Journal of Zhejiang University SCIENCE A
ISSN 1009-3095 (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

# Almost split sequences for symmetric non-semisimple Hopf algebras*

SHI Mei-hua

(*Department of Mathematics, Zhejiang Education Institute, Hangzhou 310012, China*)

E-mail: smh@zjei.net

Received Oct. 8, 2005; revision accepted Jan. 4, 2006

**Abstract:** We first prove that for a finite dimensional non-semisimple Hopf algebra $H$, the trivial $H$-module is not projective and so the almost split sequence ended with $k$ exists. By this exact sequence, for all indecomposable $H$-module $X$, we can construct a special kind of exact sequence ending with it. The main aim of this paper is to determine when this special exact sequence is an almost split one. For this aim, we restrict $H$ to be unimodular and the square of its antipode to be an inner automorphism. As a special case, we give an application to the quantum double $D(H)=(H^{op})^{*}\bowtie H)$ of any non-semisimple Hopf algebra.

**Key words:** Indecomposable, Unimodular, Almost split sequences, Symmetric non-semisimple Hopf algebras
**doi:**10.1631/jzus.2006.A1077 **Document code:** A **CLC number:** O153.3

## INTRODUCTION

Almost split sequences, also called Auslander-Reiten sequences, were discovered by Auslander *et al*.(1995) for finitely generated modules over a finite-dimensional (artin) algebra and play an important role in other settings as well. While the proof of the existence of almost split sequences for arbitrary artin algebras (see Section 1 of Chapter V in (Auslander *et al*., 1995)) is fairly constructive, it is sometimes possible to give other ways of constructing almost split sequences which are particularly well suited to special types of artin algebras. This paper is devoted to illustrating this point of view in the case for a special kind of Hopf algebras, which are unimodular and the square of their antipodes are inner automorphisms. Fortunately, the quantum double $D(H)=(H^{op})^{*}\bowtie H)$ of any finite dimensional Hopf algebra automatically satisfies these conditions (see Section 4).

In order to discuss almost split sequences, it is natural to require that $H$ is not semisimple. Therefore, in Section 2, we give an equivalent description for a Hopf algebra $H$ to be semisimple by its trivial module. It is interesting to note that we can give a new proof of the well-known Maschke Theorem for Hopf algebras (see Theorem 2.2.1 in (Montgomery, 1993)). Since our methods seem special, we always require the artin algebras to be symmetric. When is a Hopf algebra symmetric? Fortunately, by Proposition 2.5 of (Lorenz, 1997), we know that a Hopf algebra $H$ is symmetric if and only if it is unimodular and the square of its antipode is an automorphism. That is the reason why we always need our Hopf algebras in this paper to be unimodular and the square of their antipodes to be inner automorphisms.

The main result is in Section 3. Let $0\to A\to E \xrightarrow{f} k \to 0$ be the almost split sequence ending with trivial $H$-module $k$. Then for each indecomposable module $X$, we obtain the exact sequence of $H$-modules $0\to X\otimes A\to X\otimes E \xrightarrow{id_X\otimes f} X\to 0$. The results of this section show that the epimorphisms $id_X\otimes f$: $X\otimes E\to X$ are either slit epimorphisms or right almost split morphisms and to determine for which $X$ they are right almost split morphisms. In the last section, by using the results obtained in Section 3, we give an application of the case of quantum doubles.

We now fix some notations for this paper. $k$ always denotes a field where all spaces are finite di-

* Project (No. 10371107) supported by the National Natural Science Foundation of China

mensional $k$-spaces. $\otimes$ means $\otimes_k$. For a Hopf algebra $H$, $\Delta$ means its comultiplication, $\varepsilon$ its counit, $S$ its antipode and $S^{-1}$ the composition inverse of $S$. We accept Sweedler notations for $\Delta$ without sigma notations and indices, i.e., we write $\Delta(h)=h'\otimes h''$ for all $h\in H$. ${}_HM$ means left $H$-module category. For any $H$-module, $A^*=\mathrm{Hom}_k(A,k)$. In this paper, we freely use some definitions and results about almost split sequences. But for completeness, we list them in the Appendix.

## SOME RESULTS ON REPRESENTATION OF HOPF ALGEBRAS

Let $H$ be a finite dimensional Hopf algebra and $A$, $B$ two $H$-modules. Then $A\otimes B$ becomes an $H$-module through $\Delta$, the comultiplication of $H$ and $\mathrm{Hom}_k(A,B)$ becomes an $H$-module by $(h\cdot f)(a):=h''f(S^{-1}(h')a)$ for $h\in H, f\in \mathrm{Hom}_k(A,B)$ and $a\in A$. The field $k$ is clearly an $H$-module through $\varepsilon$. Unless stated to the contrary, this is the only way we will consider $A\otimes B$, $\mathrm{Hom}_k(A,B)$ and $k$ as $H$-modules. In particular, we call $k$ the "trivial" $H$-module. As a special case of these definitions, we have $(h\cdot f)(a)=f(S^{-1}(h')a)$ for all $h\in H, f\in A^*$ and $a\in A$.

### A conclusion on projective module

**Theorem 1** Let $A$ be a projective $H$-module. Then $B\otimes A$ is also projective as $H$-module for all $H$-module $B$.

Before giving the proof of this result we give an interesting application.

**Corollary 1** The following statements are equivalent for $H$.

(1) $H$ is semisimple.

(2) The trivial $H$-module is projective.

**Proof** We know that $H$ is semisimple if and only if every $H$-module is projective. Therefore (1) implies (2). Suppose $k$ is projective. Then $A\cong A\otimes k$ is projective for each $H$-module $A$ by the theorem above. Hence (2) implies (1).

In order to determine precisely when the trivial $H$-module is projective, it is convenient to make the following observations. For an $H$-module $A$ we denote by $A^H$ the $H$-submodule of $A$ consisting of all $a$ in $A$ such that $ha=\varepsilon(h)a$ for all $h$ in $H$. Moreover, if $f:A\to B$ is a morphism in ${}_HM$, then $f(A^H)\subseteq B^H$. Hence we obtain the fixed point functor $()^H:{}_HM\to {}_HM$ given by $()^H(A)\to A^H$ and for $f:A\to B$ in ${}_HM$, $()^H(f):=f_{|A^H}:A^H\to B^H$. We now give another description of the fixed point functor.

Let $A$ be an $H$-module. Then it is easily seen that an element $a$ is in $A^H$ if and only if there is an $H$-morphism $f:k\to A$ such that $f(1)=a$. Therefore for each $A$ in ${}_HM$ we have isomorphisms $\mathrm{Hom}_H(k,A)\to A^H$ functorial in $A$ given by $f\mapsto f(1)$ for all $f$ in ${}_H(k,A)$. Hence we have an isomorphism between $\mathrm{Hom}_H(k, )$ and the fixed point functor $()^H$. We now apply these considerations to determine when $k$ is a projective $H$-module.

Clearly, the counit $\varepsilon$ is an $H$-epimorphism from $H$ to $k$. Therefore $k$ is a projective $H$-module if and only if there is an $H$-morphism $f:k\to H$ such that $\varepsilon f=id_k$. By our previous remarks, this is equivalent to saying that there is an element $z$ in $H^H$ such that $\varepsilon(z)=1$.

Summarizing, we have the following theorem which gives a new proof of the well-known Maschke Theorem for Hopf algebras (see Theorem 2.2.1 in (Montgomery, 1993)).

**Theorem 2** The following are equivalent for the Hopf algebra $H$:

(1) $H$ is semisimple.

(2) The trivial $H$-module is projective.

(3) There is an element $t$ in $H^H$ such that $\varepsilon(t)\neq 0$.

We will finish the proof of Theorem 2 by giving a proof of Theorem 1. This will require two preliminary results.

**Lemma 1** (Lorenz, 1997) Let $A$, $B$ be two $H$-modules. Then

$$(\mathrm{Hom}_k(A,B))^H=\mathrm{Hom}_H(A,B). \tag{1}$$

**Lemma 2** Let $A$, $B$ and $C$ be in ${}_HM$. Then the morphism $\alpha:\mathrm{Hom}_H(A,\mathrm{Hom}_k(B,C))\to \mathrm{Hom}_H(B\otimes A,C)$ given by $\alpha(f)(b\otimes a)=f(a)(b)$ for all $f$ in $\mathrm{Hom}_H(A,\mathrm{Hom}_k(B,C))$ and all $a$ in $A$ and $b$ in $B$ are isomorphism functorials in $A$, $B$ and $C$.

**Proof** It is well known that functors Hom and $\otimes$ are adjoint to each other. Therefore we have the following isomorphism $\alpha':\mathrm{Hom}_k(A,\mathrm{Hom}_k(B,C))\to \mathrm{Hom}_k(B\otimes A,C)$ of $k$-vector spaces given by $\alpha'(f)(b\otimes a)=f(a)(b)$ for all $f$ in $\mathrm{Hom}_k(A,\mathrm{Hom}_k(B,C))$ and all $a$ in $A$ and $b$ in $B$

which are functorials in $A$, $B$ and $C$. We claim that $\alpha'$ is also an $H$-morphism. In fact, for all $f$ in $\mathrm{Hom}_k(A, \mathrm{Hom}_k(B,C))$ and $a$ in $A$ and $b$ in $B$ and all $h$ in $H$,

$$\alpha'(h{\cdot}f)(b\otimes a)=(h{\cdot}f)f(a)(b)=(h''{\cdot}f(S^{-1}(h')a))(b) =h'''f(S^{-1}(h')a)(S^{-1}(h'')b),$$
$$(h{\cdot}\alpha'(f))(b\otimes a)=h'''\alpha'(f)(S^{-1}(h'')b\otimes S^{-1}(h')a) =h'''f(S^{-1}(h')a)(S^{-1}(h'')b).$$

Therefore, $\alpha'$ induces an isomorphism $\alpha$ on fixed points, which gives our desired result by the lemma above.

As a consequence of this result we obtain the following proof of Theorem 1.

**Proof** Let $A$ be a projective $H$-module and let $B$ be an arbitrary $H$-module. We want to show that $B\otimes A$ is projective. Let $0\to C\to C'\to C\to C''\to 0$ be an exact sequence of $H$-modules. Since $\mathrm{Hom}_k(B, \ )$ is an exact functor and $A$ is a projective $H$-module, we obtain by applying Lemma 2 the following commutative exact diagram.

$$\begin{array}{ccccc} \mathrm{Hom}_H(A,\mathrm{Hom}_k(B,C')) & \to & \mathrm{Hom}_H(A,\mathrm{Hom}_k(B,C)) & \to & \mathrm{Hom}_H(A,\mathrm{Hom}_k(B,C''))\to 0 \\ \cong\downarrow & & \cong\downarrow & & \downarrow\cong \\ \mathrm{Hom}_H(B\otimes A,C') & \to & \mathrm{Hom}_H(B\otimes A,C) & \to & \mathrm{Hom}_H(B\otimes A,C'') \end{array}$$

This shows that $\mathrm{Hom}_H(B\otimes A,C)\to\mathrm{Hom}_H(B\otimes A, C'')\to 0$ is exact, which proves that $B\otimes A$ is projective. This finishes the proof of Theorem 1.

## When $S^2$ is an inner automorphism

In this subsection, we always assume the square of the antipode $S$ of $H$ is an inner automorphism, i.e., there exists an invertible element $u\in H$ such that $S^2(x)=uxu^{-1}$ for any $x\in H$. Clearly, in this case, $S^{-2}(x)=u^{-1}xu^{-1}$ for $x\in H$.

**Lemma 3** (Lorenz, 1997) Let $B$ and $C$ be two $H$-modules. Then the vector space isomorphism $\delta{:}B\otimes C\to\mathrm{Hom}_k(B^*,C)$ given by $\delta(b\otimes c)(f)=f(u^{-1}b)c$ for $f$ in $B^*$, for all $b$ in $B$ and $c$ in $C$ is an $H$-isomorphism functorial in $B$ and $C$.

As a consequence of this lemma we obtain the following result.

**Proposition 1** Let $A$, $B$ and $C$ be in ${}_HM$. Then we have the following

$$\beta{:}\mathrm{Hom}_H(A,B\otimes C)\to\mathrm{Hom}_H(B^*\otimes A,C) \qquad (2)$$

given by $\beta(f)(g\otimes a)=\eta(g)(f(a))$ for all $f\in\mathrm{Hom}_H(A, B\otimes C)$, $a\in A$ and $g\in B^*$, where $\eta(g)(b\otimes c)=g(u^{-1}b)c$ for all $b\in B$ and $c\in C$, is an isomorphism functorial in $A$, $B$ and $C$.

**Proof** By Lemma 3, we know that $\delta{:}B\otimes C\to \mathrm{Hom}_k(B^*,C)$ given by $\delta(b\otimes c)(f)=f(u^{-1}b)c$ for $f$ in $B^*$, for all $b$ in $B$ and $c$ in $C$ is an $H$-isomorphism functorial in $B$ and $C$. This induces the isomorphism $\mathrm{Hom}_H(A,\delta)$: $\mathrm{Hom}_H(A,B\otimes C)\to\mathrm{Hom}_H(A,\mathrm{Hom}_k(B^*,C))$ functorial in $A$, $B$ and $C$. By Lemma 2, we have the canonical isomorphism $\alpha{:}\mathrm{Hom}_H(A,\mathrm{Hom}_k(B^*,C))\to\mathrm{Hom}_H(B^*\otimes A, C)$ which is also functorial in $A$, $B$ and $C$. It is not difficult to check that the composition $\alpha\mathrm{Hom}_H(A,\delta)$ is our morphism $\beta{:}\mathrm{Hom}_H(A,B\otimes C)\to\mathrm{Hom}_H(B^*\otimes A,C)$. Therefore, $\beta$ is an isomorphism functorial in $A$, $B$ and $C$.

It is easy to check that the usual vector space isomorphism $\varepsilon{:}A^*\otimes A\to\mathrm{End}_k(A)$, given by $\varepsilon(f\otimes a)(x)=f(x)a$ for all $f\in A^*$ and $a$, $x\in A$, is an $H$-isomorphism whose inverse can be described as follows. Let $\{a_1, \ldots, a_d\}$ be a $k$-basis for $A$ with dual basis $\{f_1, \ldots, f_d\}$. Then it is easily seen that the map $\mu{:}\mathrm{End}_k(A)\to A^*\otimes A$ given by $\mu(g)=\sum_{i=1}^{d}(f_i\otimes g(a_i))$ for all $g\in\mathrm{End}_k(A)$ is the inverse of $\varepsilon$ and therefore an $H$-isomorphism. It is also not difficult to see that $\sum_{i=1}^{d} f_i(g(a_i))=\mathrm{tr}(g)$, the trace of $g$, for all $g\in \mathrm{End}_k(A)$.

We now use Proposition 1 with $A=B$ and $C=k$, the trivial $H$-module. Then $\beta{:}\mathrm{Hom}_H(A,A)\to\mathrm{Hom}_H(A^*\otimes A,k)$ given by $\beta(f)(g\otimes a)=g(u^{-1}f(a))$ for all $f\in \mathrm{Hom}_H(A,A)$, $g\in A^*$ and $a\in A$ is an isomorphism. From these remarks it follows that the isomorphism $v$: $\mathrm{End}_k(A)\to\mathrm{Hom}_H(\mathrm{End}_k(A),k)$ which is the composition

$$\mathrm{Hom}_H(A,A)\xrightarrow{\ \beta\ }\mathrm{Hom}_H(A^*\otimes A,k) \xrightarrow{\mathrm{Hom}_H(\mu,k)}\mathrm{Hom}_H(\mathrm{End}_k(A),k) \qquad (3)$$

is given by $v(f)(g)=(\mathrm{Hom}_H(\mu,k)\beta)(f)(g)=\beta(f)(u(g))= \beta(f)(\sum_{i=1}^{d} f_i\otimes g(a_i))=\sum_{i=1}^{d} f_i(\mu^{-1}fg(a_i))=\mathrm{tr}(u^{-1}fg)$ for all $f\in\mathrm{End}_H(A)$ and $g\in\mathrm{End}_k(A)$, where we consider $u^{-1}$ as a linear endomorphism of $A$. We are particularly interested in the following property of the isomorphism $v$.

**Lemma 4** Let $A$ be in ${}_HM$.

(1) An element $f$ in $\mathrm{End}_H(A)$ has the property that $v(f){:}\mathrm{End}_k(A)\to k$ is a split $H$-epimorphism if and only

if there is some $g \in \mathrm{End}_H(A)$ such that $\mathrm{tr}(u^{-1}fg) \neq 0$.

(2) The trivial module $k$ is an $H$-summand of $\mathrm{End}_k(A)$ if and only if there is some $g \in \mathrm{End}_H(A)$ such that $\mathrm{tr}(u^{-1}g) \neq 0$.

**Proof** (1) Obviously $v(f):\mathrm{End}_k(A) \to k$ is a split $H$-epimorphism if and only if there is an $H$-morphism $t:k \to \mathrm{End}_k(A)$ such that $v(f)t \neq 0$. The isomorphism $\mathrm{Hom}_H(k,\mathrm{End}_k(A)) \to \mathrm{End}_k(A)^H = \mathrm{End}_H(A)$ (Lemma 1) given by $t \mapsto t(1)$ shows that there is an $H$-morphism $t:k \to \mathrm{End}_k(A)$ such that $v(f)t \neq 0$ if and only if there is some $g \in \mathrm{End}_H(A)$ such that $v(f)(g) \neq 0$. Using the fact that $v(f)(g) = \mathrm{tr}(u^{-1}fg)$, we have our desired result.

(2) "If" part: This is a trivial consequence of (1).

"Only If" part: Since $k$ is an $H$-summand of $\mathrm{End}_k(A)$, there is a split $H$-epimorphism $p:\mathrm{End}_k(A) \to k$. By $v$ being an isomorphism, we have $f \in \mathrm{End}_H(A)$ such that $v(f)=p$. Using (1) again, there is some $f' \in \mathrm{End}_H(A)$ such that $\mathrm{tr}(u^{-1}ff') \neq 0$. Therefore we get the result as long as we let $g=ff'$.

Recall that a Hopf algebra $H$ is said to be "unimodular" if the set of left integrals equals that of right integrals, i.e., $H$ has a non-zero two-sided integral. An algebra $A$ is called "symmetric" if $A \cong A^*$ as $A$–$A$-bimodules. The following result was cited in (Lorenz, 1997).

**Lemma 5** A finite dimensional Hopf algebra $H$ is symmetric if and only if it is unimodular and the square of the antipode is an inner automorphism.

This lemma can also be deduced from the following consequence, which was proved in (Böhm *et al*., 1999), since a Hopf algebra must be a weak Hopf algebra and every non-zero integral in a Hopf algebra must be non-degenerate.

**Theorem 3** The weak Hopf algebra $H$ is a symmetric algebra if and only if it has non-degenerate two-sided integrals and the square of the antipode is an automorphism.

Since the group algebra $kG$ is clearly unimodular and the square of the antipode equals the identity map of $kG$, we have the following well-known result.

**Corollary 2** The group algebra $kG$ is a symmetric algebra.

## ALMOST SPLIT SEQUENCES FOR A KIND OF HOPF ALGEBRAS

In this section, we always assume the Hopf algebra $H$, which is not semisimple, is unimodular and the square of the antipode is an inner automorphism, i.e., $S^2(x)=uxu^{-1}$ for $x \in H$. Therefore $H$ is a symmetric algebra by Lemma 5.

Since $H$ is not semisimple, the theory of almost sequences can be applied to ${}_HM$. In particular, since the trivial $H$-module $k$ is not projective by Corollary 1, $k$ has an almost split sequence $0 \to A \to E \xrightarrow{f} k \to 0$ (see Appendix). Then for each indecomposable module $X$ we obtain the exact sequence of $H$-module $0 \to X \otimes A \to X \otimes E \xrightarrow{id_X \otimes f} X \to 0$. Our main aim is to show that the epimorphisms $id_X \otimes f: X \otimes E \to X$ are either split epimorphisms or right almost split morphisms, and to determine for which $X$ they are right almost split morphisms.

We know that $A \cong D\mathrm{Tr}(k)$ (see Appendix) in the almost split sequence $0 \to A \to E \xrightarrow{f} k \to 0$. Since $H$ is a symmetric algebra, we know that $D\mathrm{Tr}(k) \cong \Omega^2(k)$ (see Appendix). We now show that $X \otimes \Omega^2(k) \cong \Omega^2(k) \oplus Q$ for some projective $H$-module $Q$.

**Lemma 6** Let

$$\begin{aligned} \cdots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \\ P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} k \to 0 \end{aligned} \tag{4}$$

be a minimal projective resolution of $k$. Then for each $H$-module $X$ we have the following:

(1) The exact sequence

$$\begin{aligned} \cdots X \otimes P_n \xrightarrow{id_X \otimes d_n} X \otimes P_{n-1} \xrightarrow{id_X \otimes d_{n-1}} \cdots \\ \xrightarrow{id_X \otimes d_2} X \otimes P_1 \xrightarrow{id_X \otimes d_1} X \otimes P_0 \xrightarrow{id_X \otimes d_0} X \to 0 \end{aligned} \tag{5}$$

is a (not necessarily minimal) projective resolution of $X$.

(2) $X \otimes \Omega^n(k) \cong \Omega^n(X) \oplus Q_n$ with $Q_n$ a projective module for all $n>0$.

**Proof** By Theorem 1 the $X \otimes P_n$ are projective $H$-modules since the $P_n$ are projective $H$-modules. Part (1) follows trivially from this fact. Part (2) also follows this fact as long as we note that the definition of $n$th syzygy.

Thus we see that if $X$ is an indecomposable non-projective module, the exact sequence $0 \to X \otimes \Omega^2(k) \to X \otimes E \xrightarrow{id_X \otimes f} X \to 0$ is isomorphic to $0 \to \Omega^2(k) \oplus Q \to X \otimes E \xrightarrow{id_X \otimes f} X \to 0$ with $Q$ projective.

Since $H$ is a Frobenius algebra, $Q$ is also injective. Hence the exact sequence $0\to\Omega^2(X)\oplus Q\to X\otimes E \xrightarrow{id_X\otimes f} X\to 0$ can be written as the sum of exact sequences $0\to\Omega^2(X)\oplus Q\xrightarrow{g\oplus id_Q}(X\otimes E)_0\oplus Q \xrightarrow{((id_X\otimes f)_0,0)} X\to 0$.

Consequently $id_X\otimes f$ is a split epimorphism or a right almost split morphism according to whether $(id_X\otimes f)_0$ is a split epimorphism or a right almost split morphism. Also $id_X\otimes f$ is right almost split if and only if $0\to\Omega^2(X)\xrightarrow{g}(X\otimes E)_0\xrightarrow{(id_X\otimes f)_0} X\to 0$ is an almost split sequence (see Appendix). We use these observations and notations freely throughout this section.

Our proof that $id_X\otimes f{:}X\otimes E\to X$ is either a split epimorphism or a right almost split morphism is based on the following comparison with the morphism $\mathrm{Hom}_H(X,id_X\otimes f){:}\mathrm{Hom}_H(X,X\otimes E)\to\mathrm{Hom}_H(X,X)$.

**Lemma 7** Let $0\to\Omega^2(k)\to E\xrightarrow{f}k\to 0$ be an almost split sequence. Then the following are equivalent for an indecomposable $H$-module $X$.

(1) The Jacobson radical $rad\mathrm{End}_H(X)$ is contained in $\mathrm{ImHom}_H(X,id_X\otimes f)$, image of $\mathrm{Hom}_H(X,id_X\otimes f)$.

(2) $id_X\otimes f{:}X\otimes E\to X$ is either a split epimorphism or a right almost split morphism.

**Proof** Proof of this lemma is similar to that of Lemma 4.2 in Chapter V of (Auslander *et al*., 1995). For completeness, we write it out.

(1)⇒(2) Since $X$ is indecomposable, $\mathrm{End}_H(X)$ is a local ring. Therefore $\mathrm{ImHom}_H(X,id_X\otimes f)\supseteq rad\mathrm{End}_H(X)$ implies that $\mathrm{ImHom}_H(X,id_X\otimes f)$ is $\mathrm{End}_H(X)$ or $rad\mathrm{End}_H(X)$. If $\mathrm{ImHom}_H(X,id_X\otimes f)=\mathrm{End}_H(X)$, then $id_X\otimes f$ is a split epimorphism. Suppose now that $\mathrm{ImHom}_H(X,id_X\otimes f)=rad\mathrm{End}_H(X)$. Then the exact sequence

$$0\to\Omega^2(X)\xrightarrow{g}(X\otimes E)_0\xrightarrow{(id_X\otimes f)_0}X\to 0 \quad (6)$$

also has the property that $\mathrm{ImHom}_H(X,(id_X\otimes f)_0)=rad\mathrm{End}_H(X)$. This means that an endomorphism $X\to X$ factors through $(id_X\otimes f)_0$ if and only if it is not an isomorphism. Then it follows from Proposition 2.2 in Chapter V of (Auslander *et al*., 1995) that the exact sequence Eq.(6) is almost split since it is not split, $\Omega^2(X)\cong D\mathrm{Tr}X$ and every endomorphism $X\to X$ which is not an automorphism factor through $(id_X\otimes f)_0$. Hence $(id_X\otimes f)_0$ is right almost split and therefore $id_X\otimes f$ is right almost split.

(2)⇒(1) This is trivial.

In view of this lemma it is of interest to have a description of $\mathrm{ImHom}_H(X,id_X\otimes f)$.

**Proposition 2** Let $0\to\Omega^2(k)\to E\xrightarrow{f}k\to 0$ be an almost split sequence and $X$ be an $H$-module. Then the following are equivalent for an element $f'\in\mathrm{End}_H(X)$.

(1) $f'\in\mathrm{Im}(\mathrm{Hom}_H(X,X\otimes E)\xrightarrow{\mathrm{Hom}_H(X,id_X\otimes f)}\mathrm{Hom}_H(X,X))$.

(2) $\mathrm{tr}(u^{-1}f'g)=0$ for all $g$ in $\mathrm{End}_H(X)$.

**Proof** Recall Proposition 1, for all $A$, $B$, $C\in{}_HM$ we have $H$-isomorphism

$$\beta{:}\mathrm{Hom}_H(A,B\otimes C)\to\mathrm{Hom}_H(B^*\otimes A,C), \quad (7)$$

and it is functorial in $A$, $B$ and $C$. On the other hand, we also have another kind of $H$-isomorphism $\varepsilon{:}A^*\otimes A\to\mathrm{End}_k(A)$ which was also given in Section 2. Combining these two kinds of $H$-isomorphism, we have the following commutative exact diagram.

$$\begin{array}{ccc}
\mathrm{Hom}_H(X,X\otimes E) & \xrightarrow{\mathrm{Hom}_H(X,id_X\otimes f)} & \mathrm{Hom}_H(X,X)\\
\cong\downarrow & & \cong\downarrow\\
\mathrm{Hom}_H(X^*\otimes X,E) & \xrightarrow{\mathrm{Hom}_H(X^*\otimes X,f)} & \mathrm{Hom}_H(X^*\otimes X,k)\\
\cong\downarrow & & \cong\downarrow\\
\mathrm{Hom}_H(\mathrm{Hom}_k(X,X),E) & \xrightarrow{\mathrm{Hom}_H(\mathrm{Hom}_k(X,X),f)} & \mathrm{Hom}_H(\mathrm{Hom}_k(X,X),k)
\end{array}$$

Since $f{:}E\to k$ is right almost split, we have that $\mathrm{Im}(\mathrm{Hom}_H(\mathrm{Hom}_k(X,X),E)\to\mathrm{Hom}_H(\mathrm{Hom}_k(X,X),k))$ consists of morphisms $g{:}\mathrm{End}_k(X)\to k$ which are not split epimorphisms. Then by Lemma 4(1) we have that

$$\mathrm{Im}((\mathrm{Hom}_H(X,X\otimes E)\xrightarrow{\mathrm{Hom}_H(X,id_X\otimes f)}\mathrm{Hom}_H(X,X)) \quad (8)$$

consists of the $f'$ such that $\mathrm{tr}(u^{-1}f'g)=0$ for all $g$ in $\mathrm{End}_H(X)$. Therefore, the proof of this proposition is now complete.

We deduce some consequences from this proposition.

**Corollary 3** Let $0\to\Omega^2(k)\to E\xrightarrow{f}k\to 0$ be an almost split sequence. For each $H$-module $X$ we have the following

(1) $\mathrm{Im}((\mathrm{Hom}_H(X,X\otimes E)\xrightarrow{\mathrm{Hom}_H(X,id_X\otimes f)}\mathrm{Hom}_H$

$(X,X)$) contains $rad\mathrm{End}_H(X)$.

(2) $X\otimes E\xrightarrow{id_X\otimes f}X$ is a split epimorphism if and only if $\mathrm{tr}(u^{-1}f')=0$ for all $f'$ in $\mathrm{End}_H(X)$.

**Proof** Suppose $f'$ is in $rad\mathrm{End}_H(X)$. Then $f'g$ is in $rad\mathrm{End}_H(X)$ for all $g$ in $\mathrm{End}_H(X)$. Therefore $f'g$ is nilpotent for all $g$ in $\mathrm{End}_H(X)$ and so $u^{-1}f'g$ is also nilpotent since clearly $u^{-1}$ commutes with any $g\in\mathrm{End}_H(X)$. This means that $\mathrm{tr}(u^{-1}f'g)=0$ for all $g$ in $\mathrm{End}_H(X)$. Hence is $f'$ in $\mathrm{Im}(\mathrm{Hom}_H(X,X\otimes E)\xrightarrow{\mathrm{Hom}_H(X,id_X\otimes f)}\mathrm{Hom}_H(X,X))$ by Proposition 2.

(2) This is a trivial consequence of Proposition 2.

Combining Lemma 7 and Corollary 3 we have the following.

**Theorem 4** Let $0\to\Omega^2(k)\to E\xrightarrow{f}k\to 0$ be an almost split sequence and $X$ an indecomposable $H$-module.

(1) $id_X\otimes f{:}X\otimes E\to X$ is a split epimorphism if and only if $\mathrm{tr}(u^{-1}f')=0$ for all $f'$ in $\mathrm{End}_H(X)$. Otherwise $id_X\otimes f{:}X\otimes E\to X$ is right almost split.

(2) If $\mathrm{tr}(u^{-1})\neq 0$, then $id_X\otimes f{:}X\otimes E\to X$ is right almost split.

(3) If $k$ is algebraically closed, then $id_X\otimes f$: $X\otimes E\to X$ is right almost split if and only if $\mathrm{tr}(u^{-1})\neq 0$.

**Proof** (1) This is an easy consequence of Lemma 7 and Corollary 3.

(2) This is a direct consequence of (1) since $\mathrm{tr}(u^{-1})=\mathrm{tr}(u^{-1}id_X)\neq 0$.

(3) Suppose $k$ is algebraically closed. Then the elements of the local ring $\mathrm{End}_H(X)$ can be written as $v\cdot id_X+f'$ with $v\in k$ and $f'\in rad\mathrm{End}_H(X)$. We then get $\mathrm{tr}(u^{-1}(v\cdot id_X+f'))v(\mathrm{tr}(u^{-1}))$. Hence it follows that $\mathrm{tr}(u^{-1}(v\cdot id_X+f'))\neq 0$ if and only if $v\neq 0$ and $\mathrm{tr}(u^{-1})\neq 0$. Therefore, by part (1), we have that $id_X\otimes f{:}X\otimes E\to X$ is right almost split if and only if $\mathrm{tr}(u^{-1})\neq 0$.

The following is an immediate consequence of this theorem and seems interesting.

**Corollary 4** Let $X$ be an indecomposable projective $H$-module. Then we have $\mathrm{tr}(u^{-1}f')\neq 0$ for all $f'\in\mathrm{End}_H(X)$.

Theorem 4 can be formulated in terms of almost split sequences as follows.

**Theorem 5** Let $0\to\Omega^2(k)\to E\xrightarrow{f}k\to 0$ be an almost split sequence and $X$ an indecomposable $H$-module. Then the exact sequence $0\to\Omega^2(X)\xrightarrow{g}(X\otimes E)_0\xrightarrow{(id_X\otimes f)_0}X\to 0$ has the following properties.

(1) It is either split or almost split.

(2) It is split if and only if $\mathrm{tr}(u^{-1}f')=0$ for all $f'\in\mathrm{End}_H(X)$.

(3) It is almost split if $\mathrm{tr}(u^{-1})\neq 0$.

(4) Suppose $k$ is algebraically closed. Then the sequence is almost split if and only if $\mathrm{tr}(u^{-1})\neq 0$.

**Remark 1** In the case of group algebras, we can take $u=1$ since the square of antipodes of group algebras are identical. In this case, we recover the conclusions in Section V.4 of (Auslander *et al*., 1995).

## APPLICATION TO QUANTUM DOUBLE

Let $H$ be a non-semisimple Hopf algebra. Then its quantum double $D(H)$ is not semisimple and unimodular (Montgomery, 1993). Moreover, it is a braided Hopf algebra (Kassel, 1995) and thus its antipode $S$ satisfying $S^2$ is an inner automorphism (see Proposition VIII.4.1 in (Kassel, 1995)). In fact we can write this inner automorphism explicitly.

Let $\{e_i\}_{i\in I}$ be a basis of the vector space $H$ and $\{e^i\}_{i\in I}$ its dual basis in $(H^{\mathrm{op}})^*$. Then the universal $R$-matrix of $D(H)$ is

$$R=\sum_{i\in I}(1\otimes e_i)(e^i\otimes 1). \tag{9}$$

Let $u=\sum_{i\in I}S(e^i\otimes 1)(1\otimes e_i)$. Then $S^2(x)=uxu^{-1}$ by Proposition VIII.4.1 in (Kassel, 1995). In this case, the inverse of $R$ equals $\sum_{i\in I}(1\otimes e_i)((e^i\circ S)\otimes 1)$ and $u^{-1}=\sum_{i\in I}S^{-1}((e^i\circ S)\otimes 1)(1\otimes e_i)$.

By the above discussion and the conclusions developed in Section 2, we have the following results.

**Proposition 3** Let $H$ be a finite dimensional Hopf algebra, then its quantum double $D(H)$ is a symmetric algebra.

**Theorem 6** Let $H$ be a finite dimensional non-semisimple Hopf algebra, $\{e_i\}_{i\in I}$ be a basis of the vector space $H$ with its dual basis $\{e^i\}_{i\in I}$ in $(H^{\mathrm{op}})^*$ and $0\to\Omega^2(k)\to E\xrightarrow{f}k\to 0$ be an almost split sequence in ${}_{D(H)}M$. Assume $H$ is an indecomposable $D(H)$-module. Then the exact sequence $0\to\Omega^2(X)\xrightarrow{g}(X\otimes E)_0\xrightarrow{(id_X\otimes f)_0}X\to 0$ has the following properties:

(1) It is either split or almost split.

(2) It is split if and only if $\mathrm{tr}\left(\sum_{i\in I} S^{-1}((e^i \circ S) \otimes 1)(1\otimes e_i)f'\right)=0$ for all $f' \in \mathrm{End}_H(X)$.

(3) It is almost split if $\mathrm{tr}\left(\sum_{i\in I} S^{-1}((e^i \circ S)\otimes 1)\cdot (1\otimes e_i)\right)\neq 0$.

(4) Suppose $k$ is algebraically closed. Then the sequence is almost split if and only if $\mathrm{tr}\left(\sum_{i\in I} S^{-1}((e^i \circ S)\otimes 1)(1\otimes e_i)\right)\neq 0$.


**References**

Auslander, M., Reiten, I., Smalø, S.O., 1995. Representation Theory of Artin Algebras. Cambridge University Press, Cambridge.

Böhm, G., Nill, F., Szlachanyi, K., 1999. Weak Hopf algebras I. Integral theory and $C^*$-structure. *J. Algebra*, **221**(2): 385-438. [doi:10.1006/jabr.1999.7984]

Kassel, C., 1995. Quantum Groups. GTM 155. Springer-Verlag, p.127-128.

Lorenz, M., 1997. Representation of finite dimensinal Hopf algebras. *J. Algebra*, **188**(2):476-505. [doi:10.1006/jabr.1996.6827]

Montgomery, S., 1993. Hopf Algebras and Their Actions on Rings. CBMS, Lecture in Math, Providence, RI, **82**: 215-217.


## APPENDIX

We first fix an artin algebra $\Lambda$ and all modules in this appendix are $\Lambda$-modules.

We say that a morphism $f$:$B\to C$ is a split epimorphism if $id_C$:$C\to C$, the identity morphism of $C$, factors through $f$. Dually, we say that a morphism $g$:$A\to B$ is a split monomorphism if $id_A$ factors through $g$. A morphism $f$:$B\to C$ is right almost split if (1) it is not a split epimorphism and (2) any morphism $X\to C$ which is not a split epimorphism factors through $f$. Dually, a morphism is left almost split if (1) it is not a split monomorphism and (2) any morphism $A\to Y$ which is not a split monomorphism factors through $g$. An exact sequence $0\to A\xrightarrow{g} B\xrightarrow{f} C\to 0$ is called an almost split sequence if $g$ is left almost split and $f$ is right almost split.

If $C$ is an indecomposable non-projective module, then there is, up to isomorphisms of short exact sequence, a unique almost split sequence $0\to A\xrightarrow{g} B\xrightarrow{f} C\to 0$. (Theorem 1.15 and 1.16 in Chapter V of (Auslander *et al.*, 1995)).

Dually, for any non-injective indecomposable module $A$, we have, up to isomorphisms of short exact sequence, a unique almost split sequence $0\to A\xrightarrow{g} B\xrightarrow{f} C\to 0$.

Let $A\in {}_{\Lambda}M$ and $P_1\xrightarrow{d_1} P_0\xrightarrow{d_0} A\to 0$ be a minimal projective presentation of $A$. Then Tr($A$):= CokerHom($d_1$,$\Lambda$). Let $D$=Hom$_k$( ,$k$).

The following are equivalent for an exact sequence $0\to A\xrightarrow{g} B\xrightarrow{f} C\to 0$ (Proposition 1.14 in Chapter V of (Auslander *et al.*, 1995)):

(1) The sequence is an almost split sequence.

(2) The module $C$ is isomorphic to Tr$D$($A$) and $g$ is left almost split.

(3) The module $A$ is isomorphic to Tr$D$($C$) and $f$ is right almost split.

Suppose $\cdots\xrightarrow{d_n} P_{n-1}\xrightarrow{d_{n-1}}\cdots\xrightarrow{d_2} P_1\xrightarrow{d_1} P_0\xrightarrow{d_0} A\to 0$ the minimal projective resolution of $A$, then the $i$th syzygy of $A$ defined by $\Omega^i(A)$=Ker($d_{i-1}$) for $i\geq 1$.

If $\Lambda$ is a symmetric algebra, then $D\mathrm{Tr}\cong\Omega^2$ as functors (Proposition 3.8 in Chapter IV of (Auslander *et al.*, 1995)).