



## An effective quadrilateral mesh adaptation

KHATTRI Sanjay Kumar

(Stord/Haugesund University College, Bjørnsonsg. 45, 5528 Haugesund, Norway)

E-mail: sanjay@mi.uib.no

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**Abstract:** Accuracy of a simulation strongly depends on the grid quality. Here, quality means orthogonality at the boundaries and quasi-orthogonality within the critical regions, smoothness, bounded aspect ratios and solution adaptive behaviour. It is not recommended to refine the parts of the domain where the solution shows little variation. It is desired to concentrate grid points and cells in the part of the domain where the solution shows strong gradients or variations. We present a simple, effective and computationally efficient approach for quadrilateral mesh adaptation. Several numerical examples are presented for supporting our claim.

**Key words:** Quadrilateral mesh, Area functional, Adaptive function, Jacobian, Partial differential equations  
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### INTRODUCTION

Adaptive grids are desired for solving partial differential equations (PDEs) (Khattri, 2006a; 2007). There are various techniques for generating adaptive quadrilateral meshes. For example, solution of coupled elliptic system (Khattri, 2006d; Thompson *et al.*, 1998), minimization of a functional or variational approach (Cao *et al.*, 1999; 2003; Huang, 2001), etc. In this article, we present a simple and effective technique for generating adaptive quadrilateral meshes. Several numerical examples are presented for validating our approach. We extend the area functional for generating adaptive meshes. For a detailed description of the area functional, please refer to (Castillo, 1991; Castillo *et al.*, 1987; Thompson *et al.*, 1998; Tinoco-Ruiz *et al.*, 2001).

Let us define some quantities of interest. Fig.1 shows a quadrilateral cell. In this figure,  $\mathbf{g}_1$  and  $\mathbf{g}_2$  are the two co-variant vectors at the node  $o$ . Other interesting quantities such as the Jacobian matrix ( $\mathbf{J}$ ) and the  $\mathbf{g}$ -tensor at the node  $o$  and for the given cell can be defined from these two vectors. As can be seen in Fig.1, the columns of the Jacobian matrix are the two co-variant vectors.

An outline of the article follows. In Section 2, a discrete functional for quadrilateral mesh adaptation is presented. Section 3 presents several numerical examples. Finally, Section 4 concludes the article.

### AREA FUNCTIONAL FOR MESH ADAPTATION

The first study of the area functional was done by Castillo and Steinberg (Castillo, 1991; Castillo *et al.*, 1987). As per author's information, area functionals have not been used for generating adaptive mesh.

Let a quadrilateral mesh consist of  $n$  internal nodes, each being surrounded by four quadrilaterals (mesh can also be unstructured). According to (Castillo, 1991; Castillo *et al.*, 1987; Tinoco-Ruiz *et al.*, 2001) the area functional is given as

$$F(x, y) \doteq \sum_{k=1}^n \left[ \sum_{i=1}^4 |\mathbf{J}(k_i)|^2 \right], \quad (1)$$

here,  $\mathbf{J}(k_i)$  is the Jacobian matrix at the node  $k$  and for the quadrilateral cell  $i$ , and  $|\mathbf{J}(k_i)|$  is the determinant of

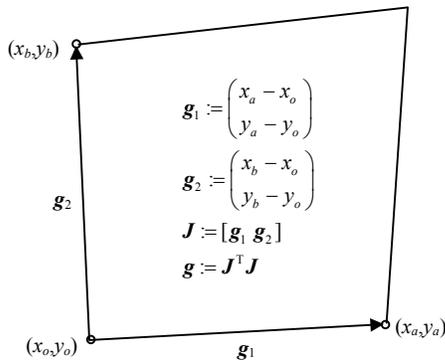


Fig.1 Quantities of interest for a quadrilateral cell

the Jacobian matrix. The Jacobian is a measure of the area of the quadrilateral cell. Thus, the optimization of the area functional is aimed at producing grids with least variation in cell areas (Tinoco-Ruiz *et al.*, 2001). Fig.2 shows a 2x2 mesh. The internal node  $k$  is surrounded by four cells. The Jacobian matrices for the four cells are given in Table 1. Tinoco-Ruiz *et al.*(2001) present some general properties of the area functional.

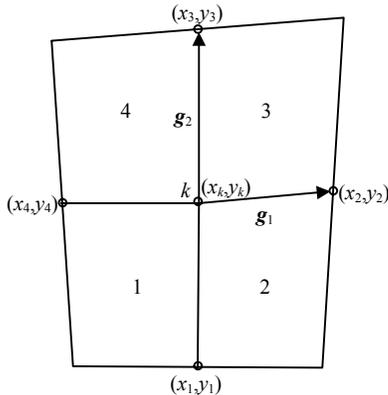


Fig.2 2D structural mesh

Table 1 Jacobian matrix at the node  $k$  for the four surrounding cells (see Fig.2)

$J(k_1) = \begin{bmatrix} (x_4 - x_k) & (x_1 - x_k) \\ (y_4 - y_k) & (y_1 - y_k) \end{bmatrix}$	$J(k_2) = \begin{bmatrix} (x_2 - x_k) & (x_1 - x_k) \\ (y_2 - y_k) & (y_1 - y_k) \end{bmatrix}$
$J(k_3) = \begin{bmatrix} (x_2 - x_k) & (x_3 - x_k) \\ (y_2 - y_k) & (y_3 - y_k) \end{bmatrix}$	$J(k_4) = \begin{bmatrix} (x_4 - x_k) & (x_3 - x_k) \\ (y_4 - y_k) & (y_3 - y_k) \end{bmatrix}$

For grid adaptation, the authors propose the following form of the area functional

$$F(x, y) := \sum_{k=1}^I \left[ \sum_{i=1}^4 |J(k_i)|^2 \Phi(k_i) \right], \quad (2)$$

here,  $\Phi$  is called the adaptive function, and  $\Phi(k_i)$  is the value of the adaptive function at the center of the cell  $i$  surrounding the node  $k$ . It is assumed that  $\Phi$  is greater than zero. The functional can be optimized by algorithms such as the Newton (Khattri, 2006a). Optimization of Eq.(2) will equi-distribute the area product of each cell and adaptive function. Roughly speaking, the cells with larger value of  $\Phi$  will have smaller area. If  $\Phi$  is the same for each cell then the optimization of Eq.(2) is aimed at generating equal area cells.

Some of the properties of the functional  $F(x,y)$  are: the critical point of the functional is a grid for which the product of cell area and the adaptive function is the same for every cell, and the Hessian is semipositive definite.

NUMERICAL EXAMPLES

Example 1

Let the adaptive functions be given as

- $\Phi(x,y) = 1.0 + \eta \operatorname{sech}[20(x-0.5)^2 + 20(y-0.5)^2 - 1.8], \quad (3)$
- $\Phi(x,y) = 1.0 + 1.0 \operatorname{sech}[\alpha(x+y-1)^2], \quad (4)$
- $\Phi(x,y) = 5.0 + \kappa [\sin(2\pi x) \sin(2\pi y)], \quad (5)$
- $\Phi(x,y) = 5.0 + \beta |\sin(2\pi x) \cos(2\pi y)|, \quad (6)$
- $\Phi(x,y) = 5.0 + 200.0 |\sin(\pi x) \sin(2\pi y)|, \quad (7)$
- $\Phi(x,y) = 1.0 + \tanh[(10(x-0.5)^2 + 50(y-0.5)^2 - 1.875)]. \quad (8)$

For different value of  $\eta$ , the adapted meshes are shown in Fig.3. Fig.4 shows the outcome of our experiments for different values of the parameter  $\alpha$ . Fig.5 are the adaptive meshes for  $\kappa=20.0$  and  $\kappa=200.0$  respectively.

For  $\beta=20.0$  and  $\beta=200.0$  the adaptive meshes are given in Fig.6. Fig.7a is adapted by Eq.(7) and Fig.7b is adapted by Eq.(8).

Example 2

We solve the Poisson problem  $-\operatorname{div}(\operatorname{grad}u) = f(x,y)$  on an adaptive and on a uniform mesh by the method

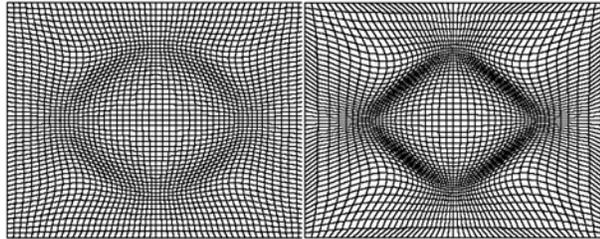
of Finite Volumes (Khattri, 2006b; 2006c; Khattri and Fladmark, 2006; Aavatsmark *et al.*, 1998). Our domain is  $\Omega=[0,1]\times[0,1]$ . Let the exact solution be  $u(x,y)=\exp\{-100[(x-0.5)^2+(y-0.5)^2]\}$ . The solution inside the domain is enforced by the Dirichlet boundary condition and source term. Table 2 shows the errors in the  $L_2$  and  $L_\infty$  norms on the adapted (Fig.8b) and uniform mesh (Fig.8a). The table shows that error (in the  $L_2$  and  $L_\infty$  norms) on the adaptive

mesh is substantially smaller.

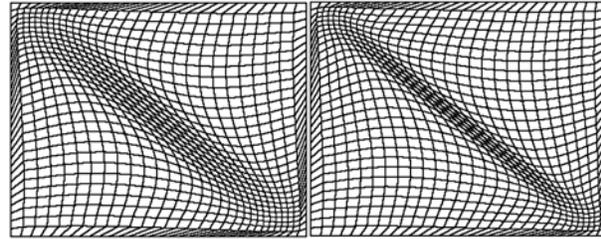
**Table 2 Error on the adaptive (Fig.8b) and uniform meshes (Fig.8a)**

Mesh	$\ p-p_h\ _{L_2}$	$\ p-p_h\ _{L_\infty}$
Uniform	0.0030	0.030
Adaptive	0.0009	0.008

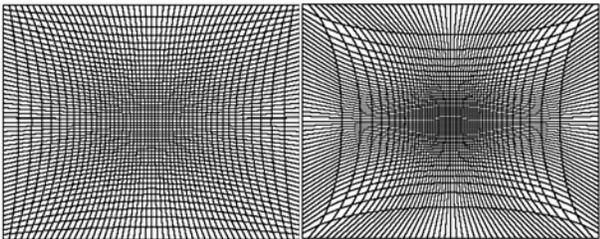
$p$  is the exact solution, and  $p_h$  is the computed solution (finite volume)



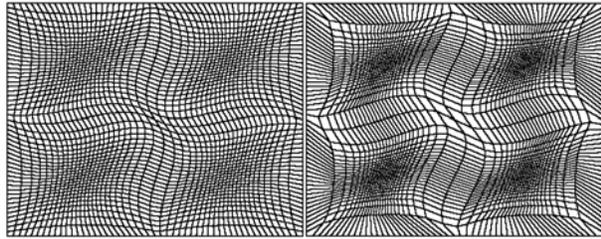
**Fig.3 Adaptive function is given by Eq.(3)**  
(a)  $\eta=1.0$ ; (b)  $\eta=5.0$



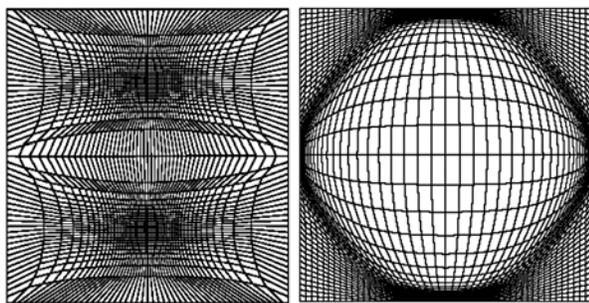
**Fig.4 Adaptive function is given by Eq.(4)**  
(a)  $\alpha=20.0$ ; (b)  $\alpha=50.0$



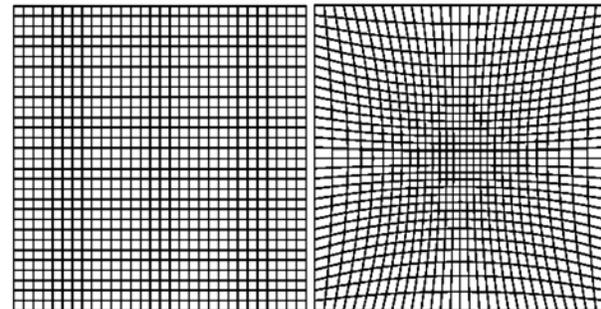
**Fig.5 Adaptive function is given by Eq.(5)**  
(a)  $\kappa=20.0$ ; (b)  $\kappa=200.0$



**Fig.6 Adaptive function is given by Eq.(6)**  
(a)  $\beta=20.0$ ; (b)  $\beta=200.0$



**Fig.7 Adapted functional is given by Eq.(7) (a) and by Eq.(8) (b)**



**Fig.8 Adapted grid for Example 2. (a) Initial grid; (b) Grid is generated by Eq.(5) with  $\kappa=10.0$**

## CONCLUSION

It is not always feasible to blindly refine the mesh in the hope of capturing the physics because of the computational resources. It is desired to adapt the grid to the requirement of the underlying problem. In this article, a simple and robust technique for generating adaptive quadrilateral meshes is presented. We have presented various examples for generating adaptive meshes. It is shown that the error on the adaptive mesh is small. The approach can be useful for solving evolutionary problems (parabolic and hyperbolic equations) on adaptive meshes.

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