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Rational offset approximation of rational Bézier curves*

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Abstract: The problem of parametric speed approximation of a rational curve is raised in this paper. Offset curves are widely used in various applications. As for the reason that in most cases the offset curves do not preserve the same polynomial or rational polynomial representations, it arouses difficulty in applications. Thus approximation methods have been introduced to solve this problem. In this paper, it has been pointed out that the crux of offset curve approximation lies in the approximation of parametric speed. Based on the Jacobi polynomial approximation theory with endpoints interpolation, an algebraic rational approximation algorithm of offset curve, which preserves the direction of normal, is presented.

Key words: Rational Bézier curve, Parametric speed, Offset, Rational approximation
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INTRODUCTION

Offset curves/surfaces, also called parallel curves/surfaces, are defined as the locus of the points which are at constant distance along the normal from the generator curves/surfaces. As for a planar generator curve Γ:$\boldsymbol{C}(t)=(x(t),y(t))$, the parametric speed and its norm $\sigma(t)$ are defined respectively as (Farouki, 1992)

$$\boldsymbol{C}'(t)=(x'(t),y'(t)),\ \sigma(t)=\sqrt{x'^2(t)+y'^2(t)}. \quad (1)$$

Subsequently the offset curve of the generator curve, which is at constant distance d along the norm can be represented as

$$\boldsymbol{C}_{\mathrm{O}}(t)=\boldsymbol{C}(t)\pm d\boldsymbol{N}(t)=\boldsymbol{C}(t)\pm d\frac{(-y'(t),x'(t))}{\sigma(t)}, \quad (2)$$

here $\boldsymbol{N}(t)$ represents the identity normal of the curve $\boldsymbol{C}(t)$.

Research on offset curves/surfaces has gradually become a hot topic, as offsets are widely used in various applications in computer graphics and numerical control machining. For example, in the area of access path design in robotics and in 3D numerical control machining generation, offset calculations are widely used (Maekawa, 1999).

From the representation of offset, it can be easily seen that apart from some certain curves/surfaces (such as line, circle, plane, sphere, circular conical surface, circular cylindrical surface, circular ring surface, etc.), Pythagorean hodograph curves (Farouki and Shah, 1996) and Offset-Rational curves (Lü, 1995), most offset curves and surfaces do not have the same representations as their generator curves/surfaces have (Farouki and Neff, 1990). This will result in excess of a certain range of representations which CAD/CAM system can present. And poor stability in calculation and unclosed representations will occur too. Thus approximation of offset curves or surfaces is of great importance.

In recent years, much work has been done in

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offset approximation of polynomial Bézier curves/ surfaces. For example, the way of moving the same distance of the control points (Cobb, 1984; Tiller and Hanson, 1984; Coquillart, 1987; Elber and Cohen, 1991), the way of approximating using the envelope curve of radix circles (Lee *et al*., 1996), the method based on interpolation or fitting (Hoschek, 1988; Pham, 1988; Sederberg and Buehler, 1992; Piegl and Tiller, 1999) and the method of avoiding self-meeting (Chiang *et al*., 1991) have all been studied. As to the problem of offset approximation of rational Bézier curves, few researches have been done on it because of the intrinsic complexity of the problem.

Taking convenience in representations and operations into consideration, polynomial curves have been most commonly used to approximate the offset curve. But this will result in high degree of the approximation curve. Otherwise, as to reducing the error, discrete operation on the generator curve should be taken frequently, which directly results in huge amount of data being stored. On the other hand, the obtained approximating curves in recent methods do not preserve the property of being a locus of the points which are at a constant distance along the normal from the generator curves. If user needs to alter the offset distance, then approximating work should be started from scratch to create a new offset curve. This is a critical disadvantage in real time alternation.

From the representation of offset curve, it can be easily seen that the reason why offset curve does not have the polynomial or rational polynomial form lies in the problem that parametric speed cannot be rationalized. Based on the Jacobi polynomials approximation theory, approximation to the parametric speed of rational Bézier curve is given in this paper. Furthermore new algorithm for rational approximation of rational Bézier curve which preserves the direction of the normal is presented. Example provided in the paper showed that this new algorithm makes effective saving in data storage, as well as provides excellent approximation results.

OFFSET APPROXIMATION OF RATIONAL BÉZIER CURVES

Let $\boldsymbol{R}(t)$ be a planar rational Bézier curve with control points $\boldsymbol{R}_i=\{x_i,y_i\}$ and weights w_i (i=0, 1, …, n). $\boldsymbol{R}(t)$ can be expressed as

$$\boldsymbol{R}(t)=(R_x(t),R_y(t))=\left(\frac{x(t)}{w(t)},\frac{y(t)}{w(t)}\right)=\frac{\sum_{i=0}^{n}\boldsymbol{R}_i w_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)},\quad 0\le t\le 1. \tag{3}$$

Then the offset curves with distance d to the normal can be represented as

$$\begin{aligned}\boldsymbol{R}_{\mathrm{O}}(t)&=\boldsymbol{R}(t)\pm d\cdot\boldsymbol{N}(t)\\&=\boldsymbol{R}(t)\pm d\cdot\frac{(y(t)w'(t)-y'(t)w(t),x'(t)w(t)-x(t)w'(t))}{\delta(t)},\\&\qquad 0\le t\le 1.\end{aligned} \tag{4}$$

where

$$\delta(t)=\sqrt{\left(x'(t)w(t)-x(t)w'(t)\right)^2+(y'(t)w(t)-y(t)w'(t))^2}. \tag{5}$$

The parametric speed of the above rational Bézier curve is

$$\sigma(t)=\sqrt{R_x^2(t)+R_y^2(t)}=\frac{\delta(t)}{w^2(t)}. \tag{6}$$

It is not difficult to see that the obtained offset curve does not preserve rational polynomial form simply because the function $\delta(t)$ cannot be rationalized. Consequently we have to approximate the offset curve.

In the following, the best least squares Jacobi approximation to the function $\delta(t)$ is provided first.

Jacobi polynomial $J_n^{(r,s)}(x)$ is a kind of orthogonal polynomial with weight function as $w^{(r,s)}(x)=(1+x)^r(1-x)^s$. Jacobi polynomial can be explicitly expressed as a linear combination of Bernstein polynomials (Szego, 1975)

$$J_n^{(r,s)}(x)=\sum_{i=0}^{n}(-1)^{n+i}\frac{\binom{n+r}{i}\binom{n+s}{n-i}}{\binom{n}{i}}B_i^n\left(\frac{x+1}{2}\right),\quad n=0,1,\ldots, \tag{7}$$

where $x\in[-1,1]$, r, $s>-1$.

In the latter part of the paper, the restricted Jacobi polynomial will be defined as

$$\tilde{J}_{n,r,s}(x)=(1+x)^{r+1}(1-x)^{s+1}J_{n-r-s-2}^{(2r+2,2s+2)}(x),$$
$$n=r+s+2,\ r+s+3,\ \ldots$$

A set of orthogonal polynomial basis is composed of the above restricted Jacobi polynomials with weight function $w(x)=1$. Besides, it has $r+1$, $s+1$ repeated roots at $x=-1$, $+1$ respectively.

From approximation theory, it can be learned that the polynomial function

$$\tilde{\sigma}_N^{\mathrm{J}}(t)=\sigma(0)B_0^N(t)+\sum_{k=2}^{N}c_k\tilde{J}_{k,0,0}(2t-1)+\sigma(1)B_N^N(t) \tag{8}$$

is the best least squares approximation to the parametric speed $\sigma(t)$ with degree N, where

$$c_k=\frac{1}{\alpha_i}\int_{-1}^{1}\left(\sigma\left(\frac{t+1}{2}\right)-\sigma(0)B_0^N\left(\frac{t+1}{2}\right)-\sigma(1)B_N^N\left(\frac{t+1}{2}\right)\right)\tilde{J}_{k,0,0}(t)\mathrm{d}t,$$

$$\alpha_i=\int_{-1}^{1}\left(\tilde{J}_{i,0,0}(t)\right)^2\mathrm{d}t.$$

At the same time it also has interpolation property at $t=0$, 1.

The coefficients c_k in restricted Jacobi series $\tilde{\sigma}_N^{\mathrm{J}}(t)$ can be calculated by Gauss-Legendre integration (Szego, 1975). This numerical calculation is simple, stable, and has small truncation error.

Then let $h(t)=\delta(0)B_0^N(t)+\delta(1)B_N^N(t)$. From the above Jacobi approximation theory, polynomial function

$$\tilde{\delta}_N^{\mathrm{J}}(t)=h(t)+\sum_{k=2}^{N}g_k\tilde{J}_{k,0,0}(2t-1) \tag{9}$$

is the best least squares approximation to the function $\sigma(t)$ with degree N on [0,1], where

$$g_k=\frac{\int_{-1}^{1}\left(\delta\left(\frac{t+1}{2}\right)-h\left(\frac{t+1}{2}\right)\right)\tilde{J}_{k,0,0}(t)\mathrm{d}t}{\int_{-1}^{1}\left(\tilde{J}_{i,0,0}(t)\right)^2\mathrm{d}t},\ k=2,3,\ldots,N.$$

At the same time, it also has interpolation property at $t=0$, 1. Calculation of the coefficients g_k can be followed by the above calculation method of Jacobi coefficients.

Applying Jacobi approximation to function $\delta(t)$ as shown in Eq.(9), the rational polynomial approximation to the parametric speed of degree $\max(N,2n)$ and the rational offset approximation curve of degree $\max(N+n,2n-1)$ can be represented respectively as

$$\tilde{\sigma}^{\mathrm{J}}(t)=\frac{\tilde{\delta}_N^{\mathrm{J}}(t)}{w^2(t)}, \tag{10}$$

and

$$\boldsymbol{R}_{\mathrm{O}}^{\mathrm{J}}(t)=\boldsymbol{R}(t)\pm d\cdot\frac{\left(y(t)w'(t)-y'(t)w(t),x'(t)w(t)-x(t)w'(t)\right)}{\tilde{\delta}_N^{\mathrm{J}}(t)}. \tag{11}$$

The corresponding error function is

$$\begin{aligned}\varepsilon_{\mathrm{O}}^{\mathrm{J}}(t)&=\left\|\boldsymbol{R}_{\mathrm{O}}(t)-\tilde{\boldsymbol{R}}_{\mathrm{O}}^{\mathrm{J}}(t)\right\|\\&=d\cdot\left|\left(\frac{1}{\delta(t)}-\frac{1}{\tilde{\delta}_N^{\mathrm{J}}(t)}\right)\right|\cdot\delta(t)\\&=d\left|1-\frac{\delta(t)}{\tilde{\delta}_N^{\mathrm{J}}(t)}\right|.\end{aligned} \tag{12}$$

Based on the convex hull property of rational Bernstein polynomials, it is easy to see that there exists a constant E satisfying the inequality $0\le\frac{\delta^2(t)}{\tilde{\delta}_N^{\mathrm{J}\,2}(t)}\le E$. Thus we get $\left|\varepsilon(t)\right|_{[0,1]}\le d\left(1-\frac{1}{\sqrt{E}}\right)$. In practice, this error bound is much larger than the real error. Therefore we suggest taking this as the error bound taken into consideration, but calculating the real error by discrete sampling.

ALGORITHM DESCRIPTION AND EXAMPLES

Based on the method mentioned above of approximation to the parametric speed of rational Bézier curves, we systematically give our new algorithm for offset approximation to rational Bézier curves in the following:

Algorithm 1 (rational offset approximation to rational Bézier curves):

Step 1: Get the generator curve, offset distance d, approximation degree N and tolerance provided by the user.

Step 2: Calculate the Jacobi approximation polynomial $\tilde{\delta}^{\mathrm{J}}(t)$ of degree N to the parametric speed function $\delta(t)$ of the rational Bézier curve. The formula is given in Eq.(9).

Step 3: Calculate the approximation error $\varepsilon_{\mathrm{O}}^{\mathrm{J}}$ as shown in Eq.(12).

Step 4: If the error obtained in Step 3 is within the given tolerance, then turn to Step 5. Otherwise, do discrete operations to the generator rational Bézier curve. Turn back to Step 2 for every discrete subcurve.

Step 5: Calculate and export every approximation offset curve by Eq.(11).

Example 1 Given a planar rational Bézier curve of degree 3, whose control points and weights are {{−3, −2; 0.1}, {−1.5, 2.5; 0.2}, {1.0, −2.5; 0.3}, {2.5, 2.25; 0.1}}. Based on Jacobi approximation to the function $\delta(t)$ with degree 5, the following Fig.1 gives the rational offset approximation curve with offset distance d equaling 0.5. The corresponding approximation errors $\varepsilon_{\mathrm{O}}^{\mathrm{J}}$ based on Jacobi approximation to the function $\delta(t)$ with degree N (offset distance is 0.5) are presented in Table 1.

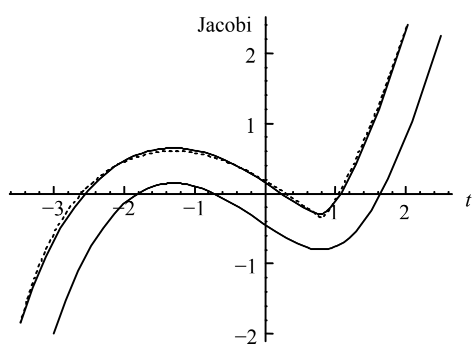

Fig.1 Offset approximation curve of rational Bézier curve of degree 3 (based on Jacobi approximation of degree 5, offset distance equals 0.5)

Table 1 Approximation errors to the curve corresponding to different approximation degrees

Degree N	Error $\varepsilon_{\mathrm{O}}^{\mathrm{J}}$
3	0.142445
5	0.0570624
8	0.00998232
10	0.0035177
13	0.000934505
15	0.000436361
18	0.000137649
20	0.000102308
23	0.0000303596
24	0.000018101

CONCLUSION

Rational offset approximation to rational Bézier curve is discussed in this paper. A new algorithm for offset approximation to rational Bézier curves is presented. This new algorithm has benefits in real time alternation as well as makes effective saving in data storage and calculations. It can be widely applied in areas such as numerical controlling, manufacturing, designing in robotics, computer graphics, and so on.

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