Journal of Zhejiang University SCIENCE A
ISSN 1009-3095
http://www.zju.edu.cn/jzus
E-mail: jzus@zju.edu.cn

On closed weak supplemented modules*

ZENG Qing-yi (曾庆怡)[†1,2], SHI Mei-hua (史美华)[3]

([1]*Department of Mathematics, Zhejiang University, Hangzhou 310027, China*)
([2]*Department of Mathematic, Shaoguan University, Shaoguan 512005, China*)
([3]*Department of Mathematics, Zhejiang Education Institute, Hangzhou 310012, China*)
[†]E-mail: zqy67@163.com

Received Nov. 22, 2004; revision accepted Jan. 12, 2005

Abstract: A module M is called closed weak supplemented if for any closed submodule N of M, there is a submodule K of M such that $M=K+N$ and $K\cap N<<M$. Any direct summand of closed weak supplemented module is also closed weak supplemented. Any nonsingular image of closed weak supplemented module is closed weak supplemented. Nonsingular V-rings in which all nonsingular modules are closed weak supplemented are characterized in Section 4.

Key words: Closed submodules, Closed weak supplemented, Small submodule
doi:10.1631/jzus.2006.A0210 **Document code:** A **CLC number:** O153.3

INTRODUCTION

In recent years theory of extending modules and rings has come to play an important role in the theory of rings and modules. A module is called an extending module (or CS-module) if every submodule is essential in a direct summand, or equivalently, every closed submodule is a direct summand. Although this generalization of injectivity is extremely useful, it does not satisfy some important properties. For example, direct sums of extending modules need not be extending; also, the image of extending module need not be extending; also, the submodule of extending module need not be extending. Much work has been done on finding necessary and sufficient conditions to ensure that the extending property is preserved under various extensions.

A submodule N of M is small in M, denoted by $N<<M$, if $N+K=M$ implies $K=M$. Let N and K be submodules of M. In (Wisbauer, 1991), N is called a supplement of K in M if it is minimal with respect to $M=N+K$, or equivalently, $M=N+K$ and $N\cap K<<N$. A module M is called supplemented if for any sub-module N of M there is submodule K of M such that $M=K+N$ and $N\cap K<<K$. A module M is called weak supplemented if for any submodule N of M there is submodule K of M such that $M=K+N$ and $N\cap K<<M$. Any supplemented module is a weak supplemented module. A module M is called $\oplus$-supplemented if for every submodule N there is a direct summand K of M which is a supplement of N in M (Harmanci *et al*., 1999).

In this paper, we replace the condition of extending modules which closed submodule is a direct summand by the condition that the closed submodule has a weak supplement. Thus we generalize both extending modules and weak supplemented modules to closed weak supplemented modules.

In Section 2, we give the definition of closed weak supplemented module and show that any direct summand of closed weak supplemented module is closed weak supplemented. Let $M=M_1\oplus M_2$ be a distributive module. Then M is closed weak supplemented if and only if each M_i is closed weak supplemented.

In Section 3, we show that any nonsingular homomorphic image of a closed weak supplemented

*Project (No. 102028) supported by the Natural Science Foundation of Zhejiang Province, China

module is a closed weak supplemented module. For a right nonsingular ring R, every projective right R-module is closed weak supplemented if and only if every nonsingular right R-module is closed weak supplemented.

In Section 4, we characterize rings in which all nonsingular modules are closed weak supplemented, see Theorem 3 and Corollary 6. The relations between closed weak supplemented modules and other kinds of (weak) supplemented modules are given in this section (Theorem 4).

Throughout this paper, unless otherwise stated, all rings are associative rings with identity and all modules are unitary right R-modules.

A submodule N of M is called an essential submodule, denoted by $N\leq_e M$, if for any nonzero submodule L of M, $L\cap N\neq 0$. A closed submodule N of M, denoted by $N\leq_c M$, is a submodule which has no proper essentail extension in M. If $L\leq_c N$ and $N\leq_c M$, then $L\leq_c M$ (Goodearl, 1976).

Let M be an R-module, we use $Rad(M)$ to denote the Jacboson radical of M and $r(m)=\{r\in R|mr=0\}$ the right annihilator of $m\in M$. First we collect some well known facts.

Lemma 1 Let M be a module and N a submodule of M. Then $N\leq_e M$ if and only if for any nonzero $m\in M$, there is $r\in R$, such that $0\neq mr\in N$.

Lemma 2 Let M be a module and let $K\leq L$ and L_i $(1\leq i\leq n)$ be submodules of M, for some positive integer n. Then the following hold:

(1) $L<<M$ if and only if $K<<M$ and $L/K<<M/K$;

(2) $L_1+L_2+\ldots+L_n<<M$ if and only if $L_i<<M$ $(1\leq i\leq n)$;

(3) If M' is a module and f:$M\rightarrow M'$ is a homomorphism, then $f(L)<<M'$, where $L<<M$;

(4) If L is a direct summand of M, then $K<<L$ if and only if $K<<M$;

(5) $K_1\oplus K_1<<L_1\oplus L_1$ if and only if $K_i<<L_i$ $(i=1,2)$.

CLOSED WEAK SUPPLEMENTED MODULES

A module M is called weak supplemented if for every submodule N of M there is a submodule K such that $M=K+N$ and $N\cap K<<M$. In (Alizade and Büyükasik, 2003), co-finitely weak supplemented modules have been studied. A module M is called co-finitely weak supplemented if every submodule N of M with M/N finitely generated has a weak supplement in M. Now we can replace the condition of extending modules that closed submodule is a direct summand by the condition which closed submodule has a weak supplement and give the definition of closed weak supplemented module as follows:

Definition 1 A module M is called closed weak supplemented if for any closed submodule N of M, there is a submodule K of M such that $M=K+N$ and $K\cap N<<M$.

Clearly, any weak supplemented module is closed weak supplemented and any extending module is closed weak supplemented. Since local modules (i.e., the sum of all proper submodules is also a proper submodule) are hollow (i.e. every proper submodule is small) and hollow modules are supplemented, hence closed weak supplemented. So we have the following implications:

Hollow$\Rightarrow$ Supplemented $\Rightarrow$ Weak supplemented $\Rightarrow$Closed weak supplemented.

But closed weak supplemented need not be weak supplemented. See:

Example 1 Let Z be the ring of all integers. Since Z is extending, it is closed weak supplemented. But Z is not weak supplemented, since, for any $n\geq 2$, nZ has no weak supplement in Z.

Any direct summand of an extending module is extending. For closed weak supplemented modules, we have:

Proposition 1 Let M be a closed weak supplemented module. Then any direct summand of M is closed weak supplemented.

Proof Let N be any direct summand of M and L any closed submodule of N. Since N is closed in M, we have that L is closed in M. Then there is a submodule K of M such that $M=K+L$ and $K\cap L<<M$. Thus $N=(N\cap K)+L$. Since N is a direct summand of M, then $N\cap K\cap L=K\cap L<<N$ by Lemma 2(4). Thus N is closed weak supplemented.

For any ring R, any finite sum of weak supplemented modules is again weak supplemented (Wisbauer, 1991). For closed weak supplemented modules, the sum of closed weak supplemented modules need not to be closed weak supplemented, even for direct sum of closed weak supplemented modules.

Example 2 Let $R=Z[x]$, where Z is the ring of all integers. Set $M=R\oplus R$, then by (Chatters and Khuri, 1980), M is not extending. As $Rad(M)=0$, we see that M is not closed weak supplemented.

Lemma 3 Let N and L be submodules of M such that $N+L$ has a weak supplement H in M and $N\cap(H+L)$ has a weak supplement G in N. Then $H+G$ is a weak supplement of L in M.

Proof See (Alizade and Büyükasik, 2003).

Proposition 2 Let $M=M_1\oplus M_2$ with each M_i (i=1,2) closed weak supplemented. Suppose that $M_i\cap(M_j+L)\leq_c M_i$ and $M_j\cap(L+K)\leq_c M_j$, where K is a weak supplement of $M_i\cap(M_j+L)$ in M_i, $i\neq j$, for any closed submodule L of M. Then M is closed weak supplemented.

Proof Let $L\leq_c M$, then $M=M_1+(M_2+L)$ has a weak supplement 0 in M. Since $M_1\cap(M_2+L)\leq_c M_1$ and M_1 is closed weak supplemented, then there is a submodule K of M_1 such that $M_1=K+M_1\cap(M_2+L)$ and $K\cap(M_1\cap(M_2+L))=K\cap(M_2+L)<<M_1$. By lemma above, K is a weak supplement of M_2+L in M, i.e., $M=K+(M_2+L)$. Since $M_2\cap(K+L)\leq_c M_1$ and M_1 is closed weak supplemented, then $M_2\cap(K+L)$ has a weak supplement J in M_2. Again by lemma above, $K+J$ is a weak supplement of L in M. Hence M is closed weak supplemented.

Proposition 3 Let $M=M_1+M_2$ with M_1 closed weak supplemented and M_2 any R-module. Suppose that for any closed submodule N of M, $N\cap M_1\leq_c M_1$. Then M is closed weak supplemented if and only if every closed submodule N of M with M_2 not contained in N has a weak supplement.

Proof ($\Rightarrow$) It is obviously.

($\Leftarrow$) Let $N\leq M$ such that $M_2\subseteq N$. Then $M=M_1+M_2=M_1+N$ and M_1+N has weak supplement 0. Since $N\cap M_1$ is closed in M_1 and M_1 is closed weak supplemented, $N\cap M_1$ has a weak supplement H in M_1. By Lemma 3, H is a weak supplement of N in M.

Let M be a nonsingular module and $N\leq_c M$, then $N\cap L\leq_c L$ for any submodule L of M. In fact, since M/N is nonsingular, so is $(L+N)/N\cong L/(L\cap N)$.

Corollary 1 Let $M=M_1+M_2$ be a nonsingular module with M_1 closed weak supplemented and M_2 any R-module. Then M is closed weak supplemented if and only if every closed submodule N of M with M_2 not contained in N has a weak supplement.

Proposition 4 Let $M=M_1\oplus M_1$ be a distributive module. Then M is closed weak supplemented if and only if each M_i is closed weak supplemented.

Proof Let L be any closed submodule of M. Then for each i, $L\cap M_i$ is closed in M_i. In fact, suppose that $L\cap M\leq_e K\leq M_1$. Since $M_2\cap L\leq_e M_2\cap L$ and M is distributive, we have that $L=(M_1\cap L)\oplus(M_2\cap L)\leq_c K\oplus(M_2\cap L)$. Hence $L=(M_1\cap L)\oplus(M_2\cap L)=K\oplus(M_2\cap L)$, because L is closed in M. So $K=L\cap M_1$ and $L\cap M_1$ is closed in M_1.

Therefore, there is a submodule K_i of M_i such that $M_i=K_i+L\cap M_i$ and $(L\cap M_i)\cap K_i=L\cap K_i<<M_i$, i=1,2. Hence $M=M_1\oplus M_2=K_1\oplus K_2+((L\cap M_1)\oplus(L\cap M_2))=K_1\oplus K_2+L$, and $L\cap(K_1\oplus K_2)=(L\cap K_1)\oplus(L\cap K_2)<<(M_1\oplus M_2)=M$. Thus M is closed weak supplemented.

The converse is Proposition 1.

Proposition 5 Let R be any ring and M a module. If any nonzero proper closed submodule of M is maximal in M, then M is extending and hence closed weak supplemented.

Proof Suppose that N is a proper closed submodule and maximal in M. Then N is not essential in M. So, there is $m\in M\backslash N$ such that for all $0\neq r\in R$ with $mr\neq 0$, then $mr\notin N$. Let $K=mR+N$, then K is a submodule of M containing N as a proper submodule. Thus $K=M$ and, since $mR\cap N=0$, we have $M=mR\oplus N$.

A module M is called a small cover of a module N if there is a small epimorphism f:$M\rightarrow N$, i.e., $Kerf<<M$. Let f:$M\rightarrow N$ be a small epimorphism, then a submodule L of M is a weak supplement in M if and only if $f(L)$ is a weak supplement in N (Lemma 2.8 of (Alizade and Büyükasik, 2003)).

Proposition 6 Let f:$M\rightarrow N$ be a small epimorphism and N a closed weak supplemented module. Suppose that every nonzero closed submodule L of M contains $Kerf$. Then M is closed weak supplemented.

Proof Let f:$M\rightarrow N$ be a small epimorphism and N a closed weak supplemented module. Let $0\neq L\leq_c M$ and suppose that $f(L)\leq_e K\leq N$. Then $L=L+Kerf=f^{-1}f(L)\leq_e f^{-1}(K)$. Hence $L=f^{-1}(K)$, and $f(L)=K$ is a closed submodule of N. Since N is closed weak supplemented, $f(L)$ has a weak supplement in N. By Lemma 2.8 of (Alizade and Büyükasik, 2003), L has a weak supplement in M, i.e., M is closed weak supplemented.

Proposition 7 Let M be a closed weak supplemented module such that $M/Rad(M)$ is semi-simple. Then $M=M_1\oplus M_2$, where M_1 is semi-simple and M_2 is a module with $Rad(M_2)\leq_e M_2$.

Proof Let M_1 be a closed submodule of M such that

$M_1 \cap Rad(M)=0$. Since M is closed weak supplemented, there is a submodule M_2 of M such that $M=M_1+M_2$ and $M_1\cap M_2<<M$. Hence $M_1\cap M_2$ is a submodule of both M_1 and $Rad(M)$. So $M_1\cap M_2=0$. Thus $M=M_1\oplus M_2$ and $Rad(M)=Rad(M_2)\leq_e M_2$. By hypothesis, $M_1\cong M/M_2$ is semi-simple.

THE HOMOMORPHIC IMAGES

In this section, we will consider the conditions for which the homomorphic images of closed weak supplemented modules are also closed weak supplemented modules. Any image of weak supplemented module is weak supplemented. But image of extending module need not to be extending. See Example 2.3 of (Chatters and Khuri, 1980). For images of closed weak supplemented modules, we first show:

Lemma 4 Let $f:M\to N$ be an epimorphism of modules and L a closed submodule of N. Suppose that N is nonsingular, then $H=f^{-1}(L)$ is a closed submodule of M.

Proof Since $N\cong M/K$, where $K=Kerf$, then $L\cong H/K$. Suppose that $K\leq H\leq_e T\leq M$. Then $T/H\cong T/K/H/K$ is singular. Since N is nonsingular, we have $H/K\leq_e T/K$. Thus $H/K=T/K$ and $H=T$, i.e., H is closed in M.

Theorem 1 Let M be a closed weak supplemented module. Then any nonsingular image of M is also closed weak supplemented.

Proof Let $f:M\to N$ be an epimorphism of modules with M a closed weak supplemented and N a nonsingular module. Let L be a closed submodule of N, then, by lemma above, $H=f^{-1}(L)$ is a closed submodule of M. Since M is closed weak supplemented, there is a submodule K of M such that $M=K+H$ and $K\cap H<<M$. Hence $N=f(K)+f(H)=f(K)+L$. By Lemma 2(3), $f(K\cap H)=f(K)\cap f(H)=f(K)\cap L<<N$, since $Kerf\leq H$. Thus N is a closed weak supplemented module.

Remark 1 This theorem's condition that N is nonsingular is not necessary. For example, Z is closed weak supplemented as Z-module, for any prime p, $Z_p=Z/pZ$ is a simple Z-module and is closed weak supplemented. But Z_p is singular.

Recall that a right R-module is called singular if $Z(M)=M$ where $Z(M)=\{m\in M|mI=0$, for some essential right ideal I of $R\}$ and non-singular if $Z(M)=0$. A ring R is called right nonsingular if R_R is nonsingular and singular if R_R is singular. Let R be a ring, then R is right nonsingular if and only if all right projective modules are nonsingular.

Corollary 2 Let M be a closed weak supplemented module such that $M/Rad(M)$ is nonsingular. Then $M/Rad(M)$ is extending.

Corollary 3 Let R be a right nonsingular ring. Then the following are equivalent:

(1) Every nonsingular right R-module is closed weak supplemented;

(2) Every projective right R-module is closed weak supplemented.

A ring R is called a right closed weak supplemented ring if R_R is closed weak supplemented module.

Corollary 4 Let R be a right nonsingular ring. Then the following are equivalent:

(1) R is right closed weak supplemented ring;

(2) Every nonsingular cyclic R-module is closed weak supplemented;

(3) Every principal right ideal of R is closed weak supplemented.

Lemma 5 Let $f:M\to N$ be an epimorphism of modules and L a closed submodule of N. Then $L\cong U/Kerf$ for some $U\leq M$. If $r(m)=r(f(m))$ for all $m\in M\backslash Kerf$, then U is closed in M.

Proof Suppose that $Kerf\leq U\leq_e K\leq M$. Then for any $k\in K\backslash Kerf$, $f(k)\neq 0$, there is $r_0\in R$ such that $0\neq kr_0\in U$. Since $r(k)=r(f(k))$, then $f(kr_0)=f(k)r_0\neq 0$, so $0\neq kr_0+Kerf\in U/Kerf$. We have that $L\cong U/Kerf\leq_e K/Kerf$. Since L is closed in N, we have that $U=K$ and that U is closed in M.

Theorem 2 Let $f:M\to N$ be an epimorphism of modules with M closed weak supplemented. If $r(m)=r(f(m))$ for all $m\in M\backslash Kerf$, then N is also closed weak supplemented.

Proof By Lemma above, for any closed submodule L of N, there is a closed submodule U of M, such that $Kerf\leq U\leq_c M$, $L\cong U/Kerf$. The rest is similar to that of Theorem 1. Hence N is closed weak supplemented.

RINGS IN WHICH ALL NONSINGULAR MODULES ARE CLOSED WEAK SUPPLEMENTED

In this section, we will characterize rings in which all nonsingular modules are closed weak supplemented. Later the relations between weak supplemented modules, ⊕-supplemented modules and closed weak supplemented modules are given. It is easy to show that the following proposition holds:

Proposition 8 Let M be an R-module with $Rad(M)=0$. Then the following are equivalent:

(1) M is a closed weak supplemented module;

(2) M is extending.

Corollary 5 Let R be a semiprimitive ring. Then the following are equivalent:

(1) R is a closed weak supplemented ring;

(2) R is an extending ring.

A ring R is called a right V-ring if every simple right R-module is injective. Equivalently, a ring R is a right V-ring if and only if $Rad(M)=0$ for all right R-modules M. Hence we have:

Theorem 3 Let R be a right nonsingular right V-ring. Then the following are equivalent:

(1) Every nonsingular right R-module M is closed weak supplement;

(2) Every projective right module is closed weak supplemented;

(3) Every nonsingular right R-module M is extending;

(4) Every nonsingular right R-module is projective.

Proof (1)⇔(2). By Corollary 3.

(1)⇔(3). This is Proposition 8.

(2)⇒(4). Let M be a nonsingular module. There is a projective module P, such that $M\cong P/N$ for some submodule N of P. Since P is nonsingular, we have that N is a closed submodule of P. By (2), P is closed weak supplemented, hence P is extending by Proposition 8. Thus N is a direct summand of P and therefore M is projective.

(4)⇒(1). Let M be a nonsingular module and N a closed submodule of M. Then M/N is nonsingular, hence, by (4), is projective. Thus N is a direct summand of M. Hence M is extending and is closed weak supplemented.

Combining with Theorem 5.23 of (Goodearl, 1976) and the corollary above, we have:

Corollary 6 The following are equivalent for a right V-ring R:

(1) R is right nonsingular and every nonsingular right module M is closed weak supplemented;

(2) R is right nonsingular and all nonsingular right R-modules are projective;

(3) R is right nonsingular and every projective right R-module is closed weak supplemented;

(4) R is right nonsingular and every nonsingular right R-module M is extending;

(5) R is left nonsingular and every nonsingular left R-module M is closed weak supplemented;

(6) R is left nonsingular and all nonsingular left R-modules are projective;

(7) R is left nonsingular and every projective left R-module is closed weak supplemented;

(8) R is left nonsingular and every nonsingular left R-module M is extending;

(9) R is right and left hereditary, right and left Artinian, and the maximal right and left quotient rings of R coincide.

Since any regular ring is left and right nonsingular, combining with exercise 5.C.17, 21 of (Goodearl, 1976), we have:

Corollary 7 Let R be a commutative regular ring such that every nonsingular module is closed weak supplemented. Then R is semi-simple and every nonsingular R-module is a direct sum of uniserial modules.

Next, we will study the relation between closed weak supplemented modules and weak supplemented modules. Let M be a module. If every submodule is closed in M, (for example, M is semi-simple), then M is closed weak supplemented if and only if M is weak supplemented. For other cases, we give:

Lemma 6 Let M be closed weak supplemented and N a closed submodule of M. Suppose that $T<<M$. Then there is a submodule K of M such that $M=K+N=K+N+T$ and $K\cap N<<M$, $K\cap(N+T)<<M$.

Proof Since M is closed weak supplemented and N is a closed submodule of M, then there is a submodule K of M such that $M=K+N$ and $K\cap N<<M$. Let $f:M\rightarrow(M/N)\oplus(M/K)$, which is defined by $f(m)=(m+N,m+K)$, and $g:(M/N)\oplus(M/K)\rightarrow(M/(N+T))\oplus(M/K)$, which is defined by $g(m+N,m'+K)=(m+N+T,m'+K)$. Since $M=N+K$, then we have that f is an epimorphism and that $Kerf=N\cap K<<M$.

Since $Kerg=((N+T)/N)\oplus 0$ and $(N+T)/N=\pi(T)<<M/N$, while $\pi:M\rightarrow M/N$ is the canonical epimorphism, we have that g is a small epimorphism. So gf is a small epimorphism. Thus $Kergf=(T+N)\cap K<<M$. Clearly, $M=(N+T)+K$.

Proposition 9 Let M be an R-module. Suppose that for any submodule N of M, there is a closed submodule L (depending on N) of M such that either $N=L+T$ or $L=N+T'$ for some T, $T'<<M$. Then M is weak supplemented if and only if M is closed weak supplemented.

Proof Suppose that M is closed weak supplemented and N any submodule of M.

Case 1: Suppose that there is a closed submodule L such that $N=L+T$ for some $T<<M$. Then this is a consequence of the lemma above.

Case 2: Suppose that there is a closed submodule L of M such that $L=N+T'$ for some $T'<<M$. Since M is closed weak supplemented, then there is a submodule K of M such that $M=K+L$ and $K\cap L<<M$. So $M=K+N+T'$, hence $M=K+N$, since $T'<<M$. $K\cap N\leq K\cap L<<M$. Thus M is weak supplemented.

The converse is trivial.

A module M is called refinable if for any submodules U, V of M with $U+V=M$, there is a direct summand U' of M with $U'\subseteq U$ and $U'+V=M$ (Wisbauer, 1996).

Combining this Proposition, we have:

Theorem 4 Let M be a refinable module. Suppose that for any submodule N of M, there is a closed submodule L (depending on N) of M such that either $N=L+T$ or $L=N+T'$ for some T, $T'<<M$. Then the following are equivalent:

(1) M is $\oplus$-supplemented;
(2) M is supplemented;
(3) M is weak supplemented;
(4) M is closed weak supplemented.

Proof It is obvious that (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4).

(4)$\Rightarrow$(1). Let N be any submodule of M.

Case 1: Suppose that there is a closed submodule L of M such that $N=L+T$ for some $T<<M$. Since M is closed weak supplemented, then there is a submodule K of M such that $M=L+K$ and $K\cap L<<M$. Hence $M=L+K=N+K$ and $N\cap K<<M$. Since M is refinable, there is a direct summand U of M such that $U\leq N$ and $M=U+K$. So $U\cap K\leq N\cap K<<M$. As U is a direct summand of M, we have that $U\cap K<<U$, which shows that M is $\oplus$-supplemented.

Case 2: Suppose that there is a closed submodule L of M such that $L=N+T'$ for some $T'<<M$. Since M is closed weak supplemented, there is a submodule K of M such that $M=L+K$ and $L\cap K<<M$. Thus $M=L+K=N+T'+K=N+K$ and $N\cap K<<M$. As M is refinable, then there is a direct summand U of M such that $U\leq N$ and $M=U+K$. Therefore, $U\cap K\leq N\cap K<<U$, since U is a direct summand of M. Thus M is $\oplus$-supplemented.

Corollary 8 Let R be a V-ring and M a refinable module. Suppose that for any submodule N of M, there is a closed submodule L (depending on N) of M such that either $N=L+T$ or $L=N+T'$ for some T, $T'<<M$. Then the following are equivalent:

(1) M is $\oplus$-supplemented;
(2) M is supplemented;
(3) M is weak supplemented;
(4) M is closed weak supplemented;
(5) M is extending.

Lemma 7 Let U and K be submodules of M such that K is a weak supplement of a maximal submodule N of M. If $K+U$ has a weak supplement X in M, then U has a weak supplement in M.

Proof Since X is a weak supplement of $K+U$ in M, then $M=K+U+X$ and $X\cap(K+U)<<M$. If $K\cap(X+U)\subseteq K\cap N<<M$, then $U\cap(K+X)\leq X\cap(K+U)+K\cap(X+U)<<M$, hence $K+X$ is a supplement of U in M.

Suppose that $K\cap(X+U)$ is not contained in $K\cap N$. Since $K/(K\cap N)\cong(K+N)/N=M/N$, $K\cap N$ is a maximal submodule of K. Therefore, $K\cap N+K\cap(X+U)=K$ and since $K\cap N<<M$, we have $M=X+U+K=X+U+K\cap N+K\cap(X+U)=X+U$. Since $U\cap X\leq(K+U)\cap X<<M$, then X is a weak supplement of U in M.

The following theorem is an immediate consequence of this lemma:

Theorem 5 Suppose that for any submodule U of M, there is a submodule K of M, which is a weak supplement of some maximal submodule N of M, such that $K+U$ is closed in M. Then M is closed weak supplemented if and only if M is weak supplemented.

References

Alizade, R., Büyükasik, E., 2003. Cofinitely weak supplemented modules. *Comm. Alg.*, **31**(11):5377-5390. [doi:10.1081/AGB-120023962]

Chatters, A.W., Khuri, S.M., 1980. Endomorphism rings of modules over nonsingular CS rings. *J. London Math. Soc.(2)*, **21**:434-444.

Goodearl, K.R., 1976. Ring Theory. New York and Basel.

Harmanci, A., Keskin, D., Smith, P.F., 1999. On $\oplus$-supplemented modules. *Acta Math. Hungar.*, **83**(1/2):161-169. [doi:10.1023/A:1006627906283]

Wisbauer, R., 1991. Foundations of Modules and Rings Theory. Gordon and Brench.

Wisbauer, R., 1996. Modules and Algebras: Bi-module Structure and Group Actions on Algebras. Pitman Monographs and Surveys in Pure and Applied Mathematics 81.