Journal of Zhejiang University SCIENCE
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

Rudiment of weak Doi-Hopf π-modules*

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Received Mar. 21, 2005; revision accepted June 24, 2005

Abstract: The notion of weak Doi-Hopf π-datum and weak Doi-Hopf π-module are given as generalizations of an ordinary weak Doi-Hopf datum and weak Doi-Hopf module introduced in (Böhm, 2000), also as a generalization of a Doi-Hopf π-module introduced in (Wang, 2004). Then we also show that the functor forgetting action or coaction has an adjoint. Furthermore we explain how the notion of weak Doi-Hopf π-datum is related to weak smash product. This paper presents our preliminary results on weak Doi-Hopf group modules.

Key words: Weak semi-Hopf π-coalgebra, Weak Doi-Hopf π-modules
doi:10.1631/jzus.2005.AS0144 **Document code:** A **CLC number:** O153.3

INTRODUCTION

Turaev recently introduced the notion of group coalgebra as a generalization of an ordinary coalgebra for the study of Homotopy quantum field theory. Naturally, as a generalization of Hopf algebra, the notion of Hopf group coalgebra was defined and used to construct Hennings-like and Kuperberg-like invariants of principal π-bundles over link complements and over 3-manifolds. It has been shown that some theories of Hopf algebras can be extended to the setting of Hopf group-coalgebras which one can find in (Virelizier, 2002).

One of the attractive aspects of weak Doi-Hopf module (Böhm, 2000; Böhm *et al.*, 1999) is that many notions of modules studied appear as its special cases. Motivated by the ideas of weak Hopf algebras and Doi-Hopf group modules, we want to generalize the results of weak Doi-Hopf modules to weak Doi-Hopf group modules.

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* Project supported by the Program for New Century Excellent Talents in University (No. 04-0522), the National Science Foundation of Zhejiang Province of China (No. 102028), and the Cultivation Fund of the Key Scientific and Technical Innovation Project, Ministry of Education of China (No. 704004)

The organization of the paper is as follows: in Section 1 we recall some basic definitions and properties; in Section 2 we introduce the notion of weak Doi-Hopf π-datum and give some examples; in Section 3 we first define the notion of weak Doi-Hopf π-modules and then state the functor forgetting action or coaction has an adjoint; in Section 4 we explain how the notion of weak Doi-Hopf π-datum is related to weak smash product.

Conventions We work over a ground field k. We denote by I the unit of the group π. We use the standard algebra and coalgebra notation, i.e., Δ is a coproduct, ε is a counit, m is a product and 1 is a unit. The identity map V from any k-space to itself is denoted by id_V. We write a_α for any element in A_α and $[a]$ for an element in $\overline{A}=A/\ker f$, where f is a k-linear map. For a right C-comodule M, we write $\rho(m)=\sum m_{[0]}\otimes m_{[1]}$. For a left C-comodule M, we write $\rho(m)=\sum m_{[1]}\otimes m_{[0]}$. For a weak bialgebra B, we have the notations $b^l=\sum\varepsilon(1_1^B\, b)1_2^B$ and $b^r=\sum\varepsilon(b1_2^B\,)1_1^B$ for any $b\in B$.

We let ${}^CM(H)_A$ denote the category which has the finite dimensional left weak Doi-Hopf modules over the left weak Doi-Hopf datum (H,A,C) as objects and morphisms being both left C-colinear and right A-linear as arrows.

Definition 1 A π-coalgebra over k is a family of $C=\{C_\alpha\}_{\alpha\in\pi}$ of k-spaces endowed with a family k-linear maps $\Delta=\{\Delta_{\alpha,\beta}:C_{\alpha\beta}\rightarrow C_\alpha\otimes C_\beta\}_{\alpha,\beta\in\pi}$ and a k-linear map $\varepsilon:C_I\rightarrow k$ such that for any $\alpha,\beta\in\pi$,

$$(1)\ (\Delta_{\alpha,\beta}\otimes id_{C_\gamma})\Delta_{\alpha\beta,\gamma}=(id_{C_\alpha}\otimes\Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}. \quad (1)$$

$$(2)\ (id_{C_\alpha}\otimes\varepsilon)\Delta_{\alpha,I}=(\varepsilon\otimes id_{C_\alpha})\Delta_{I,\alpha}=id_{C_\alpha}. \quad (2)$$

Here we extend the Sweedler notation for comultiplication, we write $\Delta_{\alpha,\beta}(c)=\sum c_{\alpha\beta 1\alpha}\otimes c_{\alpha\beta 2\beta}$ for any α, $\beta\in\pi$ and $c\in C_{\alpha\beta}$.

Remark 1 $(C_I, \Delta_{I,I}, \varepsilon)$ is a coalgebra in the usual sense of the word.

We say a π-C-coalgebra is finite dimensional if C_α is finite dimensional for any $\alpha\in\pi$.

Let $C=\{C_\alpha, \Delta, \varepsilon\}_{\alpha\in\pi}$ be a π-coalgebra and A be an algebra with multiplication m_A and unit element 1_A. For any $f\in\mathrm{Hom}_k(C_\alpha, A)$ and $g\in\mathrm{Hom}_k(C_\beta, A)$, we define their convolution product by

$$f*g=m_A(f\otimes g)\Delta_{\alpha,\beta}\in\mathrm{Hom}_k(C_{\alpha,\beta}, A).$$

In particular, for $A=k$, the π-graded algebra $\mathrm{Conv}(C, k)=\oplus_\alpha C_\alpha^*$ is called dual to C and is denoted by C^*.

Definition 2 A weak semi-Hopf π-H-coalgebra is a family of algebras $\{H_\alpha,m_\alpha,1_\alpha\}_{\alpha\in\pi}$ and also a π-coalgebra $\{H_\alpha,\Delta_{\alpha,\beta},\varepsilon\}_{\alpha,\beta\in\pi}$ satisfying the following conditions for any $\alpha,\beta\in\pi$,

$$(1)\quad \Delta(hg)=\Delta(h)\Delta(g),\ \varepsilon(1_I)=1, \quad (3)$$

$$\begin{aligned}(2)\quad \Delta^2(1_{\varepsilon\beta\gamma})&=\sum 1_{\varepsilon\beta\gamma 1\alpha\beta 1\alpha}\otimes 1_{\alpha\beta\gamma 1\alpha\beta 2\beta}\otimes 1_{\alpha\beta\gamma 2\gamma}\\ &=\sum 1_{\alpha\beta 1\alpha}\otimes 1_{\beta\gamma 1\gamma}1_{\alpha\beta 2\beta}\otimes 1_{\beta\gamma 2\gamma}\\ &=\sum 1_{\alpha\beta 1\alpha}\otimes 1_{\alpha\beta 2\beta}1_{\beta\gamma 1\beta}\otimes 1_{\beta\gamma 2\gamma}, \quad (4)\end{aligned}$$

$$\begin{aligned}(3)\quad \varepsilon(x_Iy_Iz_I)&=\sum\varepsilon(x_Iy_{I1})\varepsilon(y_{I2}z_I)\\ &=\sum\varepsilon(x_Iy_{I2})\varepsilon(y_{I1}z_I). \quad (5)\end{aligned}$$

We call a weak semi-Hopf π-H-coalgebra finite type if H_α is finite dimensional for any $\alpha\in\pi$.

Remark 2 Eqs.(4) and (5) imply that the unit preserving property of Δ and the multiplication preserving property of ε are not required. Eq.(4) can be regarded as a generalization of $(\Delta(1)\otimes 1)(1\otimes\Delta(1))=(1\otimes\Delta(1))(\Delta(1)\otimes 1)=\Delta^2(1)$ for a weak bialgebra. Obviously a semi-Hopf group coalgebra is a weak semi-Hopf group coalgebra and when the group π is trivial it is just an ordinary weak bialgebra.

Definition 3 Let $C=\{C_\alpha,\Delta,\varepsilon\}_{\alpha\in\pi}$ be a π-coalgebra. A right π-C-comodule over C is a family of k-spaces $M=\{M_\alpha\}_{\alpha\in\pi}$ endowed with a family of k-linear maps $\{\rho_{\alpha,\beta}:M_{\alpha\beta}\rightarrow M_\alpha\otimes C_\beta\}_{\alpha,\beta\in\pi}$ such that the following holds:

$$(1)\quad (\rho_{\alpha,\beta}\otimes id_{C_\gamma})\rho_{\alpha\beta,\gamma}=(id_{M_\alpha}\otimes\Delta_{\beta,\lambda})\rho_{\alpha,\beta\gamma},\ \text{for any } \alpha,\beta,\gamma\in\pi. \quad (6)$$

$$(2)\quad (id_{M_\alpha}\otimes\varepsilon)\rho_{\alpha,I}=id_{M_\alpha},\ \text{for any } \alpha\in\pi. \quad (7)$$

THE WEAK DOI-HOPF GROUP DATUM

Definition 4 Let $C=\{C_\alpha, \Delta, \varepsilon\}_{\alpha\in\pi}$ be a π-coalgebra and M a k-vector space. A right π-C-comodulelike object is a couple $(M,\{\rho_\alpha^M\}_{\alpha\in\pi})$, where $\rho_\alpha^M:M\rightarrow M\otimes C_\alpha$ is a k-linear map and written by $\rho_\alpha^M(m)=\sum m_{((0),M)}\otimes m_{((1),\alpha)}$ for any $\alpha\in\pi$, such that the following holds:

$$(1)\quad (\rho_\alpha^M\otimes id_{C_\beta})\rho_\beta^M=(id_M\otimes\Delta_{\alpha,\beta})\rho_{\alpha\beta}^M,\ \text{for any } \alpha,\beta\in\pi. \quad (8)$$

$$(2)\quad (id_M\otimes\varepsilon)\rho_I^M=id_M. \quad (9)$$

Similarly, a left π-C-comodulelike object is a couple $(M,\{\rho_\alpha^M\}_{\alpha\in\pi})$, where $\rho_\alpha^M:M\rightarrow C_\alpha\otimes M$ is a k-linear map and written by $\rho_\alpha^M(m)=\sum m_{((1),\alpha)}\otimes m_{((0),M)}$, for any $\alpha\in\pi$. Such that the following holds:

$$(1)\quad (id_{C_\alpha}\otimes\rho_\beta^M)\rho_\alpha^M=(\Delta_{\alpha,\beta}\otimes id_M)\rho_{\alpha\beta}^M,\ \text{for any } \alpha,\beta\in\pi. \quad (10)$$

$$(2)\quad (\varepsilon\otimes id_M)\rho_I^M=id_M. \quad (11)$$

Let M and N be two left π-C-comodulelike objects. A k-linear map $f:M\rightarrow N$ is called a left π-C-comodulelike morphism if $\rho_\alpha^n f=(id_{C_\alpha}\otimes f)\rho_\alpha^M$ for any $\alpha\in\pi$.

Definition 5 Let $H=\{H_\alpha,m_\alpha,1_\alpha,\Delta,\varepsilon\}$ be a weak semi-Hopf π-coalgebra and let A be an algebra. A is called a right π-H-comodule algebra if A is a right π-H-comodulelike object $(A,\{\rho_\alpha^A\}_{\alpha\in\pi})$. Such that the following holds:

(1) $\rho_\alpha^A(ab)=\rho_\alpha^A(a)\rho_\alpha^A(b)$, for any $\alpha\in\pi$ and $a,b\in A$. (12)

(2) $$(1_A\otimes\Delta_{\beta,\alpha}(1_{\beta\alpha}))(\rho_\beta^A(1_A)\otimes 1_\alpha) = (\rho_\beta^A(1_A)\otimes 1_\alpha)(1_A\otimes\Delta_{\beta,\alpha}(1_{\beta\alpha})) = (id_A\otimes\Delta_{\beta,\alpha})\rho_{\beta\alpha}^A(1_A). \quad (13)$$

Similarly, a left π-H-comodule algebra is a left π-H-comodulelike object $(A,\{\rho_\alpha^A\}_{\alpha\in\pi})$, such that the following holds:

(1) $\rho_\alpha^A(ab)=\rho_\alpha^A(a)\rho_\alpha^A(b)$, for any $\alpha\in\pi$ and $a,b\in A$. (14)

(2) $$(\Delta_{\beta,\alpha}(1_{\beta\alpha})\otimes 1_A)(1_\beta\otimes\rho_\alpha^A(1_A)) = (1_\beta\otimes\rho_\alpha^A(1_A))(\Delta_{\beta,\alpha}(1_{\beta\alpha})\otimes 1_A) = (\Delta_{\beta,\alpha}\otimes id_A)\rho_{\beta\alpha}^A(1_A). \quad (15)$$

Remark 3 Eq.(15) implies that the unit preserving property of ρ is not required and generalizes $(1\otimes\Delta(1))(\rho(1)\otimes 1)=(id_A\otimes\Delta)\rho(1)$ for an ordinary right comodule algebra over a weak bialgebra introduced in (Böhm, 2000).

A endowed with ρ_I^A is an ordinary H_I-comodule algebra introduced in (Böhm, 2000).

Definition 6 Let $H=\{H_\alpha,m_\alpha,1_\alpha,\Delta,\varepsilon\}$ be a weak semi-Hopf π-H-coalgebra and $C=\{C_\alpha,\Delta,\varepsilon\}_{\alpha\in\pi}$ be a π-coalgebra. A couple $(C,\{\varphi_\alpha^C\}_{\alpha\in\pi})$ is called a left π-H-module coalgebra where $\varphi_\alpha^C: H_\alpha\otimes C_\alpha\to C_\alpha$ is a k-linear map for any $\alpha\in\pi$ if the following holds:

(1) $(C_\alpha,\varphi_\alpha)$ is a left H_α-module, for any $\alpha\in\pi$;

(2) $\Delta_{\alpha,\beta}^C(h\cdot c)=\sum h_{1\alpha}\cdot c_{1\alpha}\otimes h_{2\beta}\cdot c_{2\beta}$,
for any $\alpha,\beta\in\pi$, $c\in C_{\alpha\beta}$, $h\in H_{\alpha\beta}$; (16)

(3) $\varepsilon(h\cdot c)=\sum\varepsilon(1_{I1}\cdot c)\varepsilon(h1_{I2})$,
for any $c\in C_I$, $h\in H_I$. (17)

Similarly, a couple $(C,\{\varphi_\alpha^C\}_{\alpha\in\pi})$ is called a right π-H-module coalgebra where $\varphi_\alpha^C: C_\alpha\otimes H_\alpha\to C_\alpha$ is a k-linear map for any $\alpha\in\pi$ if the following holds:

(1) $(C_\alpha,\varphi_\alpha)$ is a right H_α-module, for any $\alpha\in\pi$;

(2) $\Delta_{\alpha,\beta}^C(c\cdot h)=\sum c_{1\alpha}\cdot h_{1\alpha}\otimes c_{2\beta}\cdot h_{2\beta}$,
for any $\alpha,\beta\in\pi$, $c\in C_{\alpha\beta}$, $h\in H_{\alpha\beta}$; (18)

(3) $\varepsilon(c\cdot h)=\sum\varepsilon(c\cdot 1_{I2})\varepsilon(1_{I1}h)$,
for any $c\in C_I$, $h\in H_I$. (19)

Remark 4 Eq.(19) implies that the multiplication preserving property of ε is not required and C_I endowed with φ_I^C is an ordinary H_I-module coalgebra introduced in (Böhm, 2000).

Remark 5 In contrast to the case when H is a Hopf group coalgebra, the unit preserving property of $\{\rho_\alpha\}$ and the counit preserving property of $\{\varphi_\alpha\}$ are not required.

Example 1 Let $H=\{H_\alpha,m_\alpha,1_\alpha,\Delta,\varepsilon\}$ be a weak semi-Hopf π-H-coalgebra. Then $(H,\{m_\alpha^H\}_{\alpha\in\pi})$ is a right π-H-module coalgebra.

Definition 7 Let $H=\{H_\alpha,m_\alpha,1_\alpha,\Delta,\varepsilon\}$ be a weak semi-Hopf π-coalgebra. A triple (H,A,C) is called a right weak Doi-Hopf group datum or a right weak Doi-Hopf π-datum if A is a left π-H-comodule algebra and C is a right π-H-module coalgebra.

Similarly, a triple (H,A,C) is called a left weak Doi-Hopf group datum or a left weak Doi-Hopf π-datum if A is a right π-H-comodule algebra and C is a left π-H-module coalgebra.

We call a weak Doi-Hopf group datum (H,A,C) finite dimensional if H, A, C are all of finite dimension.

Remark 6 If the group π is trivial, then they are just the notions of weak Doi-Hopf datum introduced in (Böhm, 2000).

Example 2 Let $H=\{H_\alpha,m_\alpha,1_\alpha,\Delta,\varepsilon\}$ be a weak semi-Hopf π-coalgebra such that $H_{\lambda\alpha}=H_\lambda$ for a fixed element $\lambda\in\pi$ and any $\alpha\in\pi$. Let $A=H_\lambda$ together with the $\{\rho_\alpha^A: H_\lambda\to H_\lambda\otimes H_\alpha\}$ given by $\rho_\alpha^A(h)=\sum h_{1\lambda}\otimes h_{2\alpha}$ for any $h\in H_\lambda$, $\alpha\in\pi$. Let $C=H$ together with the $\{m_\alpha\}_{\alpha\in\pi}$, it is not hard to verify that the triple (H,A,C) is a left weak Doi-Hopf group datum.

Similar to 1.3.4 in (Virelizier, 2002), we have:

Lemma 1 Let $H=\{H_\alpha,m_\alpha,1_\alpha,\Delta,\varepsilon\}$ be a finite type weak semi-Hopf π-H-coalgebra. Then the π-graded algebra $H^*=\oplus_\alpha H_\alpha^*$ dual to H inherits a weak bialgebra structure by setting $\Delta^*(f)=m_\alpha^*(f)$,

$\varepsilon^*(f) = f(1_\alpha)$ for any $\alpha\in\pi$, $f\in H_\alpha^*$.

Theorem 1 For a finite dimensional right weak Doi-Hopf π-datum (H,A,C), the triple (H^*,C^*,A^*) is a left weak Doi-Hopf datum which we call the dual of (H,A,C) with the right coaction on every summand $\rho_\alpha: C_\alpha^* \to C_\alpha^* \otimes H_\alpha^*$ given by $\rho_\alpha(f) = \sum f_{[0]} \otimes f_{[1]} = \sum x_{\alpha K} \triangleright f \otimes X_{\alpha K}$ for any $\alpha\in\pi$, $f\in C_\alpha^*$, where $(x_{\alpha K}, X_{\alpha K})$ is a dual basis in H_α and H_α^*, and $(h\triangleright f)(c)=f(c\cdot h)$ for any $c\in C_\alpha$, $h\in H_\alpha$; the left A-module structure is given by $(f\cdot g)(a)=\sum f\{a_{((1),\alpha)}\}g\{a_{((0),A)}\}$ for any $f\in C_\alpha^*$, $g\in A^*$, $a\in A$.

Similarly, for a finite dimensional left weak Doi-Hopf π-datum (H,A,C), the triple (H^*,C^*,A^*) is a right weak Doi-Hopf datum which we call the dual of (H,A,C) with the left coaction on every summand $\rho_\alpha: C_\alpha^* \to H_\alpha^* \otimes C_\alpha^*$ given by $\rho_\alpha(f)=\sum f_{(1)}\otimes f_{(0)} =\sum X_{\alpha K}\otimes f\triangleleft x_{\alpha K}$ for any $\alpha\in\pi$, $f\in C_\alpha^*$, where $(x_{\alpha K}, X_{\alpha K})$ is a dual basis in H_α and H_α^*, and $(f\triangleleft h)(c)=f(h\cdot c)$ for any $c\in C_\alpha$, $h\in H_\alpha$, the right A-module structure is given by $(g\cdot f)(a)=\sum f\{a_{((1),\alpha)}\}g\{a_{((0),A)}\}$ for any $f\in H_\alpha^*$, $g\in A^*$, $a\in A$.

Proof Firstly we claim that $\rho_\alpha(f)=\sum x_{\alpha K}\triangleright f\otimes X_{\alpha K}$ for any $\alpha\in\pi$ and $f\in C_\alpha^*$ exactly defines a right coaction. In fact,

$$(id_{C_\alpha^*}\otimes\Delta_\alpha^*)\rho_\alpha(f) = \sum x_{\alpha K}\triangleright f\otimes X_{\alpha K1}\otimes X_{\alpha K2}, \quad (20)$$

$$(\rho_\alpha\otimes id_{H_\alpha^*})\rho_\alpha(f) = \sum x_{\alpha K}\triangleright(x_{\alpha l}\triangleright f)\otimes X_{\alpha K}\otimes X_{\alpha l}. \quad (21)$$

For any h, $g\in H_\alpha$, $c\in C_\alpha$, from Eqs.(20) and (21) we get

$$\begin{aligned}&\sum(x_{\alpha K}\triangleright f)(c)X_{\alpha K1}(h)X_{\alpha K2}(g)\\ &\quad= f(c\cdot hg) = f((c\cdot h)\cdot g)\\ &\quad=\sum(x_{\alpha K}\triangleright(x_{\alpha l}\triangleright f))(c)X_{\alpha K}(h)X_{\alpha l}(g).\end{aligned}$$

So $(id_{C_\alpha^*}\otimes\Delta_\alpha^*)\rho_\alpha = (\rho_\alpha\otimes id_{H_\alpha^*})\rho_\alpha$.

It is very easy to verify $(id_{C_\alpha^*}\otimes\varepsilon^*)\rho_\alpha = id_{C_\alpha^*}$.

Secondly we claim that C^* is a right H^*-comodule algebra.

For any α, $\beta\in\pi$, $f\in C_\alpha^*$, $g\in C_\beta^*$,

$$\rho_{\alpha\beta}(fg) = \sum x_{\alpha\beta K}\triangleright fg\otimes X_{\alpha\beta K}. \quad (22)$$

$$\rho_\alpha(f)\rho_\beta(g) = \sum(x_{\alpha K}\triangleright f)(x_{\beta n}\triangleright g)\otimes X_{\alpha K}X_{\beta n}. \quad (23)$$

From Eqs.(22) and (23) we have

$$\begin{aligned}&\sum((x_{\alpha K}\triangleright f)(x_{\beta n}\triangleright g))(c)(X_{\alpha K}X_{\beta n})(h)\\ &=\sum(x_{\alpha K}\triangleright f)(c_{1\alpha})(x_{\beta n}\triangleright g)(c_{2\beta})X_{\beta n}(h_{2\beta})X_{\alpha K}(h_{1\alpha})\\ &=\sum f(c_{1\alpha}\cdot h_{1\alpha})g(c_{2\beta}\cdot h_{2\beta}) = (fg)(c\cdot h)\\ &=\left(\sum x_{\alpha\beta K}\triangleright fg\right)(c)X_{\alpha\beta K}(h).\end{aligned}$$

So Eq.(22)=Eq.(23).

Now we show

$$(id_{C^*}\otimes\Delta_{H^*}^*)\rho(\varepsilon^C) = (id_{C^*}\otimes\Delta_{H^*}^*(\varepsilon^H))(\rho(\varepsilon^C)\otimes\varepsilon^H). \quad (24)$$

For any $c\in C_I$, h, $g\in H_I$, we have

$$\begin{aligned}&(id_{C^*}\otimes\Delta_{H^*}^*(\varepsilon^H))(\rho(\varepsilon^C)\otimes\varepsilon^H)(c\otimes h\otimes g)\\ &\quad=\sum(x_{IK}\triangleright\varepsilon^C)(c)(\varepsilon_1^H X_{IK})(h)(\varepsilon_2^H)(g)\\ &\quad=\sum\varepsilon^C(c\cdot x_{IK})\varepsilon^H(h_1 g)X_{IK}(h_2)\\ &\quad=\sum\varepsilon^C(c\cdot h_2)\varepsilon^H(h_1 g) = \varepsilon^C(c\cdot hg^I) = \varepsilon^C(c\cdot hg)\\ &\quad=(id_{C^*}\otimes\Delta_{H^*}^*)\rho(\varepsilon^C)(c\otimes h\otimes g).\end{aligned}$$

Next we claim that A^* is a left H^*-module. In fact, for any $f\in H_\alpha^*$, $g\in H_\beta^*$, $a^*\in A^*$, $a\in A$,

$$\begin{aligned}&(f\cdot(g\cdot a^*))(a)\\ &\quad=\sum f(a_{((1),\alpha)})g(a_{((0),A)((1),\beta)})a^*(a_{((0),A)((0),A)})\\ &\quad=\sum f(a_{((1),\alpha\beta)1\alpha})g(a_{((1),\alpha\beta)2\beta})a^*(a_{((0),A)}) = (fg\cdot a^*)(a),\\ &(\varepsilon\cdot a^*)(a) = \sum\varepsilon(a_{((1),I)})a^*(a_{((0),A)}) = a^*(a).\end{aligned}$$

Finally we claim that A^* is a left H^*-module coalgebra.

For any $f\in H_\alpha^*$, $a^*\in A^*$, a, $b\in A$,

$$\begin{aligned}&(\Delta(f\cdot a^*))(a\otimes b) = (f\cdot a^*)(ab)\\ &\quad=\sum f((ab)_{((1),\alpha)})a^*((ab)_{((0),A)})\\ &\quad=\sum f_1(a_{((1),\alpha)})f_2(b_{((1),\alpha)})a_1^*(a_{((0),A)})a_2^*(b_{((0),A)})\\ &\quad=\sum(f_1\cdot a_1^*\otimes f_2\cdot a_2^*)(a\otimes b).\end{aligned}$$

For any $f\in H_\alpha^*$, $a^*\in A^*$, $\alpha\in\pi$,

$$\begin{aligned}\varepsilon^*(f^r\cdot a^*)&=\sum\varepsilon^*(1_{H_l^*1}\cdot a^*)\varepsilon^*(f1_{H_l^*2})\\&=\sum f(1_{\alpha 1\alpha})\varepsilon(1_{A((1),\alpha)}1_{\alpha 2l})a^*(1_{A((0),A)})\\&=\sum f(1_{A((1),\alpha)1\alpha})\varepsilon(1_{A((1),\alpha)2l})a^*(1_{A((0),A)})\\&=\sum f(1_{A((1),\alpha)})a^*(1_{A((0),A)})=\varepsilon^*(f\cdot a^*).\end{aligned}$$

THE WEAK DOI-HOPF MODULES

Definition 8 A k-space M is called a right weak Doi-Hopf π-module over the right weak Doi-Hopf π-datum (H,A,C) if it is a right A-module and at the same time a left π-C-comodulelike object such that for any $m\in M$, $a\in A$, $\alpha\in\pi$,

$$\rho_\alpha^M(m\cdot a)=\sum m_{((0),M)}\cdot a_{((0),A)}\otimes m_{((1),\alpha)}\cdot a_{((1),\alpha)}. \quad (25)$$

Similarly, A k-space M is called a left weak Doi-Hopf π-module over the left weak Doi-Hopf π-datum (H,A,C) if it is a left A-module and at the same time a right π-C-comodulelike object such that for any $m\in M$, $a\in A$, $\alpha\in\pi$,

$$\rho_\alpha^M(a\cdot m)=\sum a_{([0],A)}\cdot m_{([0],M)}\otimes a_{([1],\alpha)}\cdot m_{([1],\alpha)}. \quad (26)$$

Definition 9 Let M and N be two right weak Doi-Hopf π-modules over the right weak Doi-Hopf π-datum (H,A,C), a k-linear map f:$M\to N$ is called a right weak Doi-Hopf π-module morphism if the following holds:

(1) f is a right A-module map;

(2) f is a π-C-comodulelike map, i.e.,

$$\rho_\alpha^N f=(id_{C_\alpha}\otimes f)\rho_\alpha^M, \text{ for any } \alpha\in\pi. \quad (27)$$

Similarly, a k-linear map f:$M\to N$ is called a left weak Doi-Hopf π-module morphism if the following holds:

(1) f is a left A-module map;

(2) f is a π-C-comodulelike map, i.e.,

$$\rho_\alpha^N f=(f\otimes id_{C_\alpha})\rho_\alpha^M, \text{ for any } \alpha\in\pi. \quad (28)$$

${}^C M(\pi\!-\!H)_A$ denotes the category which has finite dimensional right weak Doi-Hopf π-modules over the right weak Doi-Hopf π-datum as objects and right weak Doi-Hopf π-module morphisms as arrows.

Similarly, the category ${}_A M(\pi\!-\!H)^C$ which has the finite dimensional left weak Doi-Hopf π-modules over the left weak Doi-Hopf π-datum as objects and left weak Doi-Hopf π-module morphisms as arrows.

Let ${}^{\pi-C}M$ denote the category which has left π-C-comodulelike object as objects and left comodulelike maps as morphisms.

Example 3 Let $H=\{H_\alpha,m_\alpha,1_\alpha,\Delta,\varepsilon\}$ be a weak semi-Hopf π-coalgebra such that $H_\lambda=H_{\lambda\alpha}$ for a fixed element $\lambda\in\pi$ and $\alpha\in\pi$. Let (H,H_λ,H) be the same as Example 2 and let $M=H_\lambda$, $\rho_\alpha^M=\Delta_{\lambda,\alpha}^H$, and we define the left action of H_λ on M by the left multiplication of H, it is not hard to verify that M is a left weak Doi-Hopf π-module.

Theorem 2 Let (H,A,C) be a finite dimensional right weak Doi-Hopf π-datum and (H^*,C^*,A^*) its dual. Then the category ${}^C M(\pi\!-\!H)_A$ and ${}_{C^*}M(H^*)^{A^*}$ are equivalent.

Proof Firstly we define a functor G:${}^C M(\pi\!-\!H)_A\to {}_{C^*}M(H^*)^{A^*}$. For any $M\in{}^C M(\pi\!-\!H)_A$, we put $G(M)=M^*$ with the right coaction of A^* on M^* given by $\rho(f)=\sum f_{[0]}\otimes f_{[1]}=\sum a_k\triangleright f\otimes a^k$, where $\{a_k, a^k\}$ is a dual basis in A and A^*, and the left C^*-module structure is given by $(g\cdot u^*)(m)=\sum g(m_{((1),\alpha)})\times u^*(m_{((0),M)})$ for any $\alpha\in\pi$, $g\in C_\alpha^*$, $u^*\in M^*$, $m\in M$.

Obviously M^* is a right A^*-comodule. Now we only show that M^* is a left C^*-module. For any $\alpha,\beta\in\pi$, $g\in C_\beta^*$, $f\in C_\alpha^*$, $u^*\in M^*$, $m\in M$,

$$\begin{aligned}(g\cdot(f\cdot u^*))(m)&=\sum g(m_{((1),\beta)})(f\cdot u^*)(m_{((0),M)})\\&=\sum g(m_{((1),\beta)})f(m_{((0),M)((1),\alpha)})u^*(m_{((0),M)((0),M)})\\&=\sum g(m_{((1),\beta\alpha)1\beta})f(m_{((1),\beta\alpha)2\alpha})u^*(m_{((0),M)})\\&=(gf\cdot u^*)(m).\end{aligned}$$

And, we claim that the compatibility condition holds, i.e., $\rho(f\cdot u^*)=\sum f_{(0)}\cdot u_{[0]}^*\otimes f_{(1)}\cdot u_{[1]}^*$. In fact, for any $f\in C_\alpha^*$, $u^*\in M^*$, $m\in M$, $a\in A$,

$$\sum(f_{(0)}\cdot u_{[0]}^*\otimes f_{(1)}\cdot u_{[1]}^*)(m\otimes a)$$

$$=\sum f_{(0)}(m_{((1),\alpha)})f_{(1)}(a_{((1),\alpha)})u^*_{[0]}(m_{((0),M)})u^*_{[1]}(a_{((0),A)})$$
$$=\sum f(m_{((1),\alpha)}\cdot a_{((1),\alpha)})u^*(m_{((0),M)}\cdot a_{((0),A)})$$
$$=\sum f((m\cdot a)_{((1),\alpha)})u^*((m\cdot a)_{((0),M)})$$
$$=\rho(f\cdot u^*)(m\otimes a).$$

For any f:$M\to N\in {}^CM(\pi\text{−}H)_A$, we define $G(f)=f^t$, where f^t means the transposition of linear map. It is easy to prove $f^t\in {}_{C^*}M(H^*)^{A^*}$.

Next, we define a functor F:${}_{C^*}M(H^*)^{A^*}\to {}^CM(\pi\text{−}H)_A$. Let $F(M)=M^*$ for any $M\in {}_{C^*}M(H^*)^{A^*}$, and we define the right action of A on M^* by $(u^*\cdot a)(m)=\sum m_{[1]}(a)\,u^*(m_{[0]})$ for any $a\in A$, $u^*\in M^*$, where $\rho(m)=\sum m_{[0]}\otimes m_{[1]}$ and a comodulelike structure $\{\rho_\alpha^{M^*}:M^*\to C_\alpha\otimes M^*\}$ by $\rho_\alpha^{M^*}(u^*)=\sum x_{\alpha k}\otimes u^*\lhd X_{\alpha k}$ for any $u^*\in M^*$, $m\in M$, $\alpha\in\pi$, $a\in A$, where $\{x_{\alpha K},X_{\alpha K}\}$ is a dual basis in C_α and C_α^*.

Obviously M^* is a right A-module, here we only show M^* is a right π-C-comodulelike object. For any $g\in C_\beta^*$, $f\in C_\alpha^*$, $m\in M$,

$$((\Delta_{\beta,\alpha}\otimes id_{M^*})\rho_{\beta\alpha}^{M^*}(u^*))(f\otimes g\otimes m)$$
$$=\sum f(x_{\beta\alpha K1\beta})g(x_{\beta\alpha K2\alpha})u^*(X_{\beta\alpha K}\cdot m)$$
$$=\sum f(x_{\beta l})g(x_{\alpha K})u^*(X_{\beta l}\cdot(X_{\alpha K}\cdot m))$$
$$=\{((id_{C_\beta}\otimes\rho_\alpha^{M^*})\rho_\beta^{M^*}(u^*))(f\otimes g\otimes m)$$
$$\cdot((\varepsilon\otimes id_{M^*})\rho_I^{M^*})(u^*)(m)\}$$
$$=\sum\varepsilon(x_{Ij})u^*(X_{Ij}\cdot m)=(u^*)(m).$$

And, we claim that the compatibility condition holds, i.e., $\rho(u^*\cdot a)=\sum u^*_{((0),M^*)}\cdot a_{((0),A)}\otimes u^*_{((1),\alpha)}\cdot a_{((1),\alpha)}$.

$$\sum(u^*_{((0),M^*)}\cdot a_{((0),A)}\otimes u^*_{((1),\alpha)}\cdot a_{((1),\alpha)})(m\otimes a)$$
$$=\sum((u^*\lhd X_{\alpha k})\cdot a_{((0),A)})(m)f(x_{\alpha k}\cdot a_{((1),\alpha)})$$
$$=\sum u^*((a_{((1),\alpha)}\rhd f)\cdot m_{[0]})m_{[1]}(a_{((0),A)})$$
$$=(u^*\cdot a)(f\cdot m)=\rho(u^*\cdot a)(m\otimes a),$$
for any $u^*\in M^*$, $m\in M$, $a\in A$, $f\in C_\alpha^*$.

For any $f:M\to N\in {}_{C^*}M(H^*)^{A^*}$, let $F(f)=f^t$. One can easily verify that ${}^CM(\pi\text{−}H)_A$ and ${}_{C^*}M(H^*)^{A^*}$ are equivalent via the functors F and G.

Theorem 3 Let (H,A,C) be a right weak Doi-Hopf π-datum. Then the forgetful functor F:${}^CM(\pi\text{−}H)_A\to M_A$ has a right adjoint functor.

Proof Before defining a functor G:$M_A\to {}^CM(\pi\text{−}H)_A$, we first set $\varpi_\alpha^M:C_\alpha\otimes M\to C_\alpha\otimes M$,

$$\varpi_\alpha^M(c\otimes m)=\sum c\cdot 1_{A((1),\alpha)}\otimes m\cdot 1_{A((0),A)},$$
for any $\alpha\in\pi$, $M\in M_A$, $c\in C_\alpha$, $m\in M$. (29)

Then we claim $(\varpi_\alpha^M)^2=\varpi_\alpha^M$.

In fact,

$$(\varpi_\alpha^M)^2(c\otimes m)$$
$$=\sum c\cdot 1_{A((1),\alpha)}1'_{A((1),\alpha)}\otimes m\cdot 1_{A((0),A)}1'_{A((0),A)}$$
$$=\sum c\cdot 1_{A((1),\alpha)}\otimes m\cdot 1_{A((0),A)}=\varpi_\alpha^M(c\otimes m).$$

So we can define $G(M)=\oplus_\alpha G(M)_\alpha$, where $G(M)_\alpha=(C_\alpha\otimes M)/\ker\varpi_\alpha^M$. As a k-space, the right action of A on $G(M)_\alpha$ given by $[c\otimes m]\cdot a=\sum[c\cdot a_{((1),\alpha)}\otimes m\cdot a_{((0),A)}]$ for any $a\in A$, $\alpha\in\pi$, $m\in M$, $c\in C_\alpha$, and the left π-C-comodulelike structure is defined by $\{\rho_\beta^{G(M)_{\beta\alpha}}:C_{\beta\alpha}\otimes M\to C_\beta\otimes C_\alpha\otimes M\}$,

$$\rho_\beta^{G(M)_{\beta\alpha}}([c\otimes m])=\sum c_{1\beta}\otimes[c_{2\alpha}\otimes m],$$
for any $m\in M$, $\alpha,\beta\in\pi$, $c\in C_{\beta\alpha}$. (30)

Firstly we claim that the above action is well-defined. In fact, for any $m\in M$, $c\in C_\alpha$, $a,b\in A$,

$$(\varpi_\alpha^M(c\otimes m)-c\otimes m)\cdot a$$
$$=\sum\{(c\cdot 1_{A((1),\alpha)})\cdot a_{((1),\alpha)}\otimes(m\cdot 1_{A((0),A)})\cdot a_{((0),A)}$$
$$-c\cdot a_{((1),\alpha)}\otimes m\cdot a_{((0),A)}\}$$
$$=\sum\{c\cdot a_{((1),\alpha)}\otimes m\cdot a_{((0),A)}-c\cdot a_{((1),\alpha)}\otimes m\cdot a_{((0),A)}\}=0,$$
$$([c\otimes m]\cdot a)\cdot b$$
$$=\sum(c\cdot a_{((1),\alpha)})\cdot b_{((1),\alpha)}c\cdot a_{((1),\alpha)}\otimes(m\cdot a_{((0),A)})\cdot b_{((0),A)}$$
$$=\sum c\cdot(ab)_{((1),\alpha)}\otimes m\cdot(ab)_{((0),A)}$$
$$=[c\otimes m]\cdot ab.$$

So $G(M)$ is a right A-module.

Secondly we claim that the above comodulelike structure is well-defined. In fact, for any $c\in C_{\beta\alpha}$, $m\in M$,

$$
\begin{aligned}
&(id_{C_\beta}\otimes\varpi_\alpha^M)\rho_\beta^{G(M)_{\beta\alpha}}(\varpi_{\beta\alpha}^M(c\otimes m)-c\otimes m)\\
&=\sum\{c_{1\beta}\cdot 1_{A((1),\beta\alpha)1\beta}\otimes c_{2\alpha}\cdot 1_{A((1),\beta\alpha)2\alpha}1'_{A((1),\alpha)}\\
&\quad\otimes m\cdot 1_{A((0),A)}1'_{A((0),A)}-c_{1\beta}\otimes c_{2\alpha}\cdot 1_{A((1),\alpha)}\otimes m\cdot 1_{A((0),A)}\}\\
&=\sum\{c_{1\beta}\cdot 1_{A((1),\beta)}\otimes c_{2\alpha}\cdot 1_{A((0),A)((1),\alpha)}1'_{A((1),\alpha)}\\
&\quad\otimes m\cdot 1_{A((0),A)((0),A)}1'_{A((0),A)}-c_{1\beta}\otimes c_{2\alpha}\cdot 1_{A((1),\alpha)}\otimes m\cdot 1_{A((0),A)}\\
&=\sum\{c_{1\beta}\cdot 1_{A((1),\beta\alpha)1\beta}\otimes c_{2\alpha}\cdot 1_{A((1),\beta\alpha)2\alpha}\otimes m\cdot 1_{A((0),A)}\\
&\quad-c_{1\beta}\otimes c_{2\alpha}\cdot 1_{A((1),\alpha)}\otimes m\cdot 1_{A((0),A)}\}\\
&=\sum\{c_{1\beta}\otimes c_{2\alpha}\cdot 1_{A((1),\alpha)}\otimes m\cdot 1_{A((0),A)}\\
&\quad-c_{1\beta}\otimes c_{2\alpha}\cdot 1_{A((1),\alpha)}\otimes m\cdot 1_{A((0),A)}\}\\
&=0.
\end{aligned}
$$

For any $c\in C_{\beta\gamma\alpha}$, $m\in M$,

$$
\begin{aligned}
(id_{C_\beta}\otimes\rho_\gamma^{G(M)_{\gamma\alpha}})\rho_\beta^{G(M)_{\beta\gamma\alpha}}([c\otimes m])
&=\sum c_{1\beta}\otimes c_{2\gamma\alpha1\gamma}\otimes[c_{2\gamma\alpha2\alpha}\otimes m]\\
&=\sum c_{1\beta\gamma1\beta}\otimes c_{1\beta\gamma2\gamma}\otimes[c_{2\alpha}\otimes m]\\
&=(\Delta_{\beta,\alpha}\otimes id_{G(M)_\alpha})\rho_{\beta\gamma}^{G(M)_{\beta\gamma\alpha}}([c\otimes m]).
\end{aligned}
$$

So $G(M)$ is a left π-C-comodulelike object.

Thirdly we claim that the compatibility condition holds. For any $m\in M$, $c\in C_{\beta\alpha}$, $a\in A$,

$$
\begin{aligned}
&\sum[c\otimes m]_{((1),\beta)}\cdot a_{((1),\beta)}\otimes[c\otimes m]_{((0),G(M)_\alpha)}\cdot a_{((0),A)}\\
&=\sum c_{1\beta}\cdot a_{((1),\beta)}\otimes[c_{2\alpha}\otimes m]\cdot a_{((0),A)}\\
&=\sum c_{1\beta}\cdot a_{((1),\beta)}\otimes[c_{2\alpha}\cdot a_{((0),A)((1),\alpha)}\otimes m\cdot a_{((0),A)((0),A)}]\\
&=\sum(c\cdot a_{((1),\beta\alpha)})_{2\alpha}\otimes[(c\cdot a_{((1),\beta\alpha)})_{1\beta}\otimes m\cdot a_{((0),A)}]\\
&=\rho_\beta([c\otimes m]\cdot a).
\end{aligned}
$$

Therefore $G(M)\in{}^CM(\pi\text{-}H)_A$.

Finally for any $f: M\rightarrow N\in M_A$, $G(f)=\oplus_\alpha G(f)_\alpha$, where $G(f)_\alpha:G(M)_\alpha\rightarrow G(N)_\alpha$,

$G(f)_\alpha([c\otimes m])=[c\otimes f(m)]$, for any $m\in M$, $c\in C_\alpha$. (31)

We claim $G(f)_\alpha$ is well-defined. In fact, for any $m\in M$, $c\in C_\alpha$,

$$
\begin{aligned}
&\varpi_\alpha^N(G(f)_\alpha(\varpi_\alpha^M(c\otimes m)-c\otimes m))\\
&=\sum\{c\cdot 1_{A((1),\alpha)}1'_{A((1),\alpha)}\otimes f(m\cdot 1_{A((0),A)})\cdot 1'_{A((0),A)}\\
&\quad-c\cdot 1_{A((1),\alpha)}\otimes f(m\cdot 1_{A((0),A)})\}\\
&=\sum\{c\cdot 1_{A((1),\alpha)}\otimes f(m\cdot 1_{A((0),A)})\\
&\quad-c\cdot 1_{A((1),\alpha)}\otimes f(m\cdot 1_{A((0),A)})\}\\
&=0.
\end{aligned}
$$

And, we also claim that $G(f)_\alpha$ is a right A-module map and a left π-C-comodulelike map. In fact, for any $m\in M$, $c\in C_\alpha$, $a\in A$,

$$
\begin{aligned}
G(f)_\alpha([c\otimes m]\cdot a)&=\sum[c\cdot a_{((1),\alpha)}\otimes f(m\cdot a_{((0),A)})]\\
&=\sum[c\cdot a_{((1),\alpha)}\otimes f(m)\cdot a_{((0),A)}]\\
&=G(f)_\alpha([c\otimes m])\cdot a.
\end{aligned}
$$

For any $m\in M$, $c\in C_{\alpha\beta}$,

$$
\begin{aligned}
&\rho_\alpha^{G(N)_{\alpha\beta}}(G(f)_{\alpha\beta}([c\otimes m]))\\
&\quad=\sum c_{1\alpha}\otimes[c_{2\beta}\otimes f(m)]\\
&\quad=(id_{C_\alpha}\otimes G(f)_\beta)\rho_\alpha^{G(N)_{\alpha\beta}}([c\otimes m]).
\end{aligned}
$$

We still need to prove that F and G are adjoint functors. We define the unit natural Homomorphism $\vartheta: id_{{}^CM(\pi-H)_A}\rightarrow G\circ F$ and the counit natural Homomorphism $\tau: F\circ G\rightarrow id_{M_A}$ by the following formulas:

$$
\begin{aligned}
&\vartheta_M: M\rightarrow G(M),\\
&\vartheta_M(m)=\sum m_{((1),\alpha)}\otimes m_{((0),M)}. \qquad (32)\\
&\tau_N: G(N)\rightarrow N,\\
&\tau_N(\oplus[c_\alpha\otimes n_\alpha])=(\varepsilon\otimes id_M)\big|_{G(M)I}. \qquad (33)
\end{aligned}
$$

The existence of τ_N comes from the fact that $(H_I, m_I, 1_I, \Delta_I^H, \varepsilon^H)$ is a usual weak bialgebra and $(C_I, \Delta_{I,I}^C, \varepsilon^C)$ is a usual coalgebra. We still have to show for any $M\in M_A$, $N\in{}^CM(\pi\text{-}H)_A$, $G(\tau_M)\vartheta_{G(M)}=id_{G(M)}$ and $\tau_{F(N)}F(\vartheta_N)=id_{F(N)}$.

In fact, for any $\oplus_\alpha(C_\alpha\otimes M_\alpha)\in G(M)$,

$$\begin{aligned}&G(\tau_M)\vartheta_{G(M)}(\oplus_\alpha(c_\alpha\otimes m_\alpha))\\&=G(\tau_M)\sum_{\beta\gamma=\alpha}(c_{1\beta}\otimes\oplus_\gamma(c_{2\gamma}\otimes m_\alpha))\\&=\sum\varepsilon(c_{1I})(\oplus_\alpha(c_{2\alpha}\otimes m_\alpha))\\&=\oplus_\alpha(c_\alpha\otimes m_\alpha).\end{aligned}$$

For any $n\in N$,

$$\tau_{F(N)}F(\vartheta_N)(n)=\sum\varepsilon(n_{((1),I)})n_{((0),N)}=n.$$

Thus we complete the proof.

Theorem 4 Let (H,A,C) be a right weak Doi-Hopf π-datum. Then the forgetful functor $F:{}^C M(\pi\text{–}H)_A\to{}^{\pi\text{–}C}M$ has a left adjoint functor.

Proof Its proof is dual to Theorem 3, here we only give the construction of the right adjoint functor of F. Before defining the functor we first define a k-linear map for any $M\in{}^{\pi\text{–}C}M$, $\delta:M\otimes A\to M\otimes A$,

$$\delta(m\otimes a)=\sum\varepsilon(m_{((1),I)}\cdot a_{((1),I)})m_{((0),M)}\otimes a_{((0),A)}. \quad (34)$$

We claim $\delta^2=\delta$. In fact, for any $m\in M$, $a\in A$,

$$\begin{aligned}&\delta^2(m\otimes a)\\&=\sum\{\varepsilon(m_{((1),I)}\cdot a_{((1),I)})\varepsilon(m_{((0),M)((1),I)}\cdot a_{((0),A)((1),I)})\\&\qquad\times m_{((0),M)((0),M)}\otimes a_{((0),A)((0),A)}\}\\&=\sum\{\varepsilon(m_{((1),I)1I}\cdot a_{((1),I)1I})\varepsilon(m_{((1),I)2I}\cdot a_{((1),I)2I})\\&\qquad\times m_{((0),M)}\otimes a_{((0),A)}\}\\&=\sum\varepsilon(m_{((1),I)}\cdot a_{((1),I)})m_{((0),M)}\otimes a_{((0),A)}\\&=\delta(m\otimes a).\end{aligned}$$

So we can define the adjoint functor $G:{}^{\pi\text{–}C}M\to{}^C M(\pi\text{–}H)_A$ by $G(M)=(M\otimes A)/\ker\delta$, $G(f)=f\otimes id_A$ for an object M and a morphism f in ${}^{\pi\text{–}C}M$. The action of A on $G(M)$ is given by $[m\otimes a]\cdot b=[m\otimes ab]$. The left π-C-comodulelike structure is given by

$$\rho_\alpha([m\otimes a])=\sum m_{((1),\alpha)}\cdot a_{((1),\alpha)}\otimes[m_{((0),M)}\otimes a_{((0),A)}]. \quad (35)$$

We claim that the above action is well-defined. In fact, for any $m\in M$, a, $b\in A$,

$$\begin{aligned}&\delta((m\otimes a)\cdot b-\delta(m\otimes a)\cdot b)\\&=\sum\varepsilon(m_{((1),I)}\cdot(ab)_{((1),I)})m_{((0),M)}\otimes(ab)_{((0),A)}\\&\quad-\sum\{\varepsilon(m_{((1),I)}\cdot a_{((1),I)})\varepsilon(m_{((0),M)((1),I)}\\&\quad\cdot(a_{((0),A)}b)_{((1),I)})m_{((0),M)((0),M)}\otimes(a_{((0),A)}b)_{((0),A)}\}\\&=\sum\varepsilon(m_{((1),I)}\cdot a_{((1),I)}b_{((1),I)})m_{((0),M)}\otimes a_{((0),A)}b_{((0),A)}\\&\quad-\sum\{\varepsilon(m_{((1),I)}\cdot a_{((1),I)})\varepsilon(m_{((0),M)((1),I)}\\&\quad\cdot a_{((0),A)((1),I)}b_{((1),I)})m_{((0),M)((0),M)}\otimes a_{((0),A)((0),A)}b_{((0),A)}\}\\&=\sum\varepsilon(m_{((1),I)}\cdot a_{((1),I)}b_{((1),I)})m_{((0),M)}\otimes a_{((0),A)}b_{((0),A)}\\&\quad-\sum\{\varepsilon(m_{((1),I)2I}\cdot a_{((1),I)2I})\varepsilon(m_{((1),I)1I}\\&\quad\cdot a_{((1),I)1I}b_{((1),I)})m_{((0),M)}\otimes a_{((0),A)}b_{((0),A)}\}\\&=0.\end{aligned}$$

We claim that the above comodulelike structure is well-defined. In fact, for any $a\in A$, $m\in M$, $\alpha\in\pi$,

$$\begin{aligned}&\rho_\alpha(m\otimes a-\delta(m\otimes a))\\&=\sum m_{((1),\alpha)}\cdot a_{((1),\alpha)}\otimes m_{((0),M)}\otimes a_{((0),A)}\\&\quad-\sum\{\varepsilon(m_{((1),I)}\cdot a_{((1),I)})m_{((0),M)((1),\alpha)}\\&\quad\cdot a_{((0),A)((1),\alpha)}\otimes m_{((0),M)((0),M)}\otimes a_{((0),A)((0),A)}\}\\&=\sum m_{((1),\alpha)}\cdot a_{((1),\alpha)}\otimes m_{((0),M)}\otimes a_{((0),A)}\\&\quad-\sum\{\varepsilon(m_{((1),\alpha)1I}\cdot a_{((1),\alpha)1I})m_{((1),\alpha)2\alpha}\\&\quad\cdot a_{((1),\alpha)2\alpha}\otimes m_{((0),M)}\otimes a_{((0),A)}\}\\&=0,\end{aligned}$$

$$\begin{aligned}&(\Delta_{\beta,\alpha}\otimes id_M)\rho^M_{\beta\alpha}([m\otimes a])\\&=\sum\{(m_{((1),\beta\alpha)}\cdot a_{((1),\beta\alpha)})_{1\beta}\otimes(m_{((1),\beta\alpha)}\\&\quad\cdot a_{((1),\beta\alpha)})_{2\alpha}\otimes m_{((0),M)}\otimes a_{((0),A)}\}\\&=\sum\{m_{((1),\beta\alpha)1\beta}\cdot a_{((1),\beta\alpha)1\beta}\otimes m_{((1),\beta\alpha)2\alpha}\\&\quad\cdot a_{((1),\beta\alpha)2\alpha}\otimes m_{((0),M)}\otimes a_{((0),A)}\}\\&=\sum\{m_{((1),\beta)}\cdot a_{((1),\beta)}\otimes m_{((0),M)((1),\alpha)}\\&\quad\cdot a_{((0),A)((1),\alpha)}\otimes m_{((0),M)((0),M)}\otimes a_{((0),A)((0),A)}\}\\&=(id_{C_\beta}\otimes\rho^M_\alpha)\rho^M_\beta([m\otimes a]).\end{aligned}$$

We still claim that the compatibility condition holds. In fact, for any $\alpha\in\pi$, $m\in M$, a, $b\in A$,

$$\begin{aligned}&\sum[m\otimes a]_{((1),\alpha)}\cdot b_{((1),\alpha)}\otimes[m\otimes a]_{((0),\overline{M\otimes A})}\cdot b_{((0),A)}\\&=\sum m_{((1),\alpha)}\cdot a_{((1),\alpha)}b_{((1),\alpha)}\otimes[m_{((0),M)}\otimes a_{((0),A)}]\cdot b_{((0),A)}\\&=\sum\{\varepsilon(m_{((0),M)((1),I)}\cdot a_{((0),A)((1),I)}b_{((0),A)((1),I)})m_{((1),\alpha)}\end{aligned}$$

$$\cdot a_{((1),\alpha)}b_{((1),\alpha)}\otimes[m_{((0),M)((0),M)}\otimes a_{((0),A)((0),A)}b_{((0),A)((0),A)}]\}$$
$$=\sum m_{((1),\alpha)}\cdot a_{((1),\alpha)}b_{((1),\alpha)}\otimes[m_{((0),M)}\otimes a_{((0),A)}b_{((0),A)}]$$
$$=\rho([m\otimes a]\cdot b).$$

The unit and counit natural Homomorphisms $\vartheta_M: id_{M^{\pi-C}}\to F\circ G$ and $\tau_N: G\circ F\to id_{^CM(\pi-H)_A}$ are given by

$$\vartheta_M(m)=\sum\varepsilon(m_{((1),I)}\cdot 1_{((1),I)})m_{((0),M)}\otimes 1_{((0),A)}, \quad (36)$$
$$\tau_N([n\otimes a])=n\cdot a. \quad (37)$$

We have to show for any $N\in{}^CM(\pi-H)_A$, $M\in M^{\pi-C}$, $F(\tau_N)\vartheta_{F(N)}=id_{F(N)}$ and $\tau_{G(M)}G(\vartheta_M)=id_{G(M)}$.

In fact, for any $[m\otimes a]\in G(M)$,

$$\tau_{G(M)}G(\vartheta_M)([m\otimes a])$$
$$=\sum\{\varepsilon(m_{((1),I)}\cdot a_{((1),I)}1_{((1),I)})\varepsilon(m_{((0),M)((1),I)}\cdot a_{((0),A)((1),I)})$$
$$\otimes[m_{((0),M)((0),M)}\otimes a_{((0),A)((0),A)}1_{((0),A)}]\}$$
$$=\sum\{\varepsilon(m_{((1),I)2}\cdot a_{((1),I)2})\varepsilon(m_{((1),I)1}\cdot a_{((1),I)1})$$
$$\otimes[m_{((0),M)}\otimes a_{((0),A)}]\}$$
$$=[m\otimes a].$$

For any $n\in N$,

$$F(\tau_N)\vartheta_{F(N)}(n)=\sum\varepsilon(n_{((1),I)}\cdot 1_{((1),I)})n_{((0),N)}\cdot 1_{((0),A)}=n.$$

Thus we complete the proof.

THE WEAK SMASH PRODUCT

Lemma 2 Let (H,A,C) be a right weak Doi-Hopf π-datum, then $\chi_\alpha: A\otimes C_\alpha^*\to A\otimes C_\alpha^*$ given by

$$\chi_\alpha(a\otimes f)=\sum 1_{A((0),A)}a\otimes 1_{A((1),\alpha)}\triangleright f, \quad (38)$$

for any $a\in A$, $f\in C_\alpha^*$ is a projection, i.e., $\chi_\alpha^2=\chi_\alpha$.

Lemma 3 Let (H,A,C) be a right weak Doi-Hopf π-datum, then

$$\sum 1_{A((0),A)}a\otimes 1^l_{A((1),I)}=\sum a_{((0),A)}\otimes a^l_{((1),I)},$$
for any $a\in A$. (39)

$$h\triangleright fg=\sum(h_{1\alpha}\triangleright f)(h_{2\beta}\triangleright g),$$
for any $h\in H_{\alpha\beta}$, $f\in C_\alpha^*$, $g\in C_\beta^*$. (40)

Proof Eq.(40) is obvious, we only prove Eq.(39).

$$\sum a_{((0),A)}\otimes a^l_{((1),I)}$$
$$=\sum 1_{A((0),A)}a_{((0),A)}\varepsilon(1_{I1}1_{A((1),I)}a_{((1),I)})1_{I2}$$
$$=\sum 1_{A((0),A)}a_{((0),A)}\otimes\varepsilon(1_{I1}1_{A((1),I)1})\varepsilon(1_{A((1),I)2}a_{((1),I)})1_{I2}$$
$$=\sum\{1_{A((0),A)((0),A)}a_{((0),A)}\otimes\varepsilon(1_{I1}1_{A((1),I)1})$$
$$\times\varepsilon(1_{A((0),A)((1),I)}a_{((1),I)})1_{I2}\}$$
$$=\sum 1_{A((0),A)((0),A)}a\otimes\varepsilon(1_{I1}1_{A((1),I)})1_{I2}$$
$$=\sum 1_{A((0),A)}\otimes 1^l_{A((1),I)}.$$

Given a right weak Doi-Hopf π-datum (H,A,C), we define $\overline{A\# C^*}=\oplus((A\otimes C_\alpha^*)/\ker\chi_\alpha)$ as a k-space and its multiplication by

$$[a\# f][b\# g]=\sum[a_{((0),A)}b\# f(a_{((1),\beta)}\triangleright g)],$$
for any a, $b\in A$, $f\in C_\alpha^*$, $g\in C_\beta^*$, $\alpha,\beta\in\pi$. (41)

Theorem 5 Let (H,A,C) be a right weak Doi-Hopf π-datum, then $\overline{A\# C^*}$ is an associative algebra with the unit $[1_A\#\varepsilon]$.

Proof First we claim that the above multiplication is well-defined. In fact, for any a, $b\in A$, $f\in C_\alpha^*$, $g\in C_\beta^*$, $\alpha,\beta\in\pi$

$$\chi_{\alpha\beta}((a\# f-\chi_\alpha(a\# f))(b\# g))$$
$$=\sum\{(a_{((0),A)}b\# f(a_{((1),\beta)}\triangleright g))((1_{A((0),A)}a)_{((0),A)}$$
$$\times b\#(1_{A((1),\alpha)}\triangleright f)((1_{A((0),A)}a)_{((1),\beta)}\triangleright g))\}$$
$$=\sum\{(a_{((0),A)}b\# f(a_{((1),\beta)}\triangleright g))-(1_{A((0),A)((0),A)}a_{((0),A)}$$
$$\times b\#(1_{A((1),\alpha)}\triangleright f)(1_{A((0),A)((1),\beta)}a_{((1),\beta)}\triangleright g))\}$$
$$=\sum\{(a_{((0),A)}b\# f(a_{((1),\beta)}\triangleright g))-(1_{A((0),A)}a_{((0),A)}$$
$$\times b\#(1_{A((1),\alpha\beta)1\alpha}\triangleright f)(1_{A((1),\alpha\beta)2\beta}a_{((1),\beta)}\triangleright g))\}$$
$$=\sum\{(a_{((0),A)}b\# f(a_{((1),\beta)}\triangleright g))$$
$$-(1_{A((0),A)}a_{((0),A)}b\# 1_{A((1),\alpha\beta)}\triangleright f((a_{((1),\beta)}\triangleright g)))\}$$
$$=0,$$

$$(a\# f)(b\# g-\chi_\beta(b\# g))$$
$$=\sum\{(a_{((0),A)}b\# f(a_{((1),\beta)}\triangleright g))$$

$$-(a_{((0),A)}1_{A((0),A)}b\#f(a_{((1),\beta)}1_{A((1),\beta)}\triangleright g))\}$$
$$=\sum\{(a_{((0),A)}b\#f(a_{((1),\beta)}\triangleright g))$$
$$-(a_{((0),A)}b\#f(a_{((1),\beta)}\triangleright g))\}$$
$$=0.$$

Therefore it is well-defined.

Next we claim that it is associative. In fact, for any $a,\ b,\ c\in A,\ f\in C_\alpha^*,\ g\in C_\beta^*,\ u\in C_\gamma^*$,

$$([a\#f][b\#g])[c\#u]$$
$$=\sum\{(a_{((0),A)}b)_{((0),A)}c\#f(a_{((1),\beta)}\triangleright g)\cdot((a_{((0),A)}b)_{((1),\gamma)}\triangleright u)\}$$
$$=\sum\{a_{((0),A)((0),A)}b_{((0),A)}c\#f(a_{((1),\beta)}\triangleright g)\cdot(a_{((0),A)((1),\gamma)}b_{((1),\gamma)}\triangleright u)\}$$
$$=\sum\{a_{((0),A)}b_{((0),A)}c\#f(a_{((1),\beta\gamma)1\beta}\triangleright g)\cdot(a_{((1),\beta\gamma)2\gamma}b_{((1),\gamma)}\triangleright u)\}$$
$$=\sum(a_{((0),A)}b_{((0),A)}c\#f(a_{((1),\beta\gamma)}\triangleright g)(b_{((1),\gamma)}\triangleright u))$$
$$=\sum[a\#f](b_{((0),A)}c\#g(b_{((1),\gamma)}\triangleright u))$$
$$=[a\#f]([b\#g][c\#u]).$$

Obviously

$$[a\#f][1_A\#\varepsilon]=[1_A\#\varepsilon][a\#f]=[a\#f].$$

Lemma 4 Let (H,A,C) be the same as mentioned above, then we have:

$$[a\#\varepsilon][b\#\varepsilon]=[ab\#\varepsilon],\ \text{for any } a,\ b\in A. \tag{42}$$
$$[1_A\#f][1_A\#g]=[1_A\#fg],\ \text{for any } f\in C_\alpha^*, g\in C_\beta^*. \tag{43}$$

Proof

$$[a\#\varepsilon][b\#\varepsilon]=\sum[a_{((0),A)}b\#a_{((1),\alpha)}\triangleright\varepsilon]$$
$$=\sum[1_{A((0),A)}ab\#1^l_{A((1),\alpha)}\triangleright\varepsilon]=[ab\#\varepsilon],$$
$$[1_A\#f][1_A\#g]=\sum[1_{A((0),A)}\#f(1_{A((1),\beta)}\triangleright g)]$$
$$=\sum[1_{A((0),A)((0),A)}\#f(1_{A((1),\beta)}\triangleright g)(1_{A((0),A)((1),I)}\triangleright\varepsilon)]$$
$$=\sum[1_{A((0),A)}\#f(1_{A((1),\beta)1\beta}\triangleright g)(1_{A((1),\beta)2I}\triangleright\varepsilon)]$$
$$=\sum[1_{A((0),A)}\#f(1_{A((1),\beta)}\triangleright g)]=[1_A\#fg].$$

Theorem 6 Let (H,A,C) be a right weak Doi-Hopf π-datum such that C is finite dimensional. Then the category of ${}^C M(\pi-H)_A$ is isomorphic to the category of $M_{\overline{A\#C^*}}$.

Proof We define the functor F: ${}^C M(\pi-H)_A\rightarrow M_{\overline{A\#C^*}}$ by $F(M)=M$, $F(f)=f$ for an object M and a morphism f in ${}^C M(\pi-H)_A$.

We define $m\cdot[a\#c_\alpha^*]=\sum c_\alpha^*(m_{((1),\alpha)})m_{((0),M)}\cdot a$ for any $m\in M$, $a\in A$, $c_\alpha^*\in C_\alpha^*$.

We claim that the above map is well-defined and define an action of $\overline{A\#C^*}$ on M. In fact, for any $a\in A$, $c_\alpha^*\in C_\alpha^*$,

$$m\cdot(a\#c_\alpha^*-\chi_\alpha(a\#c_\alpha^*))$$
$$=\sum\{c_\alpha^*(m_{((1),\alpha)})m_{((0),M)}\cdot a-c_\alpha^*(m_{((1),\alpha)}\cdot 1_{A((1),\alpha)})m_{((0),M)}\cdot 1_{A((0),A)}a\}$$
$$=\sum(c_\alpha^*(m_{((1),\alpha)})m_{((0),M)}\cdot a-c_\alpha^*(m_{((1),\alpha)})m_{((0),M)}\cdot a)$$
$$=0,$$

For any $m\in M,\ a\in A,\ c_\alpha^*\in C_\alpha^*,\ c_\beta^*\in C_\beta^*$,

$$(m\cdot(a\#c_\alpha^*))\cdot(b\#c_\beta^*)$$
$$=\sum c_\alpha^*(m_{((1),\alpha)})c_\beta^*((m_{((0),M)}\cdot a)_{((1),\beta)})(m_{((0),M)}\cdot a)_{((0),M)}\cdot b$$
$$=\sum\{c_\alpha^*(m_{((1),\alpha)})c_\beta^*(m_{((0),M)((1),\beta)}\cdot a_{((1),\beta)})m_{((0),M)((0),M)}\cdot a_{((0),A)}\cdot b\}$$
$$=\sum c_\alpha^*(m_{((1),\alpha\beta)1\alpha})c_\beta^*(m_{((1),\alpha\beta)2\beta}\cdot a_{((1),\beta)})m_{((0),M)}\cdot a_{((0),A)}b$$
$$=\sum m\cdot[a_{((0),A)}b\#c_\alpha^*(a_{((1),\beta)}\triangleright c_\beta^*)]$$
$$=\sum m\cdot[(a\#c_\alpha^*)(b\#c_\beta^*)],$$
$$m\cdot(1_A\#\varepsilon)=\sum\varepsilon(m_{((1),I)})m_{((0),M)}\cdot 1_A=m.$$

So M is a right $\overline{A\#C^*}$-module.

We also claim f is a right $\overline{A\#C^*}$-module map. In fact, for any $a\in A,\ c_\alpha^*\in C_\alpha^*,\ m\in M$,

$$f(m\cdot[a\#c_\alpha^*])=\sum f(m_{((0),M)}\cdot a)c_\alpha^*(m_{((1),\alpha)})$$
$$=\sum f(m_{((0),M)})\cdot ac_\alpha^*(m_{((1),\alpha)})$$
$$=\sum f(m)_{((0),M)}\cdot ac_\alpha^*(f(m)_{((1),\alpha)})$$
$$=f(m)\cdot[a\#c_\alpha^*].$$

Because C is finite dimensional, there exists a dual basis $\{x_{\alpha K}, X_{\alpha K}\}$ in C_α and C_α^*. It is easy to verify

$$\sum \Delta(x_{\alpha\beta K})\otimes X_{\alpha\beta K} = \sum x_{\alpha K}\otimes x_{\beta l}\otimes X_{\alpha K} * X_{\beta l}. \quad (44)$$

We define the functor $G: M_{\overline{A\# C^*}} \to {}^{C}M(\pi\text{–}H)_A$ by $G(M)=M$, $G(f)=f$ for M an object and f a morphism in $M_{\overline{A\# C^*}}$.

The right action of A on $G(M)$ is given by $m\cdot a=m\cdot[a\#\varepsilon]$ for any $a\in A$, $m\in M$.

The left comodulelike structure is given by

$$\rho_\alpha^M(m)=\sum x_{\alpha K}\otimes m\cdot[1_{A((0),A)}\#1_{A((1),\alpha)}\triangleright X_{\alpha K}],$$
$$\text{for any } m\in M. \quad (45)$$

Now we claim that it exactly defines a comodulelike structure. For any $\alpha, \beta\in\pi$ and $m\in M$,

$$\begin{aligned}
&(id_{C^\alpha}\otimes\rho_\alpha^M)\rho_\beta^M\\
&=\sum\{x_{\beta K}\otimes x_{\alpha l}\otimes m\cdot((1_{A((0),A)}\#1_{A((1),\beta)}\triangleright X_{\beta K})\\
&\qquad\cdot(1_{A((0),A)}\#1_{A((1),\alpha)}\triangleright X_{\alpha l}))\}\\
&=\sum\{x_{\beta K}\otimes x_{\alpha l}\otimes m\cdot(1_{A((0),A)((0),A)}1_{A((0),A)}\#(1_{A((1),\beta)}\triangleright X_{\beta K})\\
&\qquad\cdot(1_{A((0),A)((1),\alpha)}1_{A((1),\alpha)}\triangleright X_{\alpha l}))\}\\
&=\sum\{x_{\beta K}\otimes x_{\alpha l}\otimes m\cdot(1_{A((0),A)((0),A)}\#(1_{A((1),\beta)}\triangleright X_{\beta K})\\
&\qquad\cdot(1_{A((0),A)((1),\alpha)}\triangleright X_{\alpha l}))\}\\
&=\sum\{x_{\beta K}\otimes x_{\alpha l}\otimes m\cdot(1_{A((0),A)}\#(1_{A((1),\beta\alpha)1\beta}\triangleright X_{\beta K})\\
&\qquad\cdot(1_{A((1),\beta\alpha)2\alpha}\triangleright X_{\alpha l}))\}\\
&=\sum x_{\beta K}\otimes x_{\alpha l}\otimes m\cdot(1_{A((0),A)}\#1_{A((1),\beta\alpha)}\triangleright(X_{\beta K}*X_{\alpha l}))\\
&=\sum x_{\beta\alpha K1\beta}\otimes x_{\beta\alpha K2\alpha}\otimes m\cdot(1_{A((0),A)}\#1_{A((1),\beta\alpha)}\triangleright X_{\beta\alpha K})\\
&=(\Delta_{\beta,\alpha}\otimes id_M)\rho_{\beta\alpha}^M(m),
\end{aligned}$$

$$\begin{aligned}
&(\varepsilon\otimes id_M)\rho_I^M(m)\\
&\quad=\sum\varepsilon(x_{IK})m\cdot[1_{A((0),A)}\#1_{A((1),I)}\triangleright X_{IK}]\\
&\quad=\sum m\cdot[1_{A((0),A)}\#1_{A((1),I)}\triangleright\varepsilon]=m.
\end{aligned}$$

Obviously f is a right A-linear, and we only need to show that f is a right π-C-comodulelike map. In fact, for any $m\in M$, $\alpha\in\pi$,

$$\begin{aligned}
\rho_\alpha^N(f(m))&=\sum x_{\alpha K}\otimes f(m)\cdot[1_{A((0),A)}\#1_{A((1),\alpha)}\triangleright X_{\alpha K}]\\
&=\sum x_{\alpha K}\otimes f(m\cdot[1_{A((0),A)}\#1_{A((1),\alpha)}\triangleright X_{\alpha K}])\\
&=(id_{C^\alpha}\otimes f)\rho_\alpha^M(m).
\end{aligned}$$

ACKNOWLEDGEMENT

We are very grateful to the referees for their valuable suggestions on deeper research into weak Doi-Hopf group modules. They pointed out that we can consider the new prospective of a symmetric monoidal category which we call the Turaev's category. We are doing this work and have achieved some aims.

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