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Control mesh representation of a class of minimal surfaces*

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Abstract: Minimal surface is extensively employed in many areas. In this paper, we propose a control mesh representation of a class of minimal surfaces, called generalized helicoid minimal surfaces, which contain the right helicoid and catenoid as special examples. We firstly construct the Bézier-like basis called AHT Bézier basis in the space spanned by $\{1, t, \sin t, \cos t, \sinh t, \cosh t\}$, $t\in[0,\alpha]$, $\alpha\in[0,5\pi/2]$. Then we propose the control mesh representation of the generalized helicoid using the AHT Bézier basis. This kind of representation enables generating the minimal surfaces using the de Casteljau-like algorithm in CAD/CAGD modelling systems.

Key words: Minimal surface, Helicoid surface, Catenoid, Control mesh
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INTRODUCTION

The problem of minimal surface is an old and active problem in the field of differential geometry. The minimal surface has been employed in many areas such as architecture, material science, aviation, ship manufacture, biology, crystallogeny, and so on.

The history of minimal surface began with Lagrange in 1762 (Nitsche, 1989). Many literature on the minimal surface exist in the last two hundred years (Nitsche, 1989; Osserman, 1986), but few on the minimal surface from the point view of CAGD. Jin and Wang (1999) introduced the minimal surface into the field of CAGD. They proposed the geometric construction of Enneper's minimal surface, which is the unique cubic parametric polynomial minimal surface as presented in (Man and Wang, 2002). The approximate solution of minimal surface bounded by Bézier or B-spline curves was presented in (Man and Wang, 2003; Monterde, 2004). The catenoid is the only nonplanar minimal surface of rotation and the right helicoid is the only nonplanar ruled minimal surface. Man and Wang (2005) proposed the control mesh representation of right helicoid and catenoid using the C-Bézier basis and the H-Bézier basis presented in (Zhang, 1996; Li and Wang, 2005). Because the parametric domain of C-Bézier basis is (0, π), so we cannot represent a whole catenoid by C-Bézier basis and H-Bézier basis. Moreover, it is discommodious for design, as we must use two different kinds of bases to represent a surface patch.

The generalized helicoid is a class of important minimal surface proposed by Scherk (Nitsche, 1989). The right helicoid and the catenoid are both its special examples. So it is very meaningful to represent the generalized helicoid by the control mesh. In this paper, we propose the control mesh representation of the generalized helicoid using the AHT Bézier basis in the space spanned by $\{1, t, \sin t, \cos t, \sinh t, \cosh t\}$. So we can generate the generalized helicoid minimal surfaces using the de Casteljau-like algorithm. In particular, we can represent a whole catenoid by control mesh and implement the animated deformation from a right helicoid to a catenoid by the interpolation of control points.

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GENERALIZED HELICOID MINIMAL SURFACE

In this section, we will introduce the generalized helicoid and validate that the generalized helicoid is minimal surface.

If a twisted curve $\boldsymbol{C}$ (i.e., one with torsion τ) rotates about a fixed axis $\boldsymbol{A}$ and, at the same time, is displaced parallel to $\boldsymbol{A}$ such that the displacement rate is always proportional to the angular velocity of rotation, then $\boldsymbol{C}$ generates a generalized helicoid. The general helicoid $\boldsymbol{r}(u,v)=(x(u,v), y(u,v), z(u,v))$ has the following parametric form

$$x(u,v)=a\sinh u\cos v-b\cosh u\sin v,$$
$$y(u,v)=a\sinh u\sin v+b\cosh u\cos v,$$
$$z(u,v)=av+bu,$$

where a and b are arbitrary real numbers. For a=0, it reduces to the catenoid; for b=0, it reduces to the right helicoid. We present two examples of generalized helicoid in Fig.1.

(a)

(b)

Fig.1 The two examples of generalized helicoids
(a) a=1, b=2; (b) a=2, b=1

In the following, we will prove that the generalized helicoid is a minimal surface. The proof is based on the lemma presented in (Man and Wang, 2002).

Lemma 1 The isothermal parametric surface is minimal surface if and only if it is a harmonic surface.

Theorem 1 The generalized helicoid $\boldsymbol{r}(u,v)$ is a minimal surface.

Proof After some computations, we have

$$\begin{aligned}
\boldsymbol{r}_u&=(a\cosh u\cos v-b\sinh u\sin v,\\
&\quad a\cosh u\sin v+b\sinh u\cos v, b),\\
\boldsymbol{r}_v&=(-a\cosh u\sin v-b\sinh u\cos v,\\
&\quad a\sinh u\cos v-b\cosh u\sin v, a),\\
\boldsymbol{r}_{uu}&=(a\sinh u\cos v-b\cosh u\sin v,\\
&\quad -a\sinh u\sin v+b\cosh u\cos v, 0),\\
\boldsymbol{r}_{vv}&=(-a\sinh u\cos v+b\cosh u\sin v,\\
&\quad a\sinh u\sin v-b\cosh u\cos v, 0).
\end{aligned}\tag{1}$$

If E, F and G are coefficients of the first fundamental form of $\boldsymbol{r}(u,v)$, we can obtain

$$E=\boldsymbol{r}_u\cdot\boldsymbol{r}_u=(a^2+b^2)\cosh^2 u,$$
$$F=\boldsymbol{r}_u\cdot\boldsymbol{r}_v=0,$$
$$G=\boldsymbol{r}_v\cdot\boldsymbol{r}_v=(a^2+b^2)\cosh^2 u.$$

Hence, we have E=F, G=0. That is, $\boldsymbol{r}(u,v)$ is an isothermal parametric surface. From Eq.(1), we have $\boldsymbol{r}_{uu}+\boldsymbol{r}_{vv}=0$. So $\boldsymbol{r}(u,v)$ is also a harmonic surface. From Lemma 1, we can derive the conclusion that the generalized helicoid is a minimal surface.

AHT BÉZIER SURFACES

In this section, we will construct the Bézier-like basis called AHT Bézier basis in the space spanned by $\{1, t, \sin t, \cos t, \sinh t, \cosh t\}$, $t\in[0,\alpha]$, $\alpha\in[0,5\pi/2]$. We can construct it recursively with integral method as presented in (Chen and Wang, 2003; Li and Wang, 2005). In this paper, we will construct it explicitly from its zero property.

We denote the AHT Bézier basis by $\{B_{i,5}(t)\}_{i=0}^{5}$.

Let

$$f(t)=(\sinh t+\sin t)/2-t,$$
$$f_i=F^{(i)}(\alpha),\ i=0,\ 1,\ \ldots,\ 5,$$

and denote that

$$e=f_1^2-f_0f_2,\ g=f_0f_3-f_1f_2,$$
$$h=f_2^2-f_1f_3,$$
$$H=\frac{h}{(f_2h+f_3g+f_4e)e},$$

$$s=\sin\alpha,\quad c=\cos\alpha,$$
$$\overline{s}=\sinh\alpha,\ \overline{c}=\cosh\alpha.$$

Then the AHT Bézier basis can be defined as

$$B_{5,5}(t)=\frac{F(t)}{f_0},\quad B_{4,5}(t)=\frac{f_1}{e}\left(F^{(1)}(t)-\frac{f_1F(t)}{f_0}\right),$$
$$B_{3,5}(t)=H(hF(t)+gF^{(1)}(t)+eF^{(2)}(t)),$$
$$B_{2,5}(t)=B_{3,5}(\alpha-t),\quad B_{1,5}(t)=B_{4,5}(\alpha-t),$$
$$B_{0,5}(t)=B_{5,5}(\alpha-t).$$

We can easily obtain that $\{B_{i,5}(t)\}_{i=0}^{5}$ satisfy the following properties (Fig.2):

(1) Partition of unity:

$$B_{i,5}(t)>0,\ \sum_{i=0}^{5}B_{i,5}(t)=1.$$

(2) Properties of the endpoints:

$$B_{0,5}(0)=B_{5,5}(\alpha)=1;$$
$$B_{i,5}^{(j)}(0)=B_{i,5}^{(k)}(\alpha)=0,$$
$$j=0,\ 1,\ ...,\ i-1;\ k=0,\ 1,\ ...,\ 4-i.$$

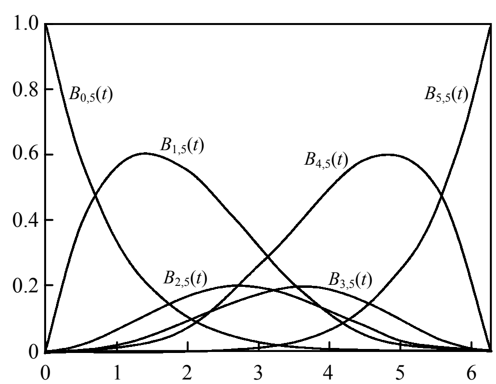


Fig.2 The AHT Bézier basis functions with α=2π

According to the above properties, we can use it for curve and surface design. The tensor product AHT Bézier surface can be defined as follows:

$$\boldsymbol{p}(u,v)=\sum_{i=0}^{5}\sum_{j=0}^{5}B_{i,5}(u)B_{j,5}(v)\boldsymbol{P}_{ij},$$

where $u\in[0,\alpha]$, $v\in[0,\beta]$ $\alpha,\beta\in(0,5\pi/2)$, $B_{i,5}(u)$, $B_{j,5}(v)$ are the AHT Bézier basis functions and $\boldsymbol{P}_{ij}$ is the control point.

In order to represent the generalized helicoid by AHT Bézier surface, we must rewrite the definition of the AHT Bézier basis in matrix form. That is,

$$(B_{0,5}, B_{1,5}, B_{2,5}, B_{3,5}, B_{4,5}, B_{5,5})^{\mathrm{T}}$$
$$=\boldsymbol{T}(1, t, \sin t, \cos t, \sinh t, \cosh t)^{\mathrm{T}},$$

where $\boldsymbol{T}$ is called the transform matrix for $\{B_{i,5}(t)\}_{i=0}^{5}$. We present it in the Appendix A. And the inverse matrix of the transform matrix $\boldsymbol{T}$ is

$$\boldsymbol{S}=\boldsymbol{T}^{-1}=(s_{ij})_{6\times 6}=\begin{pmatrix}1&1&1&1&1&1\\0&\delta&M&\alpha-M&\alpha-\delta&\alpha\\0&\delta&M&Ns-Mc&s-\delta c&s\\1&1&N&Nc+Ms&c+\delta s&c\\0&\delta&M&L\overline{s}-M\overline{c}&\overline{s}-\delta\overline{c}&\overline{s}\\1&1&L&L\overline{c}-M\overline{s}&\overline{c}-\delta\overline{s}&\overline{c}\end{pmatrix},\tag{2}$$

where

$$\delta=\frac{f_0}{f_1},\qquad L=\frac{h-2f_1g+f_0h}{h},$$
$$M=\frac{g}{h},\qquad N=\frac{h+2f_1g-f_0h}{h}.$$

CONTROL MESH REPRESENTATION OF THE GENERALIZED HELICOID

In this section, we will represent the generalized helicoid $\boldsymbol{r}(u,v)$ by AHT Bézier surface. That is, we will find the control points $\boldsymbol{P}_{ij}$ such that

$$\boldsymbol{r}(u,v)=\sum_{i=0}^{5}\sum_{j=0}^{5}B_{i,5}(u)B_{j,5}(v)\boldsymbol{P}_{ij},$$

where $u\in[0,\alpha]$, $v\in[0,\beta]$ $\alpha,\beta\in(0,5\pi/2)$.

Firstly, we can obtain the inverse matrices of the transform matrices for $\{B_{i,5}(u)\}_{i=0}^{5}$ and $\{B_{i,5}(v)\}_{i=0}^{5}$ from Eq.(2). We denote them $\boldsymbol{S}^u=(s_{ij}^u)_{6\times 6}$ and $\boldsymbol{S}^v=(s_{ij}^v)_{6\times 6}$, respectively.

From Eq.(2), we can get that

$$\sinh u=\sum_{i=0}^{5}B_{i,5}(u)s_{5i}^{u},\quad \sin v=\sum_{j=0}^{5}B_{j,5}(v)s_{3j}^{v},$$

$$\cosh u=\sum_{i=0}^{5}B_{i,5}(u)s_{6i}^{u},\quad \cos v=\sum_{j=0}^{5}B_{j,5}(v)s_{4j}^{v},$$

$$u=\sum_{i=0}^{5}B_{i,5}(u)s_{2i}^{u},\qquad v=\sum_{j=0}^{5}B_{j,5}(v)s_{2j}^{v}.$$

Thus,

$$\begin{aligned}x(u,v)&=a\sinh u\cos v-b\cosh u\sin v\\&=a\sum_{i=0}^{5}(B_{i,5}(u)s_{5i}^{u})\sum_{j=0}^{5}(B_{j,5}(v)s_{4j}^{v})\\&\quad-b\sum_{i=0}^{5}(B_{i,5}(u)s_{6i}^{u})\sum_{j=0}^{5}(B_{j,5}(v)s_{3j}^{v})\\&=\sum_{i=0}^{5}B_{i,5}(u)\left(\sum_{j=0}^{5}B_{j,5}(v)(as_{5i}^{u}s_{4j}^{v}-bs_{6i}^{u}s_{3j}^{v})\right).\end{aligned}$$

Analogously, we have

$$y(u,v)=\sum_{i=0}^{5}B_{i,5}(u)\left(\sum_{j=0}^{5}B_{j,5}(v)(as_{5i}^{u}s_{3j}^{v}+bs_{6i}^{u}s_{4j}^{v})\right),$$

$$z(u,v)=\sum_{i=0}^{5}B_{i,5}(u)\left(\sum_{j=0}^{5}B_{j,5}(v)(as_{2j}^{v}+bs_{2i}^{u})\right).$$

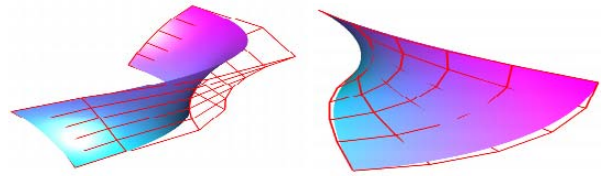
Hence, let

$$\boldsymbol{P}_{ij}=\begin{pmatrix}as_{5i}^{u}s_{4j}^{v}-bs_{6i}^{u}s_{3j}^{v}\\as_{5i}^{u}s_{3j}^{v}+bs_{6i}^{u}s_{4j}^{v}\\as_{2j}^{v}+bs_{2i}^{u}\end{pmatrix},\tag{3}$$

then the generalized helicoid $\boldsymbol{r}(u,v)$ as presented in Eq.(1) can be represented as follows:

$$\boldsymbol{r}(u,v)=\sum_{i=0}^{5}\sum_{j=0}^{5}\left(B_{i,5}(u)B_{j,5}(v)\boldsymbol{P}_{ij}\right),$$

where $u\in[0,\alpha]$, $v\in[0,\beta]$ $\alpha,\beta\in(0,5\pi/2)$. We present two examples in Fig.3.



(a) (b)

Fig.3 The control mesh representation of generalized helicoids. (a) *a*=1, *b*=2; (b) *a*=2, *b*=1

DYNAMIC DEFORMATION FROM HELICOID TO CATENOID

From Section 1, we know that for $a=0$, $b=1$, the generalized helicoid reduces to the catenoid; for $a=1$, $b=0$, it reduces to the right helicoid. So we can achieve the dynamic deformation from the right helicoid to catenoid by interpolation of the control points.

Let

$$a=f(t),\ b=g(t),\ t\in[\xi_1,\xi_2],$$

such that

$$f(\xi_1)=1,\ g(\xi_1)=0, f(\xi_2)=0,\ g(\xi_2)=1.$$

Let $t=(1-\lambda)\xi_1+\lambda\xi_2$. Then when λ varies from 0 to 1, the right helicoid can be continuously deformed into a catenoid, with each intermediate surface being minimal surface. Their control meshes can be obtained by Eq.(3). Furthermore, Eq.(3) can be rewritten as follows:

$$\boldsymbol{P}_{ij}=a\begin{pmatrix}s_{5i}^{u}s_{4j}^{v}\\s_{5i}^{u}s_{3j}^{v}\\s_{2j}^{v}\end{pmatrix}+b\begin{pmatrix}-s_{6i}^{u}s_{3j}^{v}\\s_{6i}^{u}s_{4j}^{v}\\s_{2i}^{u}\end{pmatrix}.$$

So the control meshes of the intermediate surfaces can be obtained by interpolation of the control points at time $\lambda=0$ and $\lambda=1$.

Generally speaking, there are many choices for $f(t)$ and $g(t)$. For example, we can set

$$f(t)=1-t,\ g(t)=t,\ t\in[0,1],$$
$$f(t)=\cos t,\ g(t)=\sin t,\ t\in[0,\pi/2],$$

or

$$f(t)=u_{0,n}(t), f(t)=u_{n,n}(t),\ t\in[0,1],$$

where $u_{0,n}(t)$ and $u_{n,n}(t)$ are Bernstein basis functions of degree n (Figs.4 and 5).

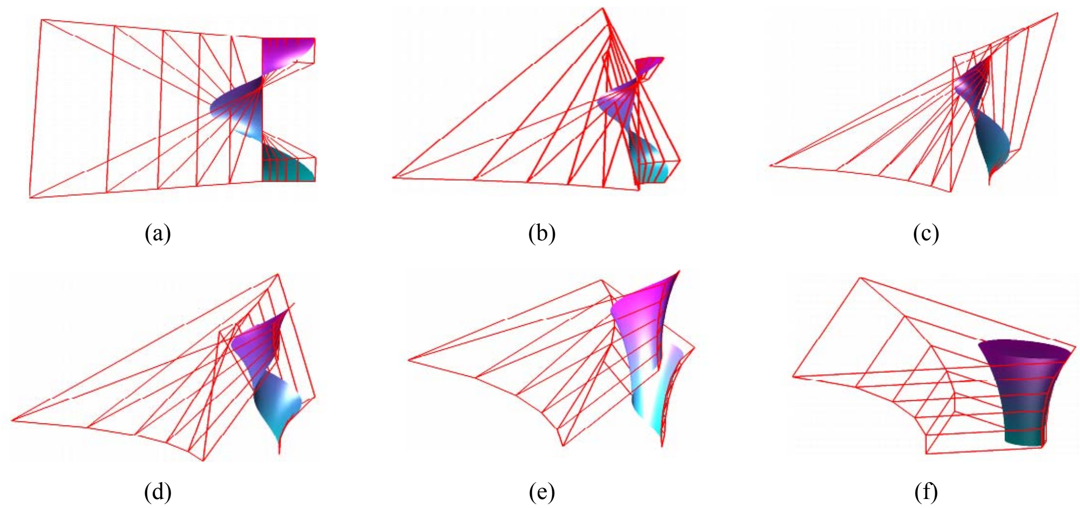


Fig.4 The dynamic deformation from right helicoid to catenoid with α=1.5, β=2π, $f(t)$=1−t, $g(t)$=t, t∈[0,1]. (a) λ=0; (a) λ=0.2; (a) λ=0.4; (a) λ=0.6; (a) λ=0.8; (a) λ=1.0

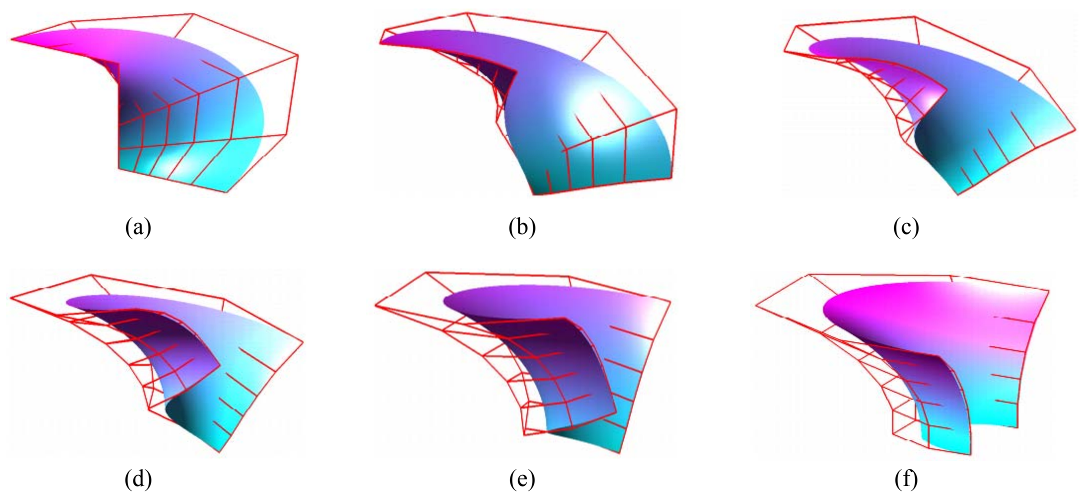


Fig.5 The dynamic deformation from right helicoid to catenoid with α=1.5, β=2π, $f(t)$=cos$t$, $g(t)$=sin$t$, t∈[0,2/π]. (a) λ=0; (a) λ=0.2; (a) λ=0.4; (a) λ=0.6; (a) λ=0.8; (a) λ=1.0

CONCLUSION AND FUTURE WORK

In this paper, we propose the control mesh representation of a class of minimal surfaces using the AHT Bézier basis in the space spanned by {1, t, sin$t$, cos$t$, sinh$t$, cosh$t$}. This kind of representation enables generating the minimal surfaces using the subdivision rules in CAD/CAGD modelling systems.

The minimal surface is very important in geometric theory and has many applications in practice. We will continue to do some research on minimal surface such as trimming, approximation and their applications in future.

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APPENDIX A

$$(B_{0,5}, B_{1,5}, B_{2,5}, B_{3,5}, B_{4,5}, B_{5,5})^{\mathrm{T}} = \boldsymbol{T}(1, t, \sin t, \cos t, \sinh t, \cosh t)^{\mathrm{T}},$$

where

$$\boldsymbol{T} = \begin{pmatrix} -\frac{\alpha}{f_0} & \frac{1}{f_0} & -\frac{c}{2f_0} & \frac{s}{2f_0} & -\frac{\overline{c}}{2f_0} & \frac{\overline{s}}{2f_0} \\ \frac{T_{21}}{ef_0} & -\frac{f_1^2}{ef_0} & \frac{T_{23}}{2ef_0} & \frac{T_{24}}{2ef_0} & \frac{T_{25}}{2ef_0} & \frac{T_{26}}{2ef_0} \\ \frac{T_{31}H}{2} & -hH & \frac{T_{33}}{2}H & \frac{T_{34}}{2}H & -\frac{T_{35}}{2}H & \frac{T_{36}}{2}H \\ \frac{-gH}{2} & hH & \frac{(h-e)H}{2} & \frac{gH}{2} & \frac{(h+e)H}{2} & \frac{gH}{2} \\ -\frac{f_1}{e} & \frac{f_1^2}{ef_0} & -\frac{f_1^2}{2ef_0} & \frac{f_1}{2ef_0} & -\frac{f_1^2}{2ef_0} & \frac{f_1}{2ef_0} \\ 0 & -\frac{1}{f_0} & \frac{1}{2f_0} & 0 & \frac{1}{2f_0} & 0 \end{pmatrix}$$

$$T_{21} = f_1(\alpha f_1 - f_0), \quad T_{23} = f_1(sf_0 + cf_1),$$
$$T_{24} = f_1(cf_0 - sf_1), \quad T_{25} = f_1(\overline{c}f_1 - \overline{s}f_0),$$
$$T_{26} = f_1(\overline{c}f_0 - \overline{s}f_1), \quad T_{31} = (h-g),$$
$$T_{33} = gs - (h-e)c, \quad T_{34} = gc + (h-e)s,$$
$$T_{35} = g\overline{s} + (h+e)\overline{c}, \quad T_{36} = g\overline{c} + (h+e)\overline{s}.$$