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A family of quasi-cubic blended splines and applications*

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Abstract: A class of quasi-cubic B-spline base functions by trigonometric polynomials are established which inherit properties similar to those of cubic B-spline bases. The corresponding curves with a shape parameter α, defined by the introduced base functions, include the B-spline curves and can approximate the B-spline curves from both sides. The curves can be adjusted easily by using the shape parameter α, where $\mathrm{d}\boldsymbol{p}_i(\alpha,t)$ is linear with respect to $\mathrm{d}\alpha$ for the fixed t. With the shape parameter chosen properly, the defined curves can be used to precisely represent straight line segments, parabola segments, circular arcs and some transcendental curves, and the corresponding tensor product surfaces can also represent spherical surfaces, cylindrical surfaces and some transcendental surfaces exactly. By abandoning positive property, this paper proposes a new C^2 continuous blended interpolation spline based on piecewise trigonometric polynomials associated with a sequence of local parameters. Illustration showed that the curves and surfaces constructed by the blended spline can be adjusted easily and freely. The blended interpolation spline curves can be shape-preserving with proper local parameters since these local parameters can be considered to be the magnification ratio to the length of tangent vectors at the interpolating points. The idea is extended to produce blended spline surfaces.

Key words: Blended spline interpolation, C^2 continuity, Global parameters, Local parameters, Quasi-cubic spline, Trigonometric polynomials

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INTRODUCTION

Bézier curves and uniform B-spline curves are powerful tools for constructing free form curves and surfaces (FFC/FFS). But they cannot represent the arcs, hyperbola, sphere, cylinders and other transcendental curves and surfaces exactly. In order to avoid the inconveniences, many bases are presented in other new spaces (Zhang, 1996; Peña, 1997; Walz, 1997; Sánchez-Reyes, 1998; Mainar *et al*., 2001). Note that, these existing methods can deal with both polynomial curves and some transcendental curves precisely due to the blending of polynomial and trigonometric functions. But the shapes of these curves based on the above methods can be adjusted with complicated procedures by the shape parameter α.

On the other hand, interpolation spline is sometimes requisite for the curves and surfaces modeling. Whereas in constructing interpolation spline we need to calculate the data points conversely using the above-mentioned methods. Additional control parameters are also desired frequently for shape manipulations freely in reverse engineering and shape representation. Many papers investigated these problems. A method (Kochanek and Bartels, 1984) for using cubic interpolating splines was presented, which used three control parameters to change the tension and continuity of the splines with each of these parameters being used for either local or global control. B-spline representation of Hermite splines via multiple knots technique was provided by

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imposing some restrictions on the differential values at segment boundaries (Grisoni *et al*., 1999). A C^2 continuous spline scheme (Tai and Loe, 1999) was proposed which called the α-spline and provided weights and tension control. The scheme was based on blending a sequence of singular reparametrized line segments with a piecewise NURBS curve, but the tangent vectors degenerated to zero when the scheme interpolated the control points. The B-spline technique was adopted for the blending of Hermite interpolants (Gfrerrer and Röchel, 2001). An alternative method of curve interpolation based on B-spline curves was offered by Piegl *et al*.(2005). The method used a base curve, which was subjected to constrained shape manipulations to achieve interpolation.

The main contribution of this paper is exhibited in two aspects:

First, we establish quasi-cubic B-spline curves by trigonometric polynomials which inherit properties similar to those of B-spline curves. The curves defined by trigonometric polynomials can be adjusted easily by using the shape parameter α, where $\mathrm{d}\boldsymbol{p}_i(\alpha,t)$ is linear with respect to the $\mathrm{d}\alpha$ for the fixed t. With the shape parameters chosen properly, the trigonometric polynomial curves can be used to represent straight lines, circular arcs and some transcendental curves precisely. The corresponding tensor product surfaces can also represent sphere and some quadratic surfaces exactly.

Second, we construct a new C^2 continuous blended interpolation spline based on the quasi-cubic B-spline curves associated with a sequence of local parameters which can be considered to be the magnification ratio to the length of tangent vectors at the interpolation points. Illustration showed that the curves and surfaces constructed by the blended spline can be adjusted easily and freely.

The rest of this paper is organized as follows. Section 2 defines the quasi-cubic B-spline basis functions, the quasi-cubic B-spline curves and discusses the related propositions. In Section 3, we present the representations of circular arcs, elliptic arcs, parabola, sine curves, etc. by using the introduced curves. We also present shape modeling by the introduced surfaces in the section. In Section 4, quasi-cubic blended interpolation spline is constructed. We detail the methods of localization and the applications of the blended interpolation spline curves and surfaces. Finally, we present some remarks and conclude the paper in Section 5.

QUASI-CUBIC B-SPLINE AND THEIR RELATED PROPOSITIONS

Given knots $u_0<u_1<\ldots<u_{n+4}$. We refer to $\boldsymbol{U}=u_0, u_1,\ldots,u_{n+4}$ as a knot vector. For an arbitrarily selected real value of α ($\alpha\in[0,1]$) and all possible $i\in\mathbb{Z}^+$, let $t_i=(u-u_i)/(u_{i+1}-u_i)$.

Definition 1 $B_0(t)$, $B_1(t)$, $B_2(t)$ and $B_3(t)$ are called quasi-cubic B-spline base functions which can be defined to be (Figs.1 and 2)

$$\begin{aligned}
B_0(t)&=\frac{1}{4}-\frac{1-\alpha}{4}t-\frac{\alpha}{3}\sin\frac{\pi}{2}t-\frac{\alpha}{12}\cos\pi t-\frac{1-\alpha}{4\pi}\sin\pi t,\\
B_1(t)&=\frac{2-\alpha}{4}-\frac{1-\alpha}{4}t+\frac{\alpha}{3}\cos\frac{\pi}{2}t+\frac{\alpha}{12}\cos\pi t+\frac{1-\alpha}{4\pi}\sin\pi t,\\
B_2(t)&=\frac{1}{4}+\frac{1-\alpha}{4}t+\frac{\alpha}{3}\sin\frac{\pi}{2}t-\frac{\alpha}{12}\cos\pi t+\frac{1-\alpha}{4\pi}\sin\pi t,\\
B_3(t)&=\frac{\alpha}{4}+\frac{1-\alpha}{4}t-\frac{\alpha}{3}\cos\frac{\pi}{2}t+\frac{\alpha}{12}\cos\pi t-\frac{1-\alpha}{4\pi}\sin\pi t.
\end{aligned}\qquad(1)$$

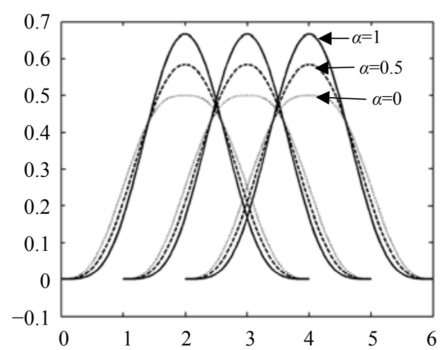


Fig.1 The figure of quasi-cubic B-spline base functions on the knot vector

Proposition 1 (Weight property and nonnegative property) The quasi-cubic B-spline bases are normalized and positive. That is,

$$B_0(t)+B_1(t)+B_2(t)+B_3(t)\equiv1,\qquad(2)$$

$$B_i(t)\geq0,\ i=0,1,2,3.\qquad(3)$$

Note: From the properties of trigonometric func-

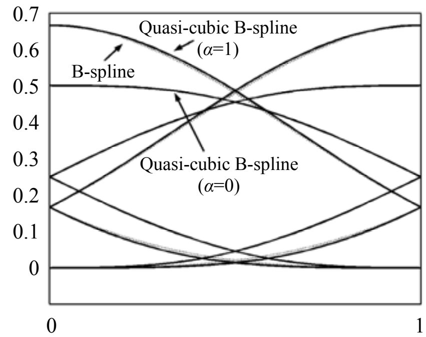


Fig.2 The figure of quasi-cubic B-spline base functions on $t\in[0, 1]$

tions, we find that when $t\in[0,1]$, $B_0(t)$ and $B_1(t)$ are monotonously decreasing while $B_2(t)$ and $B_3(t)$ are monotonously increasing. The nonnegativity of $B_i(t)$ (i=0, 1, 2, 3) can be proved simply.

Note: In view of Proposition 1, we say that the base functions form a partition of unity.

Proposition 2 (Symmetry) From Eq.(1), we know that these base functions are symmetric, namely,

$$B_{i,3}(t)=B_{3-i,3}(1-t),\ i=0, 1, 2, 3,\ t\in[0,1]. \tag{4}$$

Definition 2 Let $\boldsymbol{b}_0$, $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, …, $\boldsymbol{b}_{n+2}$ ($n\geq1$) be given control points, and take an arbitrarily selected real value of α with $\alpha\in[0, 1]$. Then quasi-cubic piecewise B-spline curves with parameter α are defined as (Fig.3)

$$\boldsymbol{p}_i(t)=\sum_{j=0}^{3}B_j(t)\boldsymbol{b}_{i+j}, \tag{5}$$

where $t\in[0,1]$, $B_0(t)$, $B_1(t)$, $B_2(t)$ and $B_3(t)$ are quasi-cubic B-spline base functions.

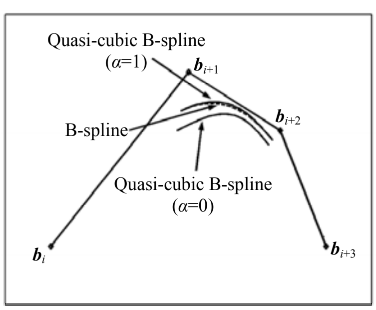


Fig.3 A family of quasi-cubic B-spline curves with fixed control points and varied α

We can also construct tensor product quasi-cubic B-spline surfaces exactly as in the construction of tensor product B-spline surfaces by

$$\boldsymbol{s}_{i,j}(u,v)=\sum_{k=i}^{i+3}\sum_{l=j}^{j+3}B_{k,3}(u)B_{l,3}(v)\boldsymbol{b}_{kl},\quad (u,v)\in[0,1]\times[0,1], \tag{6}$$

where $B_{k,3}(u)$, $B_{l,3}(v)$ are quasi-cubic B-spline base functions and $[\boldsymbol{b}_{kl}]$ ($k=i$, …, $i+3$; $l=j$, …, $j+3$; $i=1$, 2, …, n; $j=1$, 2, …, m) are the control meshes. Many properties of quasi-cubic B-spline curves can be extended to the quasi-cubic B-spline surfaces. For example, the convex hull property and the convexity preserving property also hold for the surface scheme.

Quasi-cubic B-spline curves and surfaces have the following properties:

(1) Geometric properties at the endpoints:

Proposition 3 Set $l_1(\alpha)=\dfrac{3-\alpha}{12}$, $l_2(\alpha)=\dfrac{3+\alpha}{6}$, $l_3(\alpha)=\dfrac{3+(\pi-3)\alpha}{6}$, $l_4(\alpha)=\dfrac{\pi^2\alpha}{12}$,

$$\begin{aligned}&\boldsymbol{p}_i(0)=l_1(\alpha)\boldsymbol{b}_i+l_2(\alpha)\boldsymbol{b}_{i+1}+l_1(\alpha)\boldsymbol{b}_{i+2},\\&\boldsymbol{p}_i(1)=l_1(\alpha)\boldsymbol{b}_{i+1}+l_2(\alpha)\boldsymbol{b}_{i+2}+l_1(\alpha)\boldsymbol{b}_{i+3},\\&\boldsymbol{p}_i'(0)=l_3(\alpha)(\boldsymbol{b}_{i+2}-\boldsymbol{b}_i),\\&\boldsymbol{p}_i'(1)=l_3(\alpha)(\boldsymbol{b}_{i+3}-\boldsymbol{b}_{i+1}),\\&\boldsymbol{p}_i''(0)=l_4(\alpha)(\boldsymbol{b}_i-2\boldsymbol{b}_{i+1}+\boldsymbol{b}_{i+2}),\\&\boldsymbol{p}_i''(1)=l_4(\alpha)(\boldsymbol{b}_{i+1}-2\boldsymbol{b}_{i+2}+\boldsymbol{b}_{i+3}).\end{aligned} \tag{7}$$

From the above properties, it is obvious that

$$\boldsymbol{p}_i(1)=\boldsymbol{p}_{i+1}(0),\ \boldsymbol{p}_i^{(l)}(1)=\boldsymbol{p}_{i+1}^{(l)}(0), \tag{8}$$

where l=1, 2; i=0, 1, 2, …, $n-1$.

Therefore, the continuity of quasi-cubic B-spline curves is established up to second derivatives.

(2) Symmetry: From Eq.(4) and Eq.(5), for the same α, both $\boldsymbol{b}_0$, $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, …, $\boldsymbol{b}_{n+2}$ and $\boldsymbol{b}_{n+2}$, $\boldsymbol{b}_{n+1}$, …, $\boldsymbol{b}_1$, $\boldsymbol{b}_0$ ($n\geq1$) define the same curve in a different parameterization:

$$\boldsymbol{p}_i(t, \alpha, \boldsymbol{b}_i, \boldsymbol{b}_{i+1}, \boldsymbol{b}_{i+2}, \boldsymbol{b}_{i+3})=\boldsymbol{p}_i(1-t, \alpha, \boldsymbol{b}_{i+3}, \boldsymbol{b}_{i+2}, \boldsymbol{b}_{i+1}, \boldsymbol{b}_i),\quad 0\leq t\leq1;\ 0\leq\alpha\leq1;\ i=0, 1, 2, …, n-1. \tag{9}$$

(3) Geometric invariability: The shapes of quasi-cubic B-spline curves are independent of the choice of coordinates. An affine transformation for

the curves can be achieved by carrying out the affine transformation for the control polygon.

(4) Convex hull property: The entire quasi-cubic B-spline curve segment with $\boldsymbol{p}_i(t)$ must lie inside its control polygon span by $\boldsymbol{b}_i$, $\boldsymbol{b}_{i+1}$, $\boldsymbol{b}_{i+2}$, $\boldsymbol{b}_{i+3}$.

Note: This property is the results of the partition of unity.

(5) The functions of parameter α: By introducing the parameter α, the quasi-cubic B-spline curves possess more wonderful abilities of representation than cubic B-spline curves. When the four control points are fixed, we can get a family of quasi-cubic B-spline curves which include B-spline curves by adjusting the parameter α. The quasi-cubic B-spline curve approaches control polygon as the parameter α tends to 1. On the contrary, the quasi-cubic B-spline curve approaches chord $\boldsymbol{b}_i\boldsymbol{b}_{i+3}$ (Fig.4).

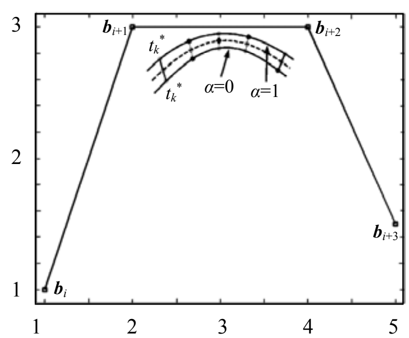


Fig.4 The functions of control parameter α

Proposition 4 When t is fixed and α varies by dα, an arbitrary point $\boldsymbol{p}_i(\alpha,t)$ on the quasi-cubic B-spline curves will produce a displacement of d$\boldsymbol{p}_i(\alpha,t)$ such that

$$\begin{aligned}\mathrm{d}\boldsymbol{p}_i(\alpha,t)=\Big[&-\frac{1}{4}(\boldsymbol{b}_{i+1}-\boldsymbol{b}_{i+3})+\frac{1}{4}t(\boldsymbol{b}_i+\boldsymbol{b}_{i+1}-\boldsymbol{b}_{i+2}-\boldsymbol{b}_{i+3})\\&-\frac{1}{3}\sin\frac{\pi}{2}t(\boldsymbol{b}_i-\boldsymbol{b}_{i+2})+\frac{1}{3}\cos\frac{\pi}{2}t(\boldsymbol{b}_{i+1}-\boldsymbol{b}_{i+3})\\&+\frac{1}{12}\cos\pi t(-\boldsymbol{b}_i+\boldsymbol{b}_{i+1}-\boldsymbol{b}_{i+2}+\boldsymbol{b}_{i+3})\\&+\frac{1}{4\pi}\sin\pi t(\boldsymbol{b}_i-\boldsymbol{b}_{i+1}-\boldsymbol{b}_{i+2}+\boldsymbol{b}_{i+3})\Big]\mathrm{d}\alpha\end{aligned} \tag{10}$$

Observing the right-hand side of Eq.(10), we find that d$\boldsymbol{p}_i(\alpha,t)$ is linear with respect to dα for the fixed t.

THE SHAPE MODELING BY THE QUASI-CUBIC B-SPLINE CURVES AND SURFACES

Let $\boldsymbol{b}_{i,x}$ denote the projection of point $\boldsymbol{b}_i$ at x-axis and $\boldsymbol{b}_{i,y}$ denote the projection of point $\boldsymbol{b}_i$ at y-axis, i=0, 1, 2, 3.

Proposition 5 Suppose $\boldsymbol{b}_i$, $\boldsymbol{b}_{i+1}$, $\boldsymbol{b}_{i+2}$, $\boldsymbol{b}_{i+3}$ are four control points. If they are collinear, or degenerate to two points with multiple knots, then their corresponding quasi-cubic B-spline curve with parameter α represents a line segment.

Note: From the definition of quasi-cubic B-spline curves and properties of corresponding base functions, we can get the result easily.

Proposition 6 Let $\boldsymbol{b}_i$=$\boldsymbol{b}_{i+2}$, $\boldsymbol{b}_{i+1}$, $\boldsymbol{b}_{i+2}$, $\boldsymbol{b}_{i+3}$ be three vertexes of some isosceles triangle and $\boldsymbol{b}_{i+1}\boldsymbol{b}_{i+3}$ be the bottom edge. Then their corresponding quasi-cubic B-spline curve with parameter α=1 represents a segment of parabola (Fig.5).

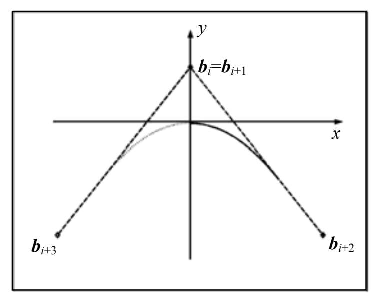


Fig.5 The representation of parabola by quasi-cubic B-spline curves

Proof Suppose $\boldsymbol{b}_i$=$\boldsymbol{b}_{i+2}$, $\boldsymbol{b}_{i+1}$, $\boldsymbol{b}_{i+2}$ and $\boldsymbol{b}_{i+3}$ are three vertexes of some isosceles triangle, $\boldsymbol{b}_{i+1}\boldsymbol{b}_{i+3}$ is the bottom edge. We may suppose the line segment joining $\boldsymbol{b}_{i+1}$ and $\boldsymbol{b}_{i+3}$ is parallel to the x-axis. Then $\boldsymbol{b}_{i+1,y}$=$\boldsymbol{b}_{i+3,y}$≠$\boldsymbol{b}_{i+2,y}$, $\boldsymbol{b}_{i+2,x}$ is the midpoint of line segment $\boldsymbol{b}_{i+1,x}\boldsymbol{b}_{i+3,x}$ in the Cartesian orthogonal coordinate system as Fig.5. So

$$\boldsymbol{p}_i(t)=[x(t)\ \ y(t)]^{\mathrm{T}}=\begin{bmatrix}\boldsymbol{b}_{i+2,x}+\dfrac{1}{3}(\boldsymbol{b}_{i+1,x}-\boldsymbol{b}_{i+3,x})\cos\dfrac{\pi}{2}t\\ \dfrac{1}{3}\boldsymbol{b}_{i+1,y}+\dfrac{2}{3}\boldsymbol{b}_{i+2,y}+\dfrac{1}{3}(\boldsymbol{b}_{i+1,y}-\boldsymbol{b}_{i+2,y})\left(\cos\dfrac{\pi}{2}t\right)^2\end{bmatrix}.$$

Let $u=\cos\dfrac{\pi}{2}t$. Then $\boldsymbol{p}_i(u)$ ($u\in[0,1]$) represents a segment of parabola.

Proposition 7 Let $\boldsymbol{b}_i$, $\boldsymbol{b}_{i+1}$, $\boldsymbol{b}_{i+2}$, $\boldsymbol{b}_{i+3}$ be four vertexes of some rhombus. Then their corresponding quasi-cubic B-spline curve with parameter $\alpha=1$ represents a segment of elliptic arc (Fig.6).

Proof Suppose $\boldsymbol{b}_i$, $\boldsymbol{b}_{i+1}$, $\boldsymbol{b}_{i+2}$, $\boldsymbol{b}_{i+3}$ are four vertexes of some rhombus and they are four control points of quasi-cubic B-spline curve with parameter $\alpha=1$. We may suppose the line segment joining $\boldsymbol{b}_{i+1}$ and $\boldsymbol{b}_{i+3}$ is parallel to the x-axis. Then $\boldsymbol{b}_{i,x}=\boldsymbol{b}_{i+2,x}$, $\boldsymbol{b}_{i+1,y}=\boldsymbol{b}_{i+3,y}$ and $\boldsymbol{b}_i-\boldsymbol{b}_{i+1}=\boldsymbol{b}_{i+3}-\boldsymbol{b}_{i+2}$ for the proper Cartesian orthogonal coordinate system. So

$$\boldsymbol{p}(t)=\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}=\begin{bmatrix} \boldsymbol{b}_{i,x}+\frac{1}{3}(\boldsymbol{b}_{i+1,x}-\boldsymbol{b}_{i+3,x})\cos\frac{\pi}{2}t \\ \boldsymbol{b}_{i+1,y}+\frac{1}{3}(\boldsymbol{b}_{i+2,y}-\boldsymbol{b}_{i,y})\sin\frac{\pi}{2}t \end{bmatrix}, t\in[0,1].$$

Therefore their corresponding quasi-cubic B-spline curve with parameter α represents a segment of elliptic arc.

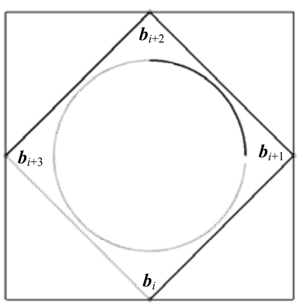


Fig.6 The representation of elliptic arc by quasi-cubic B-spline curves

Proposition 8 Let $\boldsymbol{b}_{i,x}+\boldsymbol{b}_{i+3,x}=\boldsymbol{b}_{i+1,x}+\boldsymbol{b}_{i+2,x}$, $\boldsymbol{b}_{i+1,x}\neq\boldsymbol{b}_{i+3,x}$, $\boldsymbol{b}_{i,y}+\boldsymbol{b}_{i+1,y}=\boldsymbol{b}_{i+2,y}+\boldsymbol{b}_{i+3,y}$ and $\boldsymbol{b}_{i+1,y}\neq\boldsymbol{b}_{i+3,y}$. Then their corresponding quasi-cubic B-spline curve with parameter $\alpha=0$ represents a segment of sine curve (Fig.7).

Proof Let $\boldsymbol{b}_i$, $\boldsymbol{b}_{i+1}$, $\boldsymbol{b}_{i+2}$, $\boldsymbol{b}_{i+3}$ be four control points of quasi-cubic B-spline curve with parameter $\alpha=0$, and $\boldsymbol{b}_{i,x}+\boldsymbol{b}_{i+3,x}=\boldsymbol{b}_{i+1,x}+\boldsymbol{b}_{i+2,x}$, $\boldsymbol{b}_{i+1,x}\neq\boldsymbol{b}_{i+3,x}$, $\boldsymbol{b}_{i+1,y}\neq\boldsymbol{b}_{i+3,y}$ and $\boldsymbol{b}_{i,y}+\boldsymbol{b}_{i+1,y}=\boldsymbol{b}_{i+2,y}+\boldsymbol{b}_{i+3,y}$. Then we have

$$\boldsymbol{p}(t)=[x(t)\ \ y(t)]^{\mathrm{T}}=\begin{bmatrix} \frac{1}{4}(\boldsymbol{b}_{i,x}+2\boldsymbol{b}_{i+1,x}+\boldsymbol{b}_{i+2,x})-\frac{1}{2}(\boldsymbol{b}_{i+1,x}-\boldsymbol{b}_{i+3,x})t \\ \frac{1}{4}(\boldsymbol{b}_{i,y}+2\boldsymbol{b}_{i+1,y}+\boldsymbol{b}_{i+2,y})+\frac{1}{2\pi}(\boldsymbol{b}_{i+1,y}-\boldsymbol{b}_{i+3,y})\sin\pi t \end{bmatrix}, \quad t\in[0,1].$$

So their corresponding quasi-cubic B-spline curve represents a segment of sine curve.

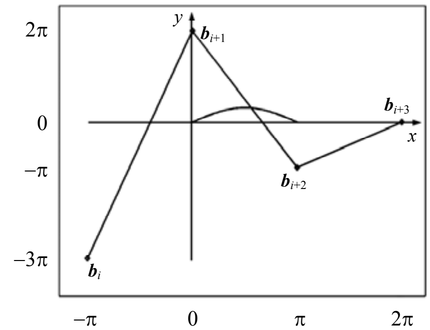


Fig.7 The representation of a segment of sine curve by quasi-cubic B-spline curves

Some examples of the shape representation by tensor product quasi-bicubic B-spline surfaces are offered as follows: Fig.8 denotes a bowl-shaped surface by four pieces of C^2 continuous quasi-bicubic B-spline surfaces, Fig.8b renders the surfaces of Fig.8a with the projection of the profile on the xy-plane. Fig.9 indicates a triarticular pipeline, where each cylinder is associated with four pieces of C^2 continuous quasi-bicubic B-spline surfaces and Fig.10 depicts a tread surface by four pieces of C^2 continuous quasi-bicubic B-spline surfaces.

QUASI-CUBIC BLENDED INTERPOLATION SPLINE

The intention of this section is to construct a quasi-cubic blended interpolation spline for interpolating given points. Suppose $\alpha=3$, we have the following results by Eqs.(1) and (7):

$$\boldsymbol{p}_i(0)=\boldsymbol{b}_{i+1},\ \boldsymbol{p}_i(1)=\boldsymbol{b}_{i+2},\ \boldsymbol{p}_i'(0)=(\pi/2-1)(\boldsymbol{b}_{i+2}-\boldsymbol{b}_i),$$
$$\boldsymbol{p}_i'(1)=(\pi/2-1)(\boldsymbol{b}_{i+3}-\boldsymbol{b}_{i+1}),\ \boldsymbol{p}_i''(0)=\pi^2(\boldsymbol{b}_i-2\boldsymbol{b}_{i+1}+\boldsymbol{b}_{i+2})/4,$$
$$\boldsymbol{p}_i''(1)=\pi^2(\boldsymbol{b}_{i+1}-2\boldsymbol{b}_{i+2}+\boldsymbol{b}_{i+3})/4. \quad (11)$$

We denote $B_i(t)$ with $B_i^3(t)$ ($t\in[0,1]$, i=0, 1, 2, 3) for $\alpha=3$ in Eq.(1). We call $B_i^3(t)$ ($t\in[0,1]$, i=0, 1, 2, 3) as interpolation spline base functions. Apparently, these base functions $B_i^3(t)$ possess all the properties of quasi-cubic B-spline base functions except for positive property. Now we define piecewise interpolation spline curves.

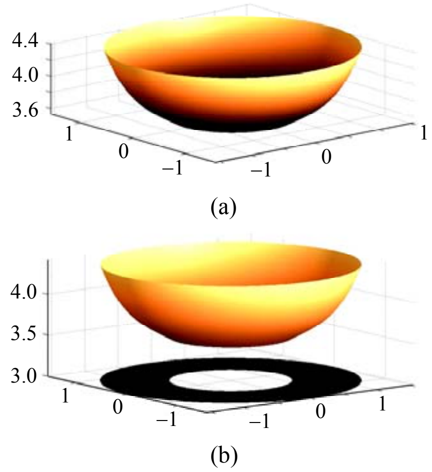


Fig.8 A bowl-shaped surfaces modeled by tensor product quasi-bicubic B-spline surfaces. (a) Four pieces of C^2 continuous quasi-bicubic B-spline surfaces; (b) The surfaces with the projection of the profile

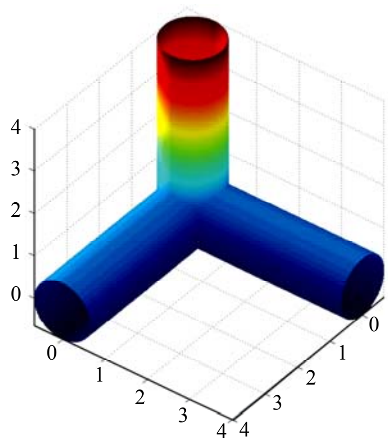


Fig.9 A triarticular pipeline modeled by twelve pieces of C^2 continuous quasi-bicubic B-spline surfaces

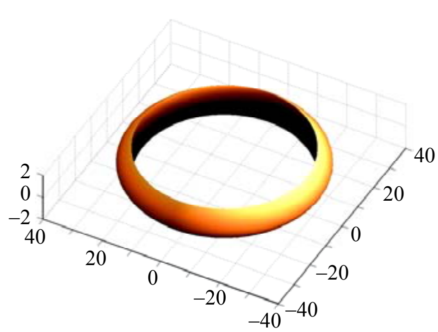


Fig.10 A tread surface modeled by four pieces of C^2 continuous quasi-bicubic B-spline surfaces

Given a set of control points $\{\boldsymbol{b}_i, i=0, 1, \ldots, n\}$, we define a sequence of curve segments as follows:

$$\tilde{\boldsymbol{p}}_i(t)=\sum_{j=0}^{3}B_j^3(t)\boldsymbol{b}_{i+j},\ t\in[0,1]. \tag{12}$$

If given two initial points $\boldsymbol{b}_{-1}$ and $\boldsymbol{b}_{n+1}$, we can get the interpolating properties with $\tilde{\boldsymbol{p}}_i(0)=\boldsymbol{b}_{i+1}$, $\tilde{\boldsymbol{p}}_i(1)=\boldsymbol{b}_{i+2}$, and $\tilde{\boldsymbol{p}}_i(t)$ ($i=-1, 0, \ldots, n-2$) are C^2 continuous from Eq.(11).

Localization of parameters

Piecewise quasi-cubic B-spline curves $\boldsymbol{p}_i(t)$ defined by Eq.(1) have a control parameter α, but the parameter must be global in order to obtain continuity as in the case of (Zhang, 1996), which means that the parameter of every segment of the curves is consistent affirmatively. On the other hand, piecewise interpolation spline curves $\tilde{\boldsymbol{p}}_i(t)$ have no parameter, which means that the solution to the interpolation problem is global and unique given an initial knot vector, i.e., if the resulting shape is not satisfactory, there is no degree of freedom that would allow the user to change this shape. So providing local control parameters at every interpolating point in $\tilde{\boldsymbol{p}}_i(t)$ ($i=-1, 0, 1, \ldots, n-2$) is necessary and useful for shape modeling. The method of localization with respect to control parameters in this section may follow the case in (Tai and Loe, 1999) analogously.

Let $k\in\mathbb{N}$, $k\geq 2$. An allocated function can be defined to be

$$s(t)=\sum_{i=0}^{k}\binom{2k+1}{i}(1-t)^i t^{2k+1-i},\ t\in[0,1]. \tag{13}$$

Simple calculation confirms that

$$s(0)=0,\ s(1)=1,\ s^{(l)}(0)=s^{(l)}(1)=0,\ 1\leq l< k. \tag{14}$$

Considering the ith segment of piecewise interpolation spline curves $\tilde{\boldsymbol{p}}_i(t)$, we construct the linear combination of interpolation points $\boldsymbol{b}_{i+1}$ and $\boldsymbol{b}_{i+2}$ by the allocated function $s(t)$,

$$\boldsymbol{L}_i(t)=(1-s(t))\boldsymbol{b}_{i+1}+s(t)\boldsymbol{b}_{i+2},$$
$$t\in[0,1],\ i=-1, 0, 1, \ldots, n-2. \tag{15}$$

From Eq.(14), we get

$$\boldsymbol{L}_i(0)=\boldsymbol{b}_{i+1},\ \boldsymbol{L}_i(1)=\boldsymbol{b}_{i+2},$$

$$\boldsymbol{L}_i^{(l)}(0) = \boldsymbol{L}_i^{(l)}(1) = 0,\ 1 \le l < k. \tag{16}$$

Given a sequence of local control parameters β_i^* associated with corresponding interpolation points $\boldsymbol{b}_i$ (i=0, 1, …, n), we define

$$\beta_i(t) = (1 - s(t))\beta_{i+1}^* + s(t)\beta_{i+2}^*, \tag{17}$$

where $\beta_i^* \in \mathbb{R}$, i=−1, 0, 1, …, n−2.

Now we define piecewise blended interpolation spline (BIS) curves via Eqs.(12), (15) and (17) as follows:

$$\boldsymbol{BIS}_i(t) = (1 - \beta_i(t))\boldsymbol{L}_i(t) + \beta_i(t)\tilde{\boldsymbol{p}}_i(t),\quad t \in [0,1]. \tag{18}$$

Note that

$$\begin{aligned}\boldsymbol{BIS}_i'(t) &= \boldsymbol{L}_i'(t) + \beta_i'(t)\left(\tilde{\boldsymbol{p}}_i(t) - \boldsymbol{L}_i(t)\right) \\ &\quad + \beta_i(t)\left(\tilde{\boldsymbol{p}}_i'(t) - \boldsymbol{L}_i'(t)\right), \\ \boldsymbol{BIS}_i''(t) &= \boldsymbol{L}_i''(t) + \beta_i''(t)\left(\tilde{\boldsymbol{p}}_i(t) - \boldsymbol{L}_i(t)\right) + 2\beta_i'(t) \\ &\quad \cdot\left(\tilde{\boldsymbol{p}}_i'(t) - \boldsymbol{L}_i'(t)\right) + \beta_i(t)\left(\tilde{\boldsymbol{p}}_i''(t) - \boldsymbol{L}_i''(t)\right),\end{aligned}$$

so

$$\begin{aligned}&\boldsymbol{BIS}_i(0) = \left(1 - \beta_{i+1}^*\right)\boldsymbol{b}_{i+1} + \beta_{i+1}^*\boldsymbol{b}_{i+1} = \boldsymbol{b}_{i+1}, \\ &\boldsymbol{BIS}_i(1) = \left(1 - \beta_{i+2}^*\right)\boldsymbol{b}_{i+2} + \beta_{i+2}^*\boldsymbol{b}_{i+2} = \boldsymbol{b}_{i+2}, \\ &\boldsymbol{BIS}_i'(0) = \beta_{i+1}^*\tilde{\boldsymbol{p}}_i'(0),\quad \boldsymbol{BIS}_i'(1) = \beta_{i+2}^*\tilde{\boldsymbol{p}}_i'(1), \\ &\boldsymbol{BIS}_i''(0) = \beta_{i+1}^*\tilde{\boldsymbol{p}}_i''(0),\quad \boldsymbol{BIS}_i''(1) = \beta_{i+2}^*\tilde{\boldsymbol{p}}_i''(1), \\ &\boldsymbol{BIS}_i^{(l)}(1) = \boldsymbol{BIS}_{i+1}^{(l)}(0),\ l = 0,1,2,\end{aligned} \tag{19}$$

i.e., the blended interpolation spline $\boldsymbol{BIS}_i(t)$ (i=−1, 0, 1, …, n−2) is C^2 continuous.

Fig.11 and Fig.12 show the effects of the varying local parameters. All the interpolating points have parameters equal to one except for $\boldsymbol{b}_{i+2}$ where β_{i+2}^* varies from 0 to 1 in the four curves with no inflexion in Fig.11. β_{i+2}^* varies from 0 to 2 in the four curves with two inflexions in Fig.12.

The idea can also be extended to construct tensor product surfaces.

Let

$$\tilde{\boldsymbol{p}}_{i,j}(u,v) = \sum_{k=i}^{i+3}\sum_{l=j}^{j+3}\left(B_k^3(u)B_l^3(v)\boldsymbol{b}_{k,l}\right), \tag{20}$$

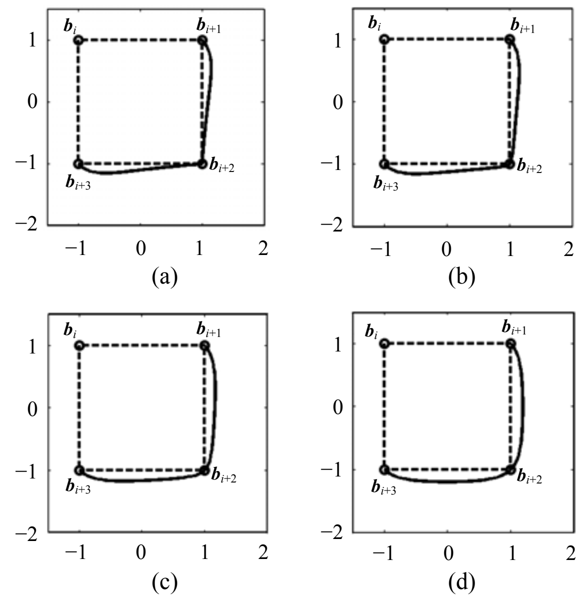


Fig.11 Effects of the local parameters: all interpolating points have local parameters equal to one except $\boldsymbol{b}_{i+2}$ where β_{i+2}^* is respectively 0, 0.3, 0.7, 1 from (a) to (d)

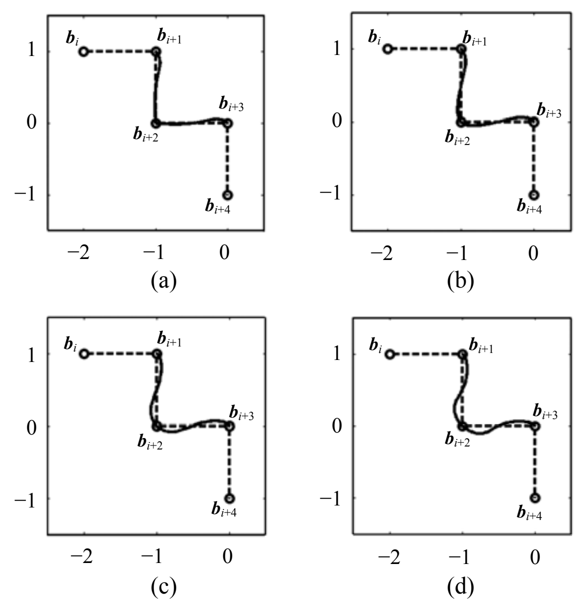


Fig.12 Effects of the local parameters: all interpolating points have local parameters equal to one except $\boldsymbol{b}_{i+2}$ where β_{i+2}^* is respectively 0, 0.7, 1.3, 1.8 from (a) to (d)

$$\begin{aligned}\boldsymbol{L}_{i,j}(u,v) &= (1 - s(u))(1 - s(v))\boldsymbol{b}_{i+1,j+1} + (1 - s(u))s(v) \\ &\quad \cdot\boldsymbol{b}_{i+1,j+2} + s(u)(1 - s(v))\boldsymbol{b}_{i+2,j+1} + s(u)s(v)\boldsymbol{b}_{i+2,j+2},\end{aligned} \tag{21}$$

$$\begin{aligned}\beta_{i,j}(u,v) &= (1 - s(u))(1 - s(v))\beta_{i+1,j+1}^* + (1 - s(u)) \\ &\cdot s(v)\beta_{i+1,j+2}^* + s(u)(1 - s(v))\beta_{i+2,j+1}^* + s(u)s(v)\beta_{i+2,j+2}^*,\end{aligned} \tag{22}$$

where $\tilde{\boldsymbol{p}}_{i,j}(u,v)$ are tensor product interpolation spline surfaces, $\boldsymbol{L}_{i,j}(u,v)$ are allocated functions, $\beta^*_{i+m,j+n}$ $(m,n=1,2)$ are local parameters corresponding to the interpolation points $\boldsymbol{b}_{i+m,j+n}$ $(m,n=1,2)$, and $(u,v)\in[0,1]\times[0,1]$.

By the properties of the allocated functions, it is easy to show that

$$\boldsymbol{L}_{i,j}(0,0)=\boldsymbol{b}_{i+1,j+1},\ \boldsymbol{L}_{i,j}(1,0)=\boldsymbol{b}_{i+2,j+1},\ \boldsymbol{L}_{i,j}(0,1)=\boldsymbol{b}_{i+1,j+2},\ \boldsymbol{L}_{i,j}(1,1)=\boldsymbol{b}_{i+2,j+2}, \tag{23}$$

$$\begin{aligned}&\partial_u^{(l)}\boldsymbol{L}_{i,j}(0,v)=\partial_u^{(l)}\boldsymbol{L}_{i,j}(1,v)=0,\\&\partial_v^{(l)}\boldsymbol{L}_{i,j}(u,0)=\partial_v^{(l)}\boldsymbol{L}_{i,j}(u,1)=0,\\&\partial_{uv}\boldsymbol{L}_{i,j}(u,0)=\partial_{uv}\boldsymbol{L}_{i,j}(u,1)=0,\\&\partial_{uv}\boldsymbol{L}_{i,j}(0,v)=\partial_{uv}\boldsymbol{L}_{i,j}(1,v)=0,\end{aligned} \tag{24}$$

$$\beta_{i,j}(0,0)=\beta^*_{i+1,j+1},\quad \beta_{i,j}(1,0)=\beta^*_{i+2,j+1},\quad \beta_{i,j}(0,1)=\beta^*_{i+1,j+2},\quad \beta_{i,j}(1,1)=\beta^*_{i+2,j+2}, \tag{25}$$

$$\begin{aligned}&\partial_u^{(l)}\beta_{i,j}(0,v)=\partial_u^{(l)}\beta_{i,j}(1,v)=0,\\&\partial_v^{(l)}\beta_{i,j}(u,0)=\partial_v^{(l)}\beta_{i,j}(u,1)=0,\\&\partial_{uv}\beta_{i,j}(u,0)=\partial_{uv}\beta_{i,j}(u,1)=0,\\&\partial_{uv}\beta_{i,j}(0,v)=\partial_{uv}\beta_{i,j}(1,v)=0,\end{aligned} \tag{26}$$

where $l=1,2$.

We define piecewise blended interpolation spline surfaces to be (Fig.13)

$$\boldsymbol{BIS}_{i,j}(u,v)=(1-\beta_{i,j}(u,v))\boldsymbol{L}_{i,j}(u,v)+\beta_{i,j}(u,v)\tilde{\boldsymbol{p}}_{i,j}(u,v),\quad (u,v)\in[0,1]\times[0,1], \tag{27}$$

so

$$\boldsymbol{BIS}_{i,j}(0,0)=\boldsymbol{b}_{i+1,j+1},\ \boldsymbol{BIS}_{i,j}(1,0)=\boldsymbol{b}_{i+2,j+1},\ \boldsymbol{BIS}_{i,j}(0,1)=\boldsymbol{b}_{i+1,j+2},\ \boldsymbol{BIS}_{i,j}(1,1)=\boldsymbol{b}_{i+2,j+2}, \tag{28}$$

$$\begin{aligned}&\partial_u^{(l)}\boldsymbol{BIS}_{i,j}(u_0,v)=\beta_{i,j}(u_0,v)\partial_u^{(l)}\tilde{\boldsymbol{p}}_{i,j}(u_0,v),\\&\partial_u^{(l)}\boldsymbol{BIS}_{i,j}(u,v_0)=\beta_{i,j}(u,v_0)\partial_u^{(l)}\tilde{\boldsymbol{p}}_{i,j}(u,v_0),\\&\partial_v^{(l)}\boldsymbol{BIS}_{i,j}(u_0,v)=\beta_{i,j}(u_0,v)\partial_v^{(l)}\tilde{\boldsymbol{p}}_{i,j}(u_0,v),\\&\partial_v^{(l)}\boldsymbol{BIS}_{i,j}(u,v_0)=\beta_{i,j}(u,v_0)\partial_v^{(l)}\tilde{\boldsymbol{p}}_{i,j}(u,v_0),\\&\partial_{uv}\boldsymbol{BIS}_{i,j}(u_0,v)=\beta_{i,j}(u_0,v)\partial_{uv}\tilde{\boldsymbol{p}}_{i,j}(u_0,v),\\&\partial_{uv}\boldsymbol{BIS}_{i,j}(u,v_0)=\beta_{i,j}(u,v_0)\partial_{uv}\tilde{\boldsymbol{p}}_{i,j}(u,v_0),\end{aligned} \tag{29}$$

where $l=1, 2$; $u_0, v_0=0, 1$.

By Eq.(22), we know

$$\beta_{i+1,j}(0,v)=(1-s(v))\beta^*_{i+2,j+1}+s(v)\beta^*_{i+2,j+2},$$

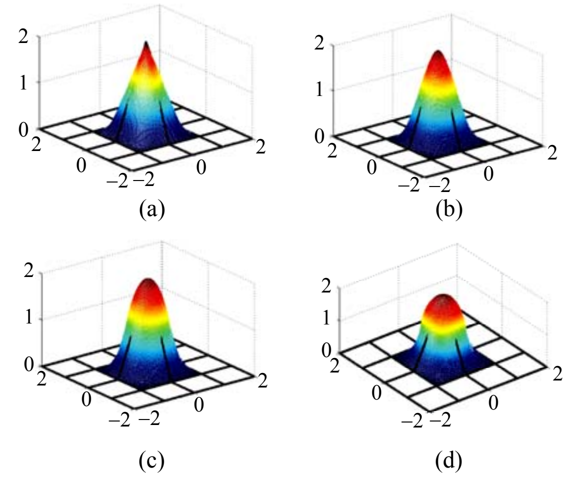


Fig.13 Effects of the local parameters associated with blended interpolation surfaces: all interpolating points have local parameters equal to one except $\boldsymbol{b}_{i+2,j+2}$ where $\beta^*_{i+2,j+2}$ is respectively 0, 0.7, 1.3, 1.8 from (a) to (d)

$$\beta_{i,j}(1,v)=(1-s(v))\beta^*_{i+2,j+1}+s(v)\beta^*_{i+2,j+2},$$

$$\beta_{i,j+1}(u,0)=(1-s(u))\beta^*_{i+1,j+2}+s(u)\beta^*_{i+2,j+2},$$

$$\beta_{i,j}(u,1)=(1-s(u))\beta^*_{i+1,j+2}+s(u)\beta^*_{i+2,j+2}.$$

So

$$\beta_{i,j}(1,v)=\beta_{i+1,j}(0,v),\quad \beta_{i,j}(u,1)=\beta_{i,j+1}(u,0). \tag{30}$$

From Eqs.(28), (29) and (30) it follows that piecewise blended interpolation spline surfaces $\boldsymbol{BIS}_{i,j}(u,v)$ are C^2 continuous.

Fairing characteristic of the blended interpolation spline curves

As mentioned above, the shape of blended interpolation spline curves and surfaces can be adjusted by local parameters. We can accommodate the length of the tangent vectors at the interpolating points without changing their directions. Adjusting these local parameters properly proves useful if fairing the curves is needed.

Let $l^*=|\boldsymbol{b}_{i+2}-\boldsymbol{b}_i|$, $c_0=\pi/2-1$, $c_1=\pi^2/4$ and $c^*=c_0c_1$. Suppose $\boldsymbol{a}_i=\boldsymbol{b}_i-\boldsymbol{b}_{i-1}$ are edge vectors, $\boldsymbol{T}_i$ are tangent vectors (Fig.14) and $a_i=|\boldsymbol{a}_i|$.

By Eq.(11) and Eq.(19), we know

$$\boldsymbol{BIS}'_i(0)=\beta^*_{i+1}\tilde{\boldsymbol{p}}'_i(0)=\beta^*_{i+1}c_0(\boldsymbol{b}_{i+2}-\boldsymbol{b}_i)$$

$$=\beta_{i+1}^{*}c_0(\boldsymbol{a}_{i+2}+\boldsymbol{a}_{i+1}),$$

$$\boldsymbol{BIS}_i''(0)=\beta_{i+1}^{*}\tilde{\boldsymbol{p}}_i''(0)=\beta_{i+1}^{*}c_1(\boldsymbol{b}_i-2\boldsymbol{b}_{i+1}+\boldsymbol{b}_{i+2})$$

$$=\beta_{i+1}^{*}c_1(\boldsymbol{a}_{i+2}-\boldsymbol{a}_{i+1}),$$

$$\boldsymbol{BIS}_i'(1)=\beta_{i+2}^{*}\tilde{\boldsymbol{p}}_i'(1)=\beta_{i+1}^{*}c_0(\boldsymbol{b}_{i+3}-\boldsymbol{b}_{i+1})$$

$$=\beta_{i+1}^{*}c_0(\boldsymbol{a}_{i+3}+\boldsymbol{a}_{i+2}),$$

$$\boldsymbol{BIS}_i''(1)=\beta_{i+2}^{*}\tilde{\boldsymbol{p}}_i''(1)=\beta_{i+1}^{*}c_1(\boldsymbol{b}_{i+1}-2\boldsymbol{b}_{i+2}+\boldsymbol{b}_{i+3})$$

$$=\beta_{i+1}^{*}c_1(\boldsymbol{a}_{i+3}-\boldsymbol{a}_{i+2}).$$

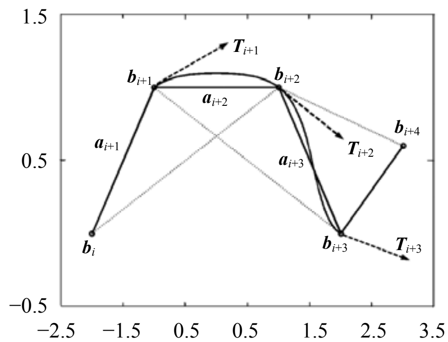


Fig.14 The figure shows two segments of blended interpolation curves in which the first segment has no inflexion and the second segment has an inflexion

Let κ_i denote the curvature of ith blended interpolation spline curve. Then

$$\kappa_i(0)=\frac{|\boldsymbol{BIS}_i'(0)\times\boldsymbol{BIS}_i''(0)|}{|\boldsymbol{BIS}_i'(0)|^3}. \tag{31}$$

So

$$\kappa_i(0)=\frac{2(\beta_{i+1}^{*})^2c^{*}|\boldsymbol{a}_{i+2}\times\boldsymbol{a}_{i+1}|}{(\beta_{i+1}^{*})^3(c_0)^3(|\boldsymbol{b}_{i+2}-\boldsymbol{b}_i|)^3}. \tag{32}$$

Let R_i be the radius of a circle which passes through the points $\boldsymbol{b}_i$, $\boldsymbol{b}_{i+1}$, $\boldsymbol{b}_{i+2}$, $K_i=1/R_i$ and Δ_i denote the directed area of triangle $\boldsymbol{b}_i\boldsymbol{b}_{i+1}\boldsymbol{b}_{i+2}$.

So

$$K_i=\frac{4\Delta_i}{|\boldsymbol{a}_{i+1}||\boldsymbol{a}_{i+2}|l^{*}}. \tag{33}$$

From Eqs.(32) and (33), we get

$$\frac{\kappa_i(0)}{K_i}=\frac{c_1a_{i+1}a_{i+2}}{(c_0)^2\beta_{i+1}^{*}(l^{*})^2}. \tag{34}$$

Suppose

$$\lambda_i=\frac{a_{i+2}}{a_{i+1}+a_{i+2}},\mu_i=\frac{a_{i+1}}{a_{i+1}+a_{i+2}},$$

then

$$\frac{\kappa_i(0)}{K_i}=\frac{c_1}{(c_0)^2\beta_{i+1}^{*}}\lambda_i\mu_i+O(\varphi_i^2), \tag{35}$$

where φ_i is the included angle of $\boldsymbol{b}_i\boldsymbol{b}_{i+1}$ and $\boldsymbol{b}_{i+1}\boldsymbol{b}_{i+2}$.

If

$$\beta_{i+1}^{*}=\frac{c_1}{(c_0)^2}\lambda_i\mu_i, \tag{36}$$

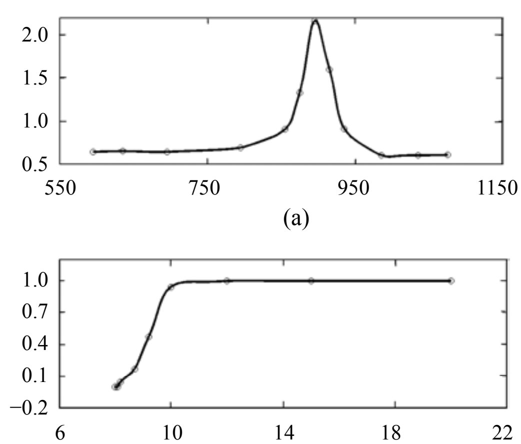
then

$$\kappa_i(0)=K_i+O(\varphi_i^2),\ i=1,2,...,n-1. \tag{37}$$

In fact, Eqs.(31)~(37) imply the following theorem:

Theorem 1 If β_{i+1}^{*} (i=1, 2, …, n−1) satisfies Eq.(36) and the sequence K_i (i=1, 2, …, n−1) of circular curvature with interpolation points is smooth, then the blended interpolation spline curves associated with local parameters β_{i+1}^{*} are also smooth.

Fig.15 indicates the shape-preserving effects by the blended interpolation spline with local parameters β_i^{*} under the two common test datasets: Titanium Heat Data sets and Fritsch-Carlson RPN 14 Radiochemical Data.



(b)

Fig.15 Effects of shape-preserving interpolation
(a) Titanium Data; (b) RPN 14 Data

CONCLUSION AND DISCUSSIONS

We have established quasi-cubic B-spline curves by trigonometric polynomials which inherit similar properties of B-spline curves. The curves and surfaces can be used to represent straight lines, circular arcs, sine curves, sphere and some quadratic surfaces exactly. Furthermore, we have proposed a new interpolation spline formulation based on the quasi-cubic B-spline curves associated with a sequence of local parameters and an allocated function $s(t)$. The interpolation spline is C^2 continuous, and the corresponding interpolation curves and surfaces constructed by the blended spline can be adjusted easily and freely.

In addition, we provide discussions about the paper by some remarks.

Remark 1 The control parameter α of the quasi-cubic B-spline base functions belongs to the domain [0, 1] for having the properties of weight property and positive property, etc. In fact, the range of α can be extended to [−3, 1] which can also enable these base functions to keep the weight property and positive property, but then the quasi-cubic B-spline curves may produce a cusp or multiple point when $\alpha<0$. When α is extended to (1, 3], we can get the interpolation curves for $\alpha=3$, but the corresponding base functions do not satisfy positive property.

Remark 2 In Section 4, we discussed the blended interpolation spline associated with $B_i^3(t)$ (i=0, 1, 2, 3). Similarly, we can also blend the quasi-cubic B-spline base functions $B_i(t)$ (i=0, 1, 2, 3, $\alpha\in[0,1]$) with the allocated functions $\boldsymbol{L}_i(t)$. Here, the blended spline will not interpolate the control points except that the local parameters β_i^* (i=1, 2, …, n−1) equal zero, which will produce a cusp at the corresponding control points as in the case of (Tai and Loe, 1999). These additional local parameters can adjust the shapes of quasi-cubic B-spline curves and surfaces freely. Fig.16 indicates the effects of the local parameters associated with the blended surfaces of $\boldsymbol{s}_{i,j}(u,v)$ in Eq.(6) and $\boldsymbol{L}_{i,j}(u,v)$ in Eq.(21): where the global parameter $\alpha=0.6$ is fixed and all control points have local parameters equal to one except $\boldsymbol{b}_{i+2,j+2}$ where $\beta_{i+2,j+2}^*$ is respectively 0, 0.3, 0.7, 0.9 from (a) to (d). Fig.17 indicates the effects of the global parameters α associated with the blended surfaces of $\boldsymbol{s}_{i,j}(u,v)$ in Eq.(6) and $\boldsymbol{L}_{i,j}(u,v)$ in Eq.(21): where all control points have accordant local parameters $\beta_{i+2,j+2}^*$ equal to 0.8 and the global parameter α is respectively 0, 0.3, 0.7, 1 from (a) to (d).

Remark 3 In Section 4, we also discussed fairing conditions of the blended interpolation spline associated with local parameters β_i^*. We can estimate it by

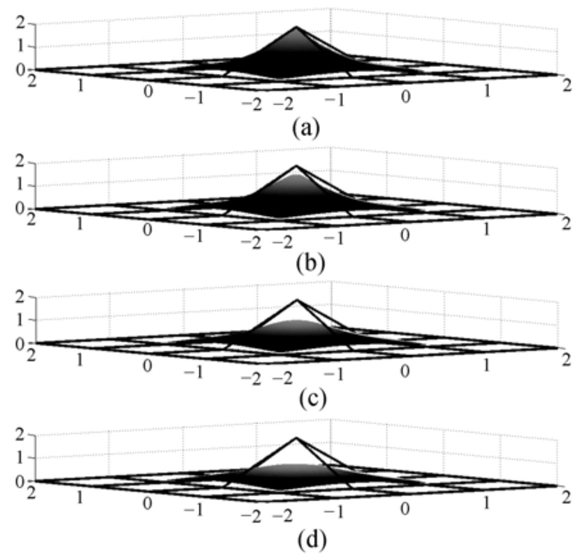


Fig.16 Effects of the local parameters associated with blended surfaces: where the global parameter α=0.6 is fixed and all control points have local parameters equal to one except $\boldsymbol{b}_{i+2,j+2}$ where $\beta_{i+2,j+2}^*$ is respectively 0, 0.3, 0.7, 0.9 from (a) to (d)

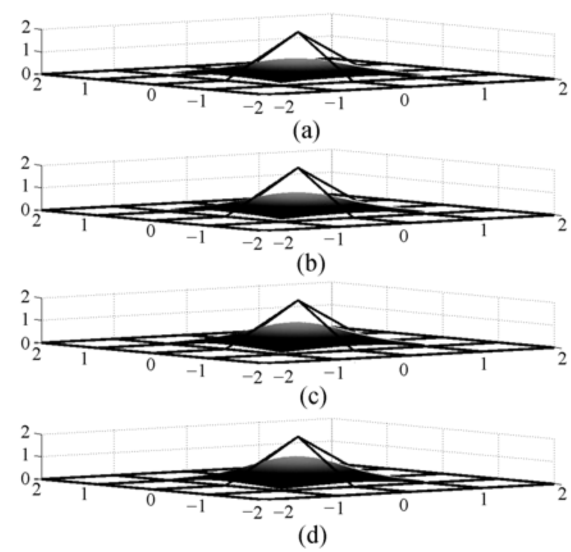


Fig.17 Effects of the global parameters associated with blended surfaces: where all control points have the accordant local parameters equal to 0.8 and the global parameter α is respectively 0, 0.3, 0.7, 1 from (a) to (d)

circular curvature sequences of corresponding interpolation points, but we do not give the explicit expressions of the curvatures of the blended interpolation spline for the shape-preserving purpose. So further investigation is still to be done in this respect.

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