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Projectively flat exponential Finsler metric*

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Abstract: In this paper, we study a class of Finsler metric in the form $F=\alpha\exp(\beta/\alpha)+\varepsilon\beta$, where α is a Riemannian metric and β is a 1-form, ε is a constant. We call F exponential Finsler metric. We proved that exponential Finsler metric F is locally projectively flat if and only if α is projectively flat and β is parallel with respect to α. Moreover, we proved that the Douglas tensor of exponential Finsler metric F vanishes if and only if β is parallel with respect to α. And from this fact, we get that if exponential Finsler metric F is the Douglas metric, then F is not only a Berwald metric, but also a Landsberg metric.

Key words: Exponential Finsler metric, Projectively flat, (α, β)-metric, Douglas tensor
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INTRODUCTION

One century ago, Hilbert announced his famous 23 problems. The Hilbert's Fourth Problem is to characterize the (not-necessarily-reversible) distance functions on an open subset in $\mathbb{R}^n$ such that straight lines are shortest paths. Distance functions induced by a Finsler metric are regarded as smooth ones. Thus Hilbert's Fourth Problem in the smooth case is to characterize Finsler metric on an open subset in $\mathbb{R}^n$ whose geodesics are straight lines. Finsler metric on an open domain in $\mathbb{R}^n$ with this property are said to be projectively flat. Hamel (1993) first found a simple system of PDE's to characterize projectively flat Finsler metric on an open subset in $\mathbb{R}^n$. That is, a Finsler metric $F=F(x,y)$ on an open subset in $\mathbb{R}^n$ is projectively flat if and only if it satisfies the following partial differential equations

$$F_{x^k y^i} y^k = F_{x^i}. \tag{1}$$

It is an important problem in Finsler geometry to study and characterize projectively flat Finsler metric on an open domain in $\mathbb{R}^n$. This problem is very difficult for general Finsler metric. Shen (2003) observed that a Randers metric $F=\alpha+\beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed. This fact is a direct consequence of Bácsó-Matsumoto's theorem on Doulas metrics (Bácsó and Matsumoto, 1997). Shen and Civi Yildirim (2005) studied the locally projectively flat metric in the form $F=(\alpha+\beta)^2/\alpha$. Senarath and Thornley (2004) gave an equation in local coordinates that characterizes projectively flat Finsler metric in the form $F=(\alpha+\beta)^2/\alpha$. These are some special forms of (α, β)-metric (see Section 2).

A natural problem is to study and characterize all (α,β)-metrics which are projectively flat. In general, this is very complicated. The fist step for us is to study some special (α,β)-metrics such as

$$F=\alpha\phi(s),\ s=\beta/\alpha,$$

where $\phi(s)=\exp(s)+\varepsilon s$, ε is a constant. We call F exponential Finsler metric.

In this paper, we shall first prove the following:

Theorem 1 Let $F=\alpha\exp(\beta/\alpha)+\varepsilon\beta$ be a Finsler met-

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ric on a manifold $\boldsymbol{M}$. F is locally projectively flat if and only if the following conditions hold:

(1) β is parallel with respect to α;

(2) α is locally projectively flat. That is, α is of constant curvature.

We say a Finsler metric on an open domain in $\mathbb{R}^n$ is "trivial", if it satisfies the conclusion of Theorem 1. Thus the above theorem tells us that in the class of exponential Finsler metric, there is no non-trivial projectively flat metrics.

A theorem due to Douglas states that a Finsler metric F is projectively flat if and only if two special curvature tensors are zero. The first is the Douglas tensor. The second is the projective Weyl tensor for $n\geq 3$, and the Berwald-Weyl tensor for $n=2$. It is known that the projective Weyl tensor vanishes if and only if the flag curvature of F has no dependence on the transverse edges (but can possibly depend on the position x and the flagpole y). If the Douglas tensor of F vanishes, we call F a Douglas metric. Bácsó and Matsumoto (1997) proved that a Randers metric $F=\alpha+\beta$ is a Douglas metric if and only if β is a closed 1-form. Matsumoto (1998) obtain that for $n=\dim\boldsymbol{M}\geq 3$, $F=(\alpha^2+\beta^2)/\alpha$ is a Douglas metric if and only if

$$b_{i|j}=\tau((1+2b^2)a_{ij}-3b_ib_j), \tag{2}$$

where $\tau=\tau(x)$ is a scalar function. In this paper, we shall also prove the following:

Theorem 2 Let $F=\alpha\exp(\beta/\alpha)+\varepsilon\beta$ be a Finsler metric on a manifold $\boldsymbol{M}$. Then the Douglas tensor of F vanishes if and only if β is parallel with respect to α.

It is known that if a Finsler metric is projectively equivalent to a Berwald metric, then it is a Douglas metric. However, it is still an open problem whether or not every Douglas metric is (locally) projectively equivalent to a Berwald metric. We have the following:

Corollary 1 Let $F=\alpha\exp(\beta/\alpha)+\varepsilon\beta$ be a Finsler metric on a manifold $\boldsymbol{M}$. If F is the Douglas metric, then

(1) F is a Berwald metric;

(2) F is a Landsberg metric.

(α, β)-METRICS

Finsler metric under our consideration are special (α, β)-metric, expressed in the following form:

$$F=\alpha\phi(s),\ s=\beta/\alpha, \tag{3}$$

where $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_iy^i$ is a 1-form. $\phi=\phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\ |s|\leq b\leq b_0. \tag{4}$$

It is known that F is a Finsler metric if and only if $||\beta_x||_\alpha<b_0$ for any $x\in\boldsymbol{M}$. Let G^i and G^i_α denote the spray coefficients of F and α, respectively, given by

$$\begin{aligned} G^i&=\frac{g^{ij}}{4}\{[F^2]_{x^ky^j}y^k-[F^2]_{x^j}\},\\ G^i_\alpha&=\frac{a^{ij}}{4}\{[\alpha^2]_{x^ky^j}y^k-[\alpha^2]_{x^j}\},\end{aligned} \tag{5}$$

where $(g^{ij}):=(g_{ij})^{-1}$, $(g_{ij}):=([F^2]_{y^iy^j}/2)$ and $(a^{ij}):=(a_{ij})^{-1}$.

By Eq.(1), we have the following:

Lemma 1 (Chern and Shen, 2005) The geodesic coefficients G^i are related to G^i_α by

$$\begin{aligned} G^i=G^i_\alpha+\alpha Qs^i_0+J\{-2Q\alpha s_0+r_{00}\}\frac{y^i}{\alpha}\\ +H\{-2Q\alpha s_0+r_{00}\}\{b^i-s\frac{y^i}{\alpha}\},\end{aligned} \tag{6}$$

where

$$Q:=\frac{\phi'}{\phi-s\phi'},$$

$$J:=\frac{\phi'(\phi-s\phi')}{2\phi((\phi-s\phi')+(b^2-s^2)\phi'')},$$

$$H:=\frac{\phi''}{2((\phi-s\phi')+(b^2-s^2)\phi'')},$$

$s=\beta/\alpha$, and $b:=||\beta_x||_\alpha$.

Lemma 2 (Shen and Civi Yildirim, 2005) An (α, β)-metric $F=\alpha\phi(s)$, where $s=\beta/\alpha$, is protectively flat on an open subset $U\subset\mathbb{R}^n$ if and only if

$$\begin{aligned}(a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha^3Qs_{l0}\\ +H\alpha\{-2Q\alpha s_0+r_{00}\}\{b_l\alpha-sy_l\}=0,\end{aligned} \tag{7}$$

where $y_m=a_{mi}y^i$.

EXPONENTIAL FINSLER METRIC

In this section, we consider a special (α, β)-metric in the following form:

$$F=\alpha\exp(\beta/\alpha)+\varepsilon\beta, \tag{8}$$

where $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_iy^i$ is a 1-form on $\boldsymbol{M}$, ε is a constant. Let $b_0>0$ be the largest number such that

$$1-s+b^2-s^2>0,\ |s|\le b<b_0, \tag{9}$$

where b_0 depends on ε such that $\exp(s)+\varepsilon s>0$.

Lemma 3 If $F=\alpha\exp(\beta/\alpha)+\varepsilon\beta$ is a Finsler metric, then $b_0\le 1$.

Proof If $F=\alpha\exp(\beta/\alpha)+\varepsilon\beta$ is a Finsler metric, then $1-s+b^2-s^2>0$, $|s|\le b<b_0$. Let $s=b$, then $\forall b<b_0$, $b_0\le 1$.

By Lemma 1,

$$Q:=\frac{\alpha(1+\varepsilon\exp(-\beta/\alpha))}{\alpha-\beta},$$
$$J:=\frac{(\exp(\beta/\alpha)+\varepsilon)(\alpha-\beta)\alpha^2}{2(\alpha\exp(\beta/\alpha)+\varepsilon\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)},$$
$$H:=\frac{\alpha^2}{2((1+b^2)\alpha^2-\alpha\beta-\beta^2)}.$$

Remark $1+\varepsilon\exp(-\beta/\alpha)\neq 0$. If $1+\varepsilon\exp(-\beta/\alpha)=0$, then $\phi(s)=\exp(s)+\varepsilon s$ is a constant, thus F is a Riemannian metric. Eq.(7) is reduced to the fowlling equation:

$$(a_{ml}\alpha^2-y_my_l)G_\alpha^m+\frac{\alpha^4(1+\varepsilon\exp(-\beta/\alpha))}{\alpha-\beta}s_{l0}-\frac{\alpha^5(1+\varepsilon\exp(-\beta/\alpha))s_0}{((1+b^2)\alpha^2-\alpha\beta-\beta^2)(\alpha-\beta)}(b_l\alpha-sy_l)+\frac{\alpha^3r_{00}}{2((1+b^2)\alpha^2-\alpha\beta-\beta^2)}(b_l\alpha-sy_l)=0. \tag{10}$$

Lemma 4 If $(a_{ml}\alpha^2-y_my_l)G_\alpha^m=0$, then α is a locally projectively flat.

Proof If $(a_{ml}\alpha^2-y_my_l)G_\alpha^m=0$, then

$$\alpha^2a_{ml}G_\alpha^m=y_mG_\alpha^my_l, \tag{11}$$

then there is a $\eta=\eta(x,y)$ such that

$$y_mG_\alpha^m=\alpha^2\eta, \tag{12}$$

we get

$$a_{ml}G_\alpha^m=\eta y_l. \tag{13}$$

Contracting Eq.(13) with a^{il} yields

$$G_\alpha^i=\eta y^i, \tag{14}$$

thus α is projectively flat.

Theorem 3 Let $F=\alpha\exp(\beta/\alpha)+\varepsilon\beta$ be a Finsler metric on a manifold $\boldsymbol{M}$. F is locally projectively flat if and only if the following conditions hold:

(1) β is parallel with respect to α;

(2) α is locally projectively flat. That is, α is of constant curvature.

Proof If F is projectively flat. We rewrite Eq.(10) as a polynomial in y^i and α, which is linear in α. This gives

$$\begin{aligned}&(\alpha-\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)(a_{ml}\alpha^2-y_my_l)G_\alpha^m\\&+\alpha^4(1+\varepsilon\exp(-\beta/\alpha))((1+b^2)\alpha^2-\alpha\beta-\beta^2)s_{l0}\\&-\alpha^5(1+\varepsilon\exp(-\beta/\alpha))(b_l\alpha-sy_l)s_0\\&+\frac{1}{2}\alpha^3(\alpha-\beta)r_{00}(b_l\alpha-sy_l)=0.\end{aligned} \tag{15}$$

Case 1 Assume that $\varepsilon\neq 0$. Contracting Eq.(15) with b^l yields

$$\begin{aligned}&(\alpha-\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)(b_m\alpha^2-y_m\beta)G_\alpha^m\\&+\frac{1}{2}\alpha^2(\alpha-\beta)(b^2\alpha^2-\beta^2)r_{00}\\&+\alpha^4(1+\varepsilon\exp(-\beta/\alpha))(\alpha^2-\alpha\beta)s_0=0.\end{aligned} \tag{16}$$

Namely

$$((1+b^2)\alpha^2-\alpha\beta-\beta^2)(b_m\alpha^2-y_m\beta)G_\alpha^m+\frac{1}{2}\alpha^2(b^2\alpha^2-\beta^2)r_{00}+(1+\varepsilon\exp(-\beta/\alpha))\alpha^5s_0=0. \tag{17}$$

Replacing y with $-y$, we get

$$((1+b^2)\alpha^2+\alpha\beta-\beta^2)(b_m\alpha^2-y_m\beta)G_\alpha^m+\frac{1}{2}\alpha^2(b^2\alpha^2-\beta^2)r_{00}-(1+\varepsilon\exp(-\beta/\alpha))\alpha^5s_0=0. \tag{18}$$

Combining Eqs.(17) and (18), we get

$$2\beta(b_m\alpha^2 - y_m\beta)G_\alpha^m = \alpha^4(2+\varepsilon\exp(-\beta/\alpha)+\varepsilon\exp(\beta/\alpha)). \quad (19)$$

Using Taylor expansion of $\exp(\beta/\alpha)$, we can find that the left side of Eq.(19) is an integral expression in y and the right side of Eq.(19) is a fractional expression in y, we get

$$s_0 = 0,\ (b_m\alpha^2 - y_m\beta)G_\alpha{}^m = 0. \quad (20)$$

Substituting Eq.(20) back into Eq.(17), we get r_{00}=0. Thus Eq.(15) becomes

$$(\alpha-\beta)(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + \alpha^4(1+\varepsilon\exp(-\beta/\alpha))s_{l0}=0. \quad (21)$$

By the same derivation, we get

$$(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m = 0,\ s_{l0} = 0. \quad (22)$$

By Lemma 4, α is locally projectively flat. By s_{l0}=0 and r_{00}=0, we get $b_{i|j}$=0, thus β is parallel with respect to α.

Case 2 ε=0, Eq.(15) becomes

$$(\alpha-\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + \alpha^4((1+b^2)\alpha^2-\alpha\beta-\beta^2)s_{l0} - \alpha^5(b_l\alpha - sy_l)s_0 + \frac{1}{2}\alpha^3(\alpha-\beta)r_{00}(b_l\alpha - sy_l) = 0. \quad (23)$$

Because α^{even} is a polynomial in y^i, then the coefficients of α and the coefficients of α^2 must be zero. We obtain

$$2(1+b^2)\alpha^3(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + \alpha^3 r_{00}(b_l\alpha^2 - \beta y_l) = 2\alpha^5\beta s_{l0}, \quad (24)$$

$$2(-(2+b^2)\alpha^2\beta+\beta^3)(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + 2((1+b^2)\alpha^6 - \alpha^4\beta^2)s_{l0} = (2\alpha^4 s_0 + \alpha^2\beta r_{00})(b_l\alpha^2 - \beta y_l). \quad (25)$$

Eq.(24)×(β/α) yields

$$2(1+b^2)\alpha^2\beta(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + \alpha^2\beta r_{00}(b_l\alpha^2 - \beta y_l) = 2\alpha^4\beta^2 s_{l0}. \quad (26)$$

Contracting Eq.(25) and Eq.(26) with b^l yields

$$2(1+b^2)\alpha^2\beta(b_m\alpha^2 - y_m\beta)G_\alpha^m + \alpha^2\beta r_{00}(b^2\alpha^2 - \beta^2) = 2\alpha^4\beta^2 s_0, \quad (27)$$

$$2(-(2+b^2)\alpha^2\beta+\beta^3)(b_m\alpha^2 - y_m\beta)G_\alpha^m + 2((1+b^2)\alpha^6 - \alpha^4\beta^2)s_0 = (2\alpha^4 s_0 + \alpha^2\beta r_{00})(b^2\alpha^2 - \beta^2). \quad (28)$$

Eq.(27)+Eq.(28) yields

$$\beta(b_m\alpha^2 - y_m\beta)G_\alpha^m = \alpha^4 s_0. \quad (29)$$

Note that the polynomial α^4 is not divisible by β, Thus $(b_m\alpha^2 - y_m\beta)G_\alpha^m$ is divisible by α^4. Therefore, there is a scalar function τ=$\tau(x)$ such that

$$(b_m\alpha^2 - y_m\beta)G_\alpha^m = \tau(x)\alpha^4, \quad (30)$$

$$s_0 = \tau(x)\beta. \quad (31)$$

Substituting Eqs.(30) and (31) back into Eq.(27), we get

$$2(1+b^2)\tau(x)\alpha^6 + \alpha^2 r_{00}(b^2\alpha^2-\beta^2) = 2\tau(x)\alpha^4\beta^2,$$

namely

$$r_{00}(b^2\alpha^2-\beta^2) = 2\tau(x)\alpha^2(\beta^2-(1+b^2)\alpha^2). \quad (32)$$

Note that the polynomial $b^2\alpha^2-\beta^2$ is not divisible by α^2, then r_{00} is divisible by α^2. Therefore, there is a scalar function λ=$\lambda(x)$ such that

$$r_{00} = \lambda\alpha^2, \quad \lambda(b^2\alpha^2-\beta^2) = 2\tau(x)(\beta^2-(1+b^2)\alpha^2). \quad (33)$$

Because the polynomial $b^2\alpha^2-\beta^2$ is not divisible by $\beta^2-(1+b^2)\alpha^2$, then λ=0, τ=0, thus we get r_{00}=0, s_0=0. Substituting r_{00}=0, and s_0=0 back into Eqs.(25) and (26), we get

$$(-(2+b^2)\alpha^2\beta+\beta^3)(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + ((1+b^2)\alpha^6 - \alpha^4\beta^2)s_{l0} = 0, \quad (34)$$

$$(1+b^2)\alpha^2\beta(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m = \alpha^4\beta^2 s_{l0}. \quad (35)$$

Because

$$\begin{vmatrix} -(2+b^2)\alpha^2\beta+\beta^3 & (1+b^2)\alpha^6-\alpha^4\beta^2 \\ (1+b^2)\alpha^2\beta & -\alpha^4\beta^2 \end{vmatrix} \neq 0,$$

we get $(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m$=0 and s_{l0}=0.

By Lemma 4, α is projectively flat. By s_{l0}=0 and r_{00}=0, we get $b_{i|j}$=0, thus β is parallel with respect to α. Conversely, if β is parallel with respect to α and α is locally projectively flat, by Lemma 1, we get

$$G^i = G_\alpha^i.$$

Because α is locally projectively flat, thus F is locally projectively flat.

$D_{jkl}^i = 0$

Definition 1 Let

$$D_{jkl}^i := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right), \tag{36}$$

$$D := D_{jkl}^i \mathrm{d}x^j \otimes \partial_i \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l.$$

It is easy to verify that D is a well-defined tensor on TM_0. We say D is the Douglas tensor.

The Douglas tensor is a non-Riemannian quantity, namely, if F is a Riemannian metric, then D_{jkl}^i=0. A Finsler metric is called a Douglas metric if D_{jkl}^i=0. Study on Douglas metrics will enhance our understanding on the geometric meaning of non-Riemannian quantities.

Lemma 5 D_{jkl}^i=0 if and only if $G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i = \gamma_{jk}^i(x) y^j y^k$, for some set of local function $\gamma_{jk}^i(x)$.

Proof It is easy to verify.

It is known that the Douglas tensor is a projective invariant, namely, if two Finsler metric F and $\bar{F}$ are projectively equivalent, that is $G^i = \bar{G}^i + Py^i$, where $P=P(x, y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is the same as that of $\bar{F}$. Thus if a Finsler metric is projectively equivalent to a Berwald metric, then it is a Douglas metric. However, it is still an open problem whether or not every Douglas metric is (locally) projectively equivalent to a Berwald metric.

For $F=\alpha\exp(\beta/\alpha)+\varepsilon\beta$, by Lemma 1, denote

$$P = \frac{\alpha^2(\varepsilon+\exp(\beta/\alpha)) - 2\alpha\beta\exp(\beta/\alpha) - \varepsilon\alpha\beta - \varepsilon\beta^2}{2(\varepsilon\beta + \alpha\exp(\beta/\alpha))((1+b^2)\alpha^2 - \alpha\beta - \beta^2)} \times\left(-2\frac{\alpha^2(1+\varepsilon\exp(-\beta/\alpha))}{\alpha-\beta}s_0 + r_{00}\right), \tag{37}$$

$$Q^i = \frac{\alpha^2(1+\varepsilon\exp(-\beta/\alpha))}{\alpha-\beta}s_0^i - \frac{\alpha^4(1+\varepsilon\exp(-\beta/\alpha))}{(\alpha-\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)}s_0 b^i + \frac{\alpha^2}{2((1+b^2)\alpha^2-\alpha\beta-\beta^2)}r_{00}b^i. \tag{38}$$

We get a new spray

$$\tilde{G} = y^i\frac{\partial}{\partial x^i} - 2\tilde{G}^i\frac{\partial}{\partial y^i}, \tag{39}$$

where

$$\tilde{G}^i := G_\alpha{}^i + Q^i. \tag{40}$$

Clearly, G and $\tilde{G}$ are projectively equivalent. So we only need to compute the Douglas tensor of $\tilde{G}$. We have the following

Theorem 4 Let $F=\alpha\exp(\beta/\alpha)+\varepsilon\beta$ be a Finsler metric on a manifold $\boldsymbol{M}$. Then the Douglas tensor of F vanishes if and only if β is parallel with respect to α.

Proof If the Douglas tensor of F vanishes. By Lemma 5, we get

$$\tilde{G}^i - \frac{1}{n+1}\frac{\partial\tilde{G}^m}{\partial y^m}y^i = \gamma_{jk}^i(x)y^j y^k, \tag{41}$$

for some set of local function $\gamma_{jk}^i(x)$.

Substituting Eq.(40) back into Eq.(36), we get

$$\tilde{D}_{jkl}^i := \bar{D}_{jkl}^i + \frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\left(Q^i - \frac{1}{n+1}\frac{\partial Q^m}{\partial y^m}y^i\right),$$

where $\tilde{D}_{jkl}^i$ is the Douglas tensor of $\tilde{G}$, $\bar{D}_{jkl}^i$ is the Douglas tensor of α, because α is a Riemannian metric, $\bar{D}_{jkl}^i$=0. We get

$$Q^i - \frac{1}{n+1}\frac{\partial Q^m}{\partial y^m} y^i = \gamma^i_{jk}(x) y^j y^k. \tag{42}$$

By Eq.(38), we get

$$\begin{aligned}\frac{\partial Q^m}{\partial y^m} &= \frac{\alpha(1+\varepsilon\exp(-\beta/\alpha))}{(\alpha-\beta)^2} s_0 \\ &-\frac{\alpha^2(b^2\alpha^2-\beta^2)(\alpha+2\beta)(1+\varepsilon\exp(-\beta/\alpha))}{(\alpha-\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)^2} s_0 \\ &+\frac{\alpha^2}{2((1+b^2)\alpha^2-\alpha\beta-\beta^2)}\left(\frac{Y}{(\alpha-\beta)^2}s_0 + r_{00}\right) \\ &+\frac{(b^2\alpha^2-\beta^2)(\alpha+2\beta)}{2((1+b^2)\alpha^2-\alpha\beta-\beta^2)^2} r_{00},\end{aligned} \tag{43}$$

where

$$Y = 2\beta^2 - \alpha\beta - b^2\alpha^2 + \varepsilon(\beta^2 - b^2\alpha\beta + \beta^3/\alpha - \alpha\beta)\exp(-\beta/\alpha).$$

Substituting Eqs.(38) and (43) back into Eq.(42), we get

$$\begin{aligned}&\frac{\alpha^2(1+\varepsilon\exp(-\beta/\alpha))}{\alpha-\beta} s_0^i \\ &-\frac{\alpha^4(1+\varepsilon\exp(-\beta/\alpha))}{2(\alpha-\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)} s_0 b^i \\ &+\frac{\alpha^2}{2((1+b^2)\alpha^2-\alpha\beta-\beta^2)} r_{00} b^i \\ &-\frac{\alpha(\alpha+\varepsilon\beta\exp(-\beta/\alpha))}{(n+1)(\alpha-\beta)^2} s_0 y^i \\ &+\frac{\alpha^2(b^2\alpha^2-\beta^2)(\alpha+2\beta)(1+\varepsilon\exp(-\beta/\alpha))}{2(n+1)(\alpha-\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)^2} s_0 y^i \\ &-\frac{(b^2\alpha^2-\beta^2)(\alpha+2\beta)}{2(n+1)((1+b^2)\alpha^2-\alpha\beta-\beta^2)^2} r_{00} y^i \\ &-\frac{\alpha^2}{2(n+1)((1+b^2)\alpha^2-\alpha\beta-\beta^2)}\Big(Y s_0/(\alpha-\beta)^2 \\ &+2r_{m0}b^m\Big) y^i = \gamma^i_{jk}(x) y^j y^k.\end{aligned} \tag{44}$$

Contracting Eq.(44) with b_i, we get

$$\begin{aligned}&\frac{(n+1)\alpha^2(1+\varepsilon\exp(-\beta/\alpha))}{\alpha-\beta} s_0 \\ &-\frac{(n+1)\alpha^4(1+\varepsilon\exp(-\beta/\alpha))}{(\alpha-\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)} s_0 b^2 \\ &+\frac{(n+1)\alpha^2}{((1+b^2)\alpha^2-\alpha\beta-\beta^2)} r_{00} b^2 \\ &-2\frac{\alpha\beta(\alpha+\varepsilon\beta\exp(-\beta/\alpha))}{(\alpha-\beta)^2} s_0 \\ &+\frac{\alpha^2\beta(b^2\alpha^2-\beta^2)(\alpha+2\beta)(1+\varepsilon\exp(-\beta/\alpha))}{(n+1)(\alpha-\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)^2} s_0 \\ &-\frac{(b^2\alpha^2-\beta^2)(\alpha+2\beta)\beta}{((1+b^2)\alpha^2-\alpha\beta-\beta^2)^2} r_{00} \\ &-\frac{\alpha^2\beta}{((1+b^2)\alpha^2-\alpha\beta-\beta^2)}\left(Y s_0\big/(\alpha-\beta)^2 + 2r_{m0}b^m\right) \\ &= 2(n+1) b_i \gamma^i_{jk}(x) y^j y^k.\end{aligned} \tag{45}$$

Denote $T = b_i\gamma^i_{jk}(x) y^j y^k$.

Case 1 Assume $\varepsilon \neq 0$, with the same discussion in Theorem 1, we get $s_0=0$.

Substituting $s_0=0$ back into Eq.(45), we get

$$\begin{aligned}&\frac{(n+1)b^2\alpha^2}{((1+b^2)\alpha^2-\alpha\beta-\beta^2)} r_{00} - \frac{(b^2\alpha^2-\beta^2)(\alpha+2\beta)\beta}{((1+b^2)\alpha^2-\alpha\beta-\beta^2)^2} r_{00} \\ &-\frac{2\alpha^2\beta r_{m0}b^m}{((1+b^2)\alpha^2-\alpha\beta-\beta^2)} = 2(n+1)T.\end{aligned} \tag{46}$$

From Eq.(45) $\times ((1+b^2)\alpha^2-\alpha\beta-\beta^2)$, we get

$$\begin{aligned}&\Big((n+1)(b^4+b^2)r_{00} - 2(n+1)T(1+b^2)^2 \\ &-2\beta(1+b^2)r_{m0}b^m\Big)\alpha^4 + \Big(-(2+n)b^2 r_{00}\beta \\ &+4(n+1)T\beta + 4b^2(n+1)T\beta + 2\beta^2 r_{m0}b^m\Big)\alpha^3 \\ &+\Big((4b^2+2)(1+n)T\beta^2 - (n+3)b^2 r_{00}\beta^2 \\ &+2\beta^3 r_{m0}b^m\Big)\alpha^2 + \Big(-4(n+1)T\beta^3 + \beta^3 r_{00}\Big)\alpha \\ &-2\Big((n+1)T - r_{00}\Big)\beta^4 = 0.\end{aligned} \tag{47}$$

(1) If r_{00} and T is divisible by α^2, namely, if there are two scalar functions $\lambda=\lambda(x)$ and $\tau=\tau(x)$ such that

$$r_{00} = \lambda(x)\alpha^2, \tag{48}$$

$$T = \tau(x)\alpha^2, \tag{49}$$

then

$$r_{m0}b^m = 2\lambda(x)\beta. \tag{50}$$

Substituting Eqs.(48)~(50) back into Eq.(47), we get

$$A_1\alpha^6 + A_2\alpha^5\beta + A_3\alpha^4\beta^2 + A_4\alpha^3\beta^3 + A_5\alpha^2\beta^4 = 0,$$

where

$$A_1 = -2(n+1)(b^2+1)^2\tau(x) + (n+1)b^2(b^2+1),$$
$$A_2 = 4(n+1)(b^2+1)\tau(x) - (2+n)b^2\lambda(x),$$
$$A_3 = 2(n+1)(2b^2+1)\tau(x) - (4+(7+n)b^2)\lambda(x),$$
$$A_4 = -4(n+1)\tau(x)+5\lambda(x),$$
$$A_5 = -2(n+1)\tau(x)+6\lambda(x).$$

We get $(A_1\alpha^2+A_3\beta^2)\alpha^2 = -A_5\beta^4$.

Because β^2 is not divisible by α^2 and $A_1\alpha^2+A_3\beta^2$, we get $A_5=0$. For the same reason, we get $A_1=A_2=A_3=A_4=0$. By $A_4=A_5=0$, we get $\lambda=\tau=0$. Thus $r_{00}=0$, $T=0$, $r_{m0}b^m=0$, substituting back into Eq.(44), we get

$$\frac{\alpha^2(1+\varepsilon\exp(-\beta/\alpha))}{\alpha-\beta}s_0^i = 0.$$

Because $1+\varepsilon\exp(-\beta/\alpha)\neq 0$, thus $s_0^i=0$. By $s_0^i=0$ and $r_{00}=0$, β is parallel with respect to α.

(2) r_{00} is not divisible by α^2 and T is divisible by α^2, namely, there is a scalar function $\tau=\tau(x)$ such that

$$T = \tau(x)\alpha^2. \tag{51}$$

Substituting Eq.(51) back into Eq.(47), we get

$$B_1\alpha^6+B_2\alpha^5+B_3\alpha^4+B_4\alpha^3+B_5\alpha^2+B_6\alpha+2\beta^4 r_{00}=0,$$

where $B_1, B_2, \ldots, B_6$ are homogeneous in y of degree 0, 1, 2, 3, 4, 5 respectively. Thus

$$(B_1\alpha^4 + B_3\alpha^2 + B_5)\alpha^2 = -2\beta^4 r_{00}.$$

Because β^2 is not divisible by α^2, thus r_{00} is divisible by α^2. This contradicts our assumption.

(3) T is not divisible by α^2, r_{00} is divisible by α^2, namely, there is a scalar function $\lambda=\lambda(x)$ such that

$$r_{00} = \lambda(x)\alpha^2. \tag{52}$$

By the same method used in (2), we find that (3) is impossible.

(4) T and r_{00} are not divisible by α^2.

By Eq.(46), we get

$$\{[(n+1)b^4 r_{00} - 2(n+1)T(1+b^2)^2+(n+1)b^2 r_{00} - 2\beta\cdot(1+b^2)r_{m0}b^m]\alpha^2+(4b^2+2)(n+1)T\beta^2 - (n+3)b^2 r_{00}\beta^2 +2\beta^3 r_{m0}b^m\}\alpha^2 = 2\left((n+1)T - r_{00}\right)\beta^4, \tag{53}$$

$$\left(-(2+n)b^2 r_{00} + 4(n+1)T+4b^2(n+1)T+2\beta r_{m0}b^m\right)\alpha^2 = [4(n+1)T - r_{00}]\beta^2. \tag{54}$$

Because β^2 is not divisible by α^2, $(n+1)T-r_{00}$ and $4(n+1)T-r_{00}$ are divisible by α^2. Therefore, there are two scalar functions $\phi=\phi(x)$ and $\varphi=\varphi(x)$ such that

$$(n+1)T - r_{00} = \phi\alpha^2,$$
$$4(n+1)T - r_{00} = \varphi\alpha^2,$$

then $T=\dfrac{\varphi-\phi}{3(n+1)}\alpha^2$. This contradicts our assumption.

Case 2 $\varepsilon=0$, by Eq.(45), we get

$$\frac{(n+1)\alpha^2}{\alpha-\beta}s_0 - \frac{(n+1)\alpha^4}{(\alpha-\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)}s_0 b^2 + \frac{(n+1)\alpha^2}{((1+b^2)\alpha^2-\alpha\beta-\beta^2)}r_{00}b^2 - 2\frac{\alpha^2\beta}{(\alpha-\beta)^2}s_0 + \frac{\alpha^2\beta(b^2\alpha^2-\beta^2)(\alpha+2\beta)}{(n+1)(\alpha-\beta)((1+b^2)\alpha^2-\alpha\beta-\beta^2)^2}s_0 - \frac{\alpha^2\beta}{((1+b^2)\alpha^2-\alpha\beta-\beta^2)}\left(\frac{Y}{(\alpha-\beta)^2}s_0 + 2r_{m0}b^m\right) - \frac{(b^2\alpha^2-\beta^2)(\alpha+2\beta)\beta}{((1+b^2)\alpha^2-\alpha\beta-\beta^2)^2}r_{00} = 2(n+1)T, \tag{55}$$

where $Y=2\beta^2-\alpha\beta-b^2\alpha^2$.

(1) If r_{00} and T is divisible by α^2, namely, if there are two scalar functions $\lambda=\lambda(x)$ and $\tau=\tau(x)$ such that

$$r_{00} = \lambda(x)\alpha^2 \tag{56}$$
$$T = \tau(x)\alpha^2, \tag{57}$$

then

$$r_{m0}b^m = 2\lambda(x)\beta. \tag{58}$$

Substituting Eqs.(56)~(58) back into Eq.(55), we get

$$C_1\alpha^8 + (C_2 s_0 + C_3\beta)\alpha^7 + (C_4\beta s_0 + C_5\beta^2)\alpha^6 +(C_6\beta^2 s_0 + C_7\beta^3)\alpha^5 + (C_8\beta^3 s_0 + C_9\beta^4)\alpha^4$$

$$+(C_{10}\beta^4 s_0 + C_{11}\beta^5)\alpha^3 + (C_{12}\beta^5 s_0 + C_{13}\beta^6)\alpha^2 = 0,$$

where C_1, C_2 , ..., C_{13} are homogeneous poloynomial in y of degree 0. Especially,

$$\begin{aligned}
C_6 &= -2n - 7 - 6b^2, \\
C_7 &= 4(n+1)(1+b^2)\tau - \left(13 + (n+12)b^2\right)\lambda, \\
C_8 &= -6n + 6b^2 - 3 - 2nb^2, \\
C_9 &= \left(8 + (n+7)b^2\right)\lambda - 4(n+1)(2+b^2)\tau, \\
C_{10} &= 2n + 7, \\
C_{11} &= 7\lambda, \\
C_{12} &= 2(n-1), \\
C_{13} &= 2(n+1)\tau - 6\lambda.
\end{aligned} \tag{59}$$

Because α^{even} is a polynomial in y^i, We obtain

$$\begin{aligned}
&C_1\alpha^8 + (C_4\beta s_0 + C_5\beta^2)\alpha^6 + (C_8\beta^3 s_0 + C_9\beta^4)\alpha^4 \\
&\quad +(C_{12}\beta^5 s_0 + C_{13}\beta^6)\alpha^2 = 0, \\
&(C_2 s_0 + C_3\beta)\alpha^7 + (C_6\beta^2 s_0 + C_7\beta^3)\alpha^5 \\
&\quad +(C_{10}\beta^4 s_0 + C_{11}\beta^5)\alpha^3 = 0.
\end{aligned} \tag{60}$$

By Eq.(60), we get

$$\left(C_1\alpha^4 + (C_4\beta s_0 + C_5\beta^2)\alpha^2 + (C_8\beta^3 s_0 + C_9\beta^4)\right)\alpha^2 = -(C_{12}s_0 + C_{13}\beta)\beta^5.$$

Because β^2 is not divisible by α^2, we get

$$C_{12}s_0 + C_{13}\beta = 0. \tag{61}$$

For the same reason, we get

$$C_6 s_0 + C_7\beta = C_8 s_0 + C_9\beta = C_{10}s_0 + C_{11}\beta = 0. \tag{62}$$

By Eqs.(61) and (62), we get

$$(-2n - 7 - 6b^2)s_0 + \{4(n+1)(1+b^2)\tau - [13 + (n+12)b^2]\lambda\}\beta = 0, \tag{63}$$

$$(-6n + 6b^2 - 3 - 2nb^2)s_0 + \{[8 + (n+7)b^2]\lambda - 4(n+1)(2+b^2)\tau\}\beta = 0, \tag{64}$$

$$(2n+7)s_0 + 7\lambda\beta = 0, \tag{65}$$

$$(n-1)s_0 + \left((n+1)\tau - 3\lambda\right)\beta = 0, \tag{66}$$

By Eqs.(65) and (66), we obtain:

$$\lambda = -\frac{2n+7}{7}\frac{s_0}{\beta}, \quad s_0 = -\frac{13(n+1)\tau\beta}{13n+14}. \tag{67}$$

Substituting Eq.(67) back into Eq.(63), we get $((4b^2-6)n-21)\tau\beta=0$. Because $b<1$, $((4b^2-6)n-21)\neq 0$, thus $\tau=0$. Substituting $\tau=0$ back into these equations, we obtain $\lambda=0$ and $r_{00}=0$, then β is parallel with respect to α. If one of r_{00} and T is not divisible by α^2 or T and r_{00} are not divisible by α^2, with the same discussion in Case 1, we get that β is parallel with respect to α.

Conversely, if β is parallel with respect to α, by Lemma 1, we get $G^i = G^i_\alpha$. Because α is a Riemannian metric, $D^i_{jkl}=0$.

Theorem 5 (Shen, 2004) For an (α, β)-metric $F=\alpha\phi(\beta/\alpha)$ on a manifold of dimension $n\geq 3$, the following hold

(1) F is a Landsberg metric;

(2) F is a Berwald metric;

(3) β is parallel with respect to α.

By Theorem 4 and Theorem 5, we have

Corollary 2 $F=\alpha\exp(\beta/\alpha)+\varepsilon\beta$ is a Finsler metric on a manifold $\boldsymbol{M}$. If F is the Douglas metric, then

(1) F is a Berwald metric;

(2) F is a Landsberg metric.

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