to N \to M \to L \to 0$ is exact. Then $h(L) \le h(M) \le h(N)+h(L)$ (3.10 (6) in (Lomp, 1996)).

(4) Let $M=\sum_I M_i$ with $M_i \neq 0$ for all $i \in I$. Then $h(M) \le \sum_I h(M_i)$ (Remarks 2 in (Lomp, 1996)).

(5) If M has finite hollow dimension and N a submodule of M, then N is a supplement submodule of $M \Leftrightarrow h(M)=h(M/N)+h(N)$ (Corollary 3.2.3 in (Lomp, 1996)).

(6) If $M=\sum_{i=1}^{n} H_i$ is an irredundant sum of hollow modules, then $h(M)=n$ (Theorem 7.10 in (Hanna and Shamsuddin, 1984); Theorem 14 in (Grezeszcuk and Puczylowski, 1984); Lemma 1 in (Rim and Takemori, 1993); 3.2.5 in (Lomp, 1996)).

Lemma 1 Let N, K be submodules of M such that K is a supplement submodule of M, $M=K+N$ and $K \cap N << M$. Then K is a supplement of N in M.

Proof Clear by Lemma 1.1 in (Keskin, 2000).

Lemma 2 Let K and L be submodules of an amply supplemented module M such that $M=K+L$. Then there exist submodules K' and L' of M such that $K' \subseteq K$, $L' \subseteq L$ and K' and L' are supplements of each other in M.

Proof Since M is amply supplemented, there exists a supplement K' of L in M with $K' \subseteq K$, namely, $M=K'+L$, $K' \cap L << K' \subseteq K$. And again there exists a supplement L' of K' in M such that $L' \subseteq L$. By Lemma 1, K' is a supplement of L' in M.

We can give the following simple example of Lemma 2 in which M, K, L, K' and L' are all constructed: Let $\boldsymbol{R}=\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ be the ring of all upper triangular $n \times n$ matrices with entries in F, where F is a field. It is well-known that $\boldsymbol{R}$ is a right perfect ring and hence $\boldsymbol{R}$ is an amply supplemented right $\boldsymbol{R}$-module by Theorem 4.41 in (Mohamed and Müller, 1990). Consider the right $\boldsymbol{R}$-modules $\boldsymbol{K}=\begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{L}=\begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$. Then $\boldsymbol{R_R}=\boldsymbol{K}+\boldsymbol{L}$. On the other hand, $\boldsymbol{R_R}=\begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$. Now we can take $\boldsymbol{K'}=\boldsymbol{K}$ and $\boldsymbol{L'}=\begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix} \le \boldsymbol{L}$. Clearly, $\boldsymbol{K'}$ and $\boldsymbol{L'}$ are supplements of each other in $\boldsymbol{R_R}$.

Lemma 3 Let K, L be submodules of a module M such that K is a supplement of L in M. Let G, H be submodules of M contained in K such that G is a supplement of H in K. Then G is a supplement of $H+L$ in M.

Proof Let $M=K+L$ with $K \cap L << K$ and $K=G+H$ with G being minimal in this property. Clearly $M=G+(H+L)$. Suppose that $M=G'+(H+L)$ for any submodule G' of M with $G' \subseteq G$. Then by modularity, $K=G'+(K \cap (H+L))=G'+H+(K \cap L)$. Since $K \cap L << K$, $K=G'+H$. By the minimality of G, $G'=G$. Therefore G is a supplement of $H+L$ in M.

Lemma 4 Let M be an R-module having finite hollow dimension and let K, L be submodules of M with $M=K+L$. Then $h(M)=h(K)+h(L)$ if and only if K and L are supplements of each other in M.

Proof Assume that $h(M)=h(K)+h(L)$. Then K and L both have finite hollow dimension. Note that there is an epimorphism from L to M/K with the kernel $K\cap L$ and so by Proposition 2 (3), $h(M)\leq h(K)+h(M/K)\leq h(K)+h(L)=h(M)$. Thus by Proposition 2 (5), K is a supplement submodule of M. Similarly L is a supplement submodule of M. Consider the isomorphism $\varphi: M/K\cap L\rightarrow[K/K\cap L]\oplus[L/K\cap L]$ defined by $\varphi(k+l+K\cap L)=(k+K\cap L, l+K\cap L)$, where $k\in K$, $l\in L$. Then by Proposition 2 (1), $h(M/K\cap L)=h([K/K\cap L]\oplus[L/K\cap L])=h(K/K\cap L)+h(L/K\cap L)=h(M/L)+h(M/K)$. Since $h(L)+h(M/L)=h(M)=h(K)+h(M/K)$, then $h(M/L)=h(M)-h(L)$ and $h(M/K)=h(M)-h(K)$. Therefore $h(M/K\cap L)=h(M)$. By Proposition 2 (2), $K\cap L<<M$. Now by Lemma 1, K and L are supplements of each other in M.

Conversely, suppose that K and L are supplements of each other in M. Since $K\cap L<<K$, $h(K)=h(K/K\cap L)$ by Proposition 2 (2). Since L is a supplement submodule of M, $h(M)=h(M/L)+h(L)$ by Proposition 2 (5). Therefore $h(M)=h(K)+h(L)$.

Lemma 5 Any supplement submodule of an amply supplemented module M is amply supplemented (41.7 (1) in (Wisbauer, 1991)).

Now we can give the main result of this paper.

Theorem 1 Let M be an amply supplemented module having finite hollow dimension and let K_i be submodules of M such that $M=K_1+...+K_n$ for some positive integer n. Then $h(M)=\sum_{i=1}^{n}h(K_i)$ if and only if K_i is a supplement of $K_1+...+K_{i-1}+K_{i+1}+...+K_n$ in M for each $1\leq i\leq n$.

Proof For $n=1$ the result is clear. Suppose that $n\geq2$ and $L_i=K_1+...+K_{i-1}+K_{i+1}+...+K_n$ for each $1\leq i\leq n$. Suppose that $h(M)=\sum_{i=1}^{n}h(K_i)$. Let $1\leq i\leq n$. By Proposition 2 (4), $h(L_i)\leq\sum_{i\neq j}h(K_j)$ and hence $h(L_i)+h(K_i)\leq\sum_{i=1}^{n}h(K_i)$ $=h(M)\leq h(L_i)+h(K_i)$. Thus $h(M)=h(K_i)+h(L_i)$. By Lemma 4, K_i is a supplement of L_i in M.

Conversely, suppose that K_i is a supplement of L_i in M for each $1\leq i\leq n$. Without loss of generality, we can assume that $K_i\neq0$ $(1\leq i\leq n)$. Then $L_i\neq M$ for all $1\leq i\leq n$. Consider the submodule K_1. Since K_1 is a supplement submodule of M, by Proposition 2 (5), K_1 has finite hollow dimension. Suppose that K_1 is not hollow. Then there exist proper submodules K_{11} and K_{12} of K_1 such that $K_1=K_{11}+K_{12}$. By Lemma 5, K_1 is amply supplemented. Therefore by Lemma 2, we can suppose without loss of generality that K_{11} and K_{12} are supplements of each other in K_1. By Lemma 3, K_{11} is a supplement of $K_{12}+L_1$ in M and K_{12} is a supplement of $K_{11}+L_1$ in M. Moreover, by Lemma 4, $h(K_1)=h(K_{11})+h(K_{12})$. Thus $\sum_{i=1}^{n}h(K_i)=h(K_{11})+h(K_{12})+\sum_{i=2}^{n}h(K_i)$. We also note that $1\leq h(K_{11})<h(K_1)$ and $1\leq h(K_{12})<h(K_1)$. If K_{11} or K_{12} is not a hollow submodule of M then the above argument can be repeated. Since $h(K_1)<\infty$, it follows that this process must stop. Thus $M=H_1+...+H_k+K_2+...+K_n$, where k is a positive integer and H_i is a hollow submodule of M for each $1\leq i\leq k$ such that H_i is a supplement of $(\sum_{i\neq j}H_j)+K_2+...+K_n$ for each $1\leq i\leq k$, K_i is a supplement of $H_1+...+H_k+K_2+...+K_{i-1}+K_{i+1}+...+K_n$ for all $2\leq i\leq n$ and $\sum_{i=1}^{n}h(K_i)=\sum_{j=1}^{k}h(H_j)+\sum_{i=2}^{n}h(K_i)$.

Now consider K_2, K_3, ..., K_n in turn using the above argument. There exist a positive integer $t\geq k$ and hollow submodules H_j $(1\leq j\leq t)$ of M such that H_j is a supplement of $H_1+...+H_{j-1}+H_{j+1}+...+H_t$ for each $1\leq j\leq t$ and $\sum_{i=1}^{n}h(K_i)=\sum_{j=1}^{t}h(H_j)$. It is easy to check that $M=\sum_{j=1}^{t}H_j$ is an irredundant sum of hollow modules H_j. By Proposition 2 (6), $h(M)=t=\sum_{i=1}^{n}h(K_i)$.

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