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A new algorithm for designing developable Bézier surfaces*

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Abstract: A new algorithm is presented that generates developable Bézier surfaces through a Bézier curve called a directrix. The algorithm is based on differential geometry theory on necessary and sufficient conditions for a surface which is developable, and on degree evaluation formula for parameter curves and linear independence for Bernstein basis. No nonlinear characteristic equations have to be solved. Moreover the vertex for a cone and the edge of regression for a tangent surface can be obtained easily. Aumann's algorithm for developable surfaces is a special case of this paper.

Key words: Bézier surfaces, Developable surfaces, Bernstein basis, Linear independence, Characteristic equations
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INTRODUCTION

A ruled surface is a curved surface which can be generated by the continuous motion of a straight line in space along a space curve called a directrix (Chen *et al*., 2001; Zheng and Sederberg, 2001). This straight line is called a generator, or ruling, of the surface. A developable surface is a special ruled surface which has the same tangent plane at all points along a generator, or to which the tangent planes along a ruling coincide. A developable surface is also the envelope of a single family of planes. There is only one developable surface that can isometrically map onto a plane. Therefore developable surfaces are particularly interesting and appealing, owing to the simplicity of the manufacturing process required to fabricate them. It can be formed by bending or rolling a planar surface without any stretching or contraction. For these reasons, developable surfaces are widely used in manufacturing items from materials that are not amenable to stretching, such as ship hulls, ducts, shoes, clothing, and automobiles parts (Tang *et al*., 1997). Thereby developable surfaces have been widely used in CAD/CAM systems. Many papers give the applications of developable surfaces in industry (Mancewicz and Frey, 1992; Frey and Bindschadler, 1993; Pottmann and Wallner, 2001; Chu and Séquin, 2002).

Many authors constructed developable free form surfaces by nonlinear characteristic equations (Aumann, 1991; Lang and Röschel, 1992; Chalfant and Maekawa, 1998; Maekawa, 1998). A different approach to designing developable surfaces is a direct surface representation in terms of geometric duality between points and planes in 3D projective space (Bodduluri and Ravani, 1993; Pottmann and Farin, 1995). Aumann (2003) presented a simple algorithm for computing a developable Bézier surface based on de Casteljau algorithm and affine transformation. Most of the known algorithms have one or more of the following restrictions: (1) The characteristic equations cannot be solved easily, except for the case in which the surfaces boundaries are made of low degree curves; (2) Only planar boundary curves are

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permitted for designing developable surfaces; (3) It is difficult to determine the vertex for a cone and the edge of regression for a tangent surface. To get rid of these disadvantages, this paper does an in-depth research on the geometric essence of developable Bézier surfaces. According to the differential geometry theory on the necessary and sufficient conditions for a surface which is developable, the degree evaluation formula for parameter curves and the linear independence for Bernstein basis, we present a new method for constructing a developable Bézier surface passing through a given space boundary Bézier curve which is as a directrix. Also we can directly determine the vertex for a cone and the edge of regression for a tangent surface. This method covers the consequence in (Aumann, 2003) as a special case.

The next section gives the representation of ruled $n\times1$ Bézier surfaces and developability constraints in Bézier surfaces. In Section 3, the three types of developable Bézier surfaces are discussed in detail, and some corresponding examples are given. The final section compares the results of this study with those of previous work and concludes this paper.

RULED BÉZIER SURFACES AND DEVELOPABILITY CONDITIONS

Given two space Bézier curves of degree n as follows:

$$\boldsymbol{\alpha}(u)=\sum_{i=0}^{n}B_i^n(u)\boldsymbol{p}_i,\qquad u\in[0,1],\tag{1}$$

$$\boldsymbol{q}(u)=\sum_{i=0}^{n}B_i^n(u)\boldsymbol{q}_i,\qquad u\in[0,1],\tag{2}$$

then we define an $n\times1$ parametric ruled Bézier surface by using the expression:

$$\begin{gathered}\boldsymbol{r}(u,v)=(1-v)\boldsymbol{\alpha}(u)+v\boldsymbol{q}(u)=\boldsymbol{\alpha}(u)+v\boldsymbol{\tau}(u),\\(u,v)\in[0,1]\otimes[0,1]\end{gathered}\tag{3}$$

to blend them, where the curves $\boldsymbol{\alpha}(u)$ and

$$\boldsymbol{\tau}(u)=\sum_{i=0}^{n}B_i^n(u)(\boldsymbol{q}_i-\boldsymbol{p}_i),\qquad u\in[0,1]\tag{4}$$

are called a directrix (or base curve) and a generator (or director curve) respectively.

According to differential geometry theory (Spivak, 1975), a ruled surface $\boldsymbol{r}(u,v)$ is developable if and only if for any $u\in[0,1]$, three vectors $\boldsymbol{\alpha}'=\boldsymbol{\alpha}'(u)$, $\boldsymbol{\tau}=\boldsymbol{\tau}(u)$, $\boldsymbol{\tau}'=\boldsymbol{\tau}'(u)$ are linearly dependent; in other words, there exist three scalar functions $\lambda=\lambda(u)$, $\mu=\mu(u)$ and $\gamma=\gamma(u)$ that are not all zeros simultaneously such that

$$\lambda\boldsymbol{\alpha}'+\mu\boldsymbol{\tau}+\gamma\boldsymbol{\tau}'=\boldsymbol{0}.\tag{5}$$

Spivak (1975) proved that there exist only three types of developable surfaces: cylinders (including planes), cones and tangent surfaces formed by the tangents of a space curve. When Eq.(5) is satisfied, different function system $\{\lambda(u), \mu(u), \gamma(u)\}$ corresponds to one of the above three types. Next we will study these three types of Bézier surfaces in detail.

ANALYTIC CONDITIONS FOR THREE TYPES OF DEVELOPABLE BÉZIER SURFACES

Case 1 $\lambda(u)\equiv0$.

In this case, $\mu(u)$, $\gamma(u)$ are not zeroes simultaneously such that $\mu\boldsymbol{\tau}+\gamma\boldsymbol{\tau}'=\boldsymbol{0}$. It demonstrates that two vectors $\boldsymbol{\tau}$, $\boldsymbol{\tau}'$ are linearly dependent. Let $\boldsymbol{e}(u)$ be a unit vector parallel to $\boldsymbol{\tau}(u)$. Hence there exist a scalar function $\omega(u)$, such that $\boldsymbol{\tau}(u)=\omega(u)\boldsymbol{e}(u)$. Thus we have $\boldsymbol{\tau}'=\omega'\boldsymbol{e}+\omega\boldsymbol{e}'$. Applying cross product to both sides of the former equation with the vector $\boldsymbol{\tau}$, we can obtain $\boldsymbol{\tau}\times\boldsymbol{\tau}'=\omega\boldsymbol{e}\times\omega\boldsymbol{e}'=\omega^2\boldsymbol{e}\times\boldsymbol{e}'=\boldsymbol{0}$. Therefore $\boldsymbol{e}\times\boldsymbol{e}'=\boldsymbol{0}$. $\|\boldsymbol{e}\|^2=1$ implies that $\boldsymbol{e}\cdot\boldsymbol{e}'=0$. From the Lagrange identity

$$(\boldsymbol{e}\cdot\boldsymbol{e})(\boldsymbol{e}'\cdot\boldsymbol{e}')-(\boldsymbol{e}\cdot\boldsymbol{e}')^2=(\boldsymbol{e}\times\boldsymbol{e}')^2,$$

it follows that $\boldsymbol{e}'\cdot\boldsymbol{e}'=0$, i.e., $\boldsymbol{e}'=\boldsymbol{0}$. This indicates that the direction of the unit vector $\boldsymbol{e}$ does not change and so does the ruling $\boldsymbol{\tau}(u)$. Consequently the developable Bézier surface is a cylinder.

In order to construct a cylinder concretely, using Eq.(4), we get $\boldsymbol{\tau}(0)=\boldsymbol{q}_0-\boldsymbol{p}_0$, hence $\boldsymbol{e}=\dfrac{\boldsymbol{q}_0-\boldsymbol{p}_0}{\|\boldsymbol{q}_0-\boldsymbol{p}_0\|}$; on the other hand,

$$\frac{\mathrm{d}\boldsymbol{\tau}(u)}{\mathrm{d}u}=\frac{\mathrm{d}\omega(u)}{\mathrm{d}u}\boldsymbol{e},$$

$$\frac{\mathrm{d}\boldsymbol{\tau}(0)}{\mathrm{d}u}=n\Delta(\boldsymbol{q}_0-\boldsymbol{p}_0)=\frac{\mathrm{d}\omega(0)}{\mathrm{d}u}\boldsymbol{e},$$

where Δ is the forward difference operator, $\Delta(\boldsymbol{q}_0-\boldsymbol{p}_0)=(\boldsymbol{q}_1-\boldsymbol{p}_1)-(\boldsymbol{q}_0-\boldsymbol{p}_0)$, $\Delta^k(\boldsymbol{q}_0-\boldsymbol{p}_0)=\Delta^{k-1}(\boldsymbol{q}_1-\boldsymbol{p}_1)-\Delta^{k-1}(\boldsymbol{q}_0-\boldsymbol{p}_0)$, k=2, 3, …. It is easy to conclude that $(\boldsymbol{q}_1-\boldsymbol{p}_1)\|\boldsymbol{e}$. In like manner, from the expression

$$\frac{\mathrm{d}^i\tau(0)}{\mathrm{d}u^i}=\frac{n!}{(n-i)!}\cdot\Delta^i(\boldsymbol{q}_0-\boldsymbol{p}_0)=\frac{\mathrm{d}^i\omega(0)}{\mathrm{d}u^i}\boldsymbol{e},$$

we obtain

$$(\boldsymbol{q}_i-\boldsymbol{p}_i)\,\|\,\boldsymbol{e},\qquad i=1,\ 2,\ ...,\ n. \tag{6}$$

Let $\boldsymbol{q}_{i+1}-\boldsymbol{p}_{i+1}=\sigma(\boldsymbol{q}_i-\boldsymbol{p}_i)$, i=0,1,…,n−1, where σ is a constant. Then the expression of the developable Bézier cylinder in (Aumann, 2003) can be obtained. Fig.1 shows two different 3×1 Bézier cylinders.

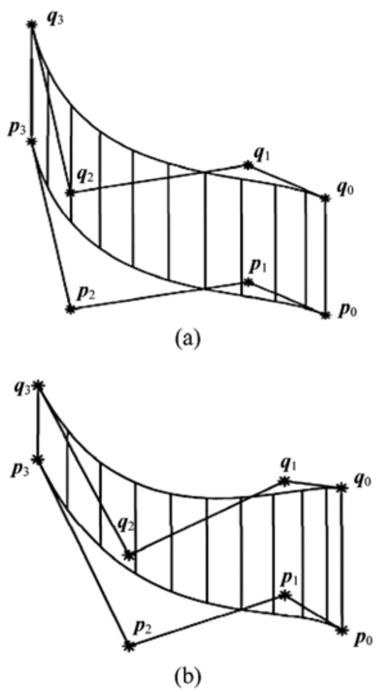

Fig.1 3×1 Bézier cylinders with $\boldsymbol{q}_{i+1}-\boldsymbol{p}_{i+1}=\sigma(\boldsymbol{q}_i-\boldsymbol{p}_i)$, i=0, 1, …, n−1. (a) σ=1.0; (b) σ=0.8

Case 2 $\lambda(u)\neq0$.

In this case, there exist two functions $a=a(u)=-\dfrac{\mu(u)}{\lambda(u)}$, $b=b(u)=-\dfrac{\gamma(u)}{\lambda(u)}$, such that $\boldsymbol{\alpha}'=a\boldsymbol{\tau}+b\boldsymbol{\tau}'$. Thus developable surfaces $\boldsymbol{r}(u,v)$ can be expressed as

$$\boldsymbol{r}(u,v)=\boldsymbol{\alpha}_1(u)+(v+b)\boldsymbol{\tau}, \tag{7}$$

$$\boldsymbol{\alpha}_1(u)=\boldsymbol{\alpha}-b\boldsymbol{\tau}, \tag{8}$$

where the derivative vector of the vector $\boldsymbol{\alpha}_1(u)$ is

$$\boldsymbol{\alpha}_1'=\boldsymbol{\alpha}'-b'\boldsymbol{\tau}-b\boldsymbol{\tau}'=(a-b')\boldsymbol{\tau}. \tag{9}$$

Case 2.1 $a=b'$.

In this case, $\boldsymbol{\alpha}_1'=\mathbf{0}$, which means that $\boldsymbol{\alpha}_1$ is a constant vector. We can make a parameter transformation

$$\hat{v}=v+b(u), \tag{10}$$

so that developable surface $\boldsymbol{r}(u,v)$ can be represented by

$$\boldsymbol{r}(u,v)=\boldsymbol{\alpha}_1+\hat{v}\boldsymbol{\tau}(u), \tag{11}$$

where $\hat{v}\in[b(u),b(u)+1]$; this denotes that the surface $\boldsymbol{r}(u,v)$ is a cone with its vertex $\boldsymbol{\alpha}_1$.

Assume that the Bézier curve $\boldsymbol{\alpha}(u)$ in Eq.(1) is known, the Bézier curve $\boldsymbol{q}(u)$ in Eq.(2) must be determined such that Bézier ruled surface $\boldsymbol{r}(u,v)$, expressed as Eq.(3), with the Bézier curve $\boldsymbol{\alpha}(u)$ in Eq.(1) as its directrix, is developable. In this case, the ruled surface $\boldsymbol{r}(u,v)$ is a cone.

Let us observe some special examples as follows.

(1) Let a=0, $b=1/(\rho-1)$ where $\rho\neq1$ is a constant.

In this case, $\boldsymbol{\alpha}'=\boldsymbol{\tau}'/(\rho-1)$. According to Eq.(1) and Eq.(4), we have

$$n\sum_{i=0}^{n-1}B_i^{n-1}(u)\Delta\boldsymbol{p}_i=n\frac{1}{\rho-1}\sum_{i=0}^{n-1}B_i^{n-1}(u)\Delta(\boldsymbol{q}_i-\boldsymbol{p}_i),$$

and based on the linear independence of Bernstein basis (Piegl and Tiller, 1997) we obtain

$$(\rho-1)(\boldsymbol{p}_{i+1}-\boldsymbol{p}_i)=(\boldsymbol{q}_{i+1}-\boldsymbol{p}_{i+1})-(\boldsymbol{q}_i-\boldsymbol{p}_i),\quad i=0,\ 1,\ ...,\ n-1;$$

that is,

$$\boldsymbol{q}_i=\boldsymbol{q}_0+\rho(\boldsymbol{p}_i-\boldsymbol{p}_0),\qquad i=1,\ 2,\ ...,\ n. \tag{12}$$

The expression of the developable Bézier cone in (Aumann, 2003) can be yielded. Fig.2 shows two different 3×1 Bézier cones.

(2) Let $a=\sigma$, $b=\sigma u$, where $\sigma\neq0$ is a constant.

In this case, according to $\boldsymbol{\alpha}'=a\boldsymbol{\tau}+b\boldsymbol{\tau}'$, we have

$$n\sum_{i=0}^{n-1}B_i^{n-1}(u)\Delta\boldsymbol{p}_i=\sigma\sum_{i=0}^{n}B_i^n(u)(\boldsymbol{q}_i-\boldsymbol{p}_i)+n\sigma u\sum_{i=0}^{n-1}B_i^{n-1}(u)\Delta(\boldsymbol{q}_i-\boldsymbol{p}_i).$$

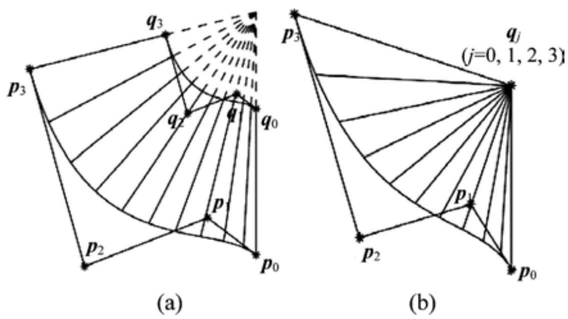

Fig.2 3×1 Bézier cones with $q_i=q_0+\rho(p_i-p_0)$, i=1, 2, …, n. (a) ρ=0.4; (b) ρ=0.0

Elevating the degree of Bézier curves on the two sides of the above equation respectively, the following expressions

$$\sum_{i=0}^{n} B_i^n(u)((n-i)\Delta \boldsymbol{p}_i + i\Delta \boldsymbol{p}_{i-1})$$
$$=\sum_{i=0}^{n} B_i^n(u)\sigma(\boldsymbol{q}_i - \boldsymbol{p}_i) + \sum_{i=1}^{n} B_i^n(u)(i\sigma\Delta(\boldsymbol{q}_{i-1} - \boldsymbol{p}_{i-1})),$$
$$\Delta \boldsymbol{p}_{-1} = \Delta \boldsymbol{p}_n = \boldsymbol{0},$$

can be derived. Again, based on the linear independence of the Bernstein basis, we obtain

$$\begin{cases} n\Delta \boldsymbol{p}_0 = \sigma(\boldsymbol{q}_0 - \boldsymbol{p}_0); \\ (n-i)\Delta \boldsymbol{p}_i + i\Delta \boldsymbol{p}_{i-1} = \sigma(\boldsymbol{q}_i - \boldsymbol{p}_i) + i\sigma\Delta(\boldsymbol{q}_{i-1} - \boldsymbol{p}_{i-1}), \\ \qquad i = 1,\ 2,\ \ldots,\ n-1; \\ n\Delta \boldsymbol{p}_{n-1} = \sigma(\boldsymbol{q}_n - \boldsymbol{p}_n) + n\sigma\Delta(\boldsymbol{q}_{n-1} - \boldsymbol{p}_{n-1}). \end{cases}$$

It follows that

$$\begin{cases} \boldsymbol{q}_0 = \boldsymbol{p}_0 + \dfrac{n}{\sigma}\Delta \boldsymbol{p}_0; \\ \boldsymbol{q}_i = \boldsymbol{p}_i + \dfrac{i\sigma(\boldsymbol{q}_{i-1} - \boldsymbol{p}_{i-1}) + (n-i)\Delta \boldsymbol{p}_i + i\Delta \boldsymbol{p}_{i-1}}{(i+1)\sigma}, \\ \qquad i = 1,\ 2,\ \ldots,\ n-1; \\ \boldsymbol{q}_n = \boldsymbol{p}_n + \dfrac{n\sigma(\boldsymbol{q}_{n-1} - \boldsymbol{p}_{n-1}) + n\Delta \boldsymbol{p}_{n-1}}{(n+1)\sigma}. \end{cases}$$

that is,

$$(i+1)\sigma(\boldsymbol{q}_i - \boldsymbol{p}_i) - i\sigma(\boldsymbol{q}_{i-1} - \boldsymbol{p}_{i-1}) = (n-i)\Delta \boldsymbol{p}_i + i\Delta \boldsymbol{p}_{i-1},$$
$$i = 1,\ 2,\ \ldots,\ n-1.$$

Using the above recursion formula, we obtain

$$\begin{cases} \boldsymbol{q}_0 = \boldsymbol{p}_0 + \dfrac{n}{\sigma}\Delta \boldsymbol{p}_0; \\ \boldsymbol{q}_i = \boldsymbol{p}_i + \dfrac{(n-i)\boldsymbol{p}_{i+1} + (i+1)\boldsymbol{p}_i - (n+1)\boldsymbol{p}_0}{(i+1)\sigma}, \\ \qquad i = 1,\ 2,\ \ldots,\ n-1; \\ \boldsymbol{q}_n = \boldsymbol{p}_n + \dfrac{\boldsymbol{p}_n - \boldsymbol{p}_0}{\sigma}. \end{cases} \quad (13)$$

Fig.3 shows two different 3×1 Bézier cones.

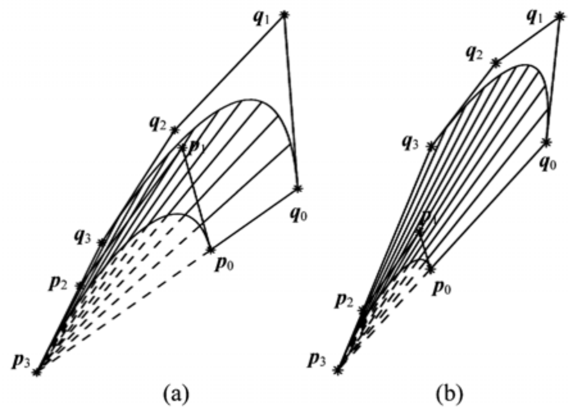

Fig.3 3×1 Bézier cones with $a=\sigma$, $b=\sigma u$. (a) σ=3.0; (b) σ=0.8

Case 2.2 $a \neq b'$.

In this case, we have $\tau = \boldsymbol{\alpha}_1'/(a-b')$. Introducing a new parameter

$$\hat{v} = \frac{v + b(u)}{a(u) - b'(u)}, \quad (14)$$

the equation of the developable surface $\boldsymbol{r}(u,v)$ can be written as

$$\boldsymbol{r}(u,v) = \boldsymbol{\alpha}_1(u) + \hat{v}\boldsymbol{\alpha}_1'(u), \quad (15)$$

where $\boldsymbol{\alpha}_1(u) = \boldsymbol{\alpha}(u) - b(u)\boldsymbol{\tau}(u)$ and $\hat{v} = v + b(u)$. $\boldsymbol{r}(u,v)$ is the tangent surface of the curve $\boldsymbol{\alpha}_1=\boldsymbol{\alpha}_1(u)$, here $\boldsymbol{\alpha}_1(u)$ is the edge of regression.

As computed in Case 2.1, the Bézier curve $\boldsymbol{q}(u)$ in Eq.(2) must be determined such that the Bézier ruled surface $\boldsymbol{r}(u,v)$, expressed by Eq.(3), with the Bézier curve $\boldsymbol{\alpha}(u)$ in Eq.(1) as its directrix, is a tangent surface. Then we also can determine its edge of regression.

In the following, we discuss some typical and important cases.

(1) a=2, b=u.

Here, according to $\alpha'=a\tau+b\tau'$ and the degree elevation formula for a Bézier curve, we get

$$\sum_{i=0}^{n} B_i^n(u)((n-i)\Delta\boldsymbol{p}_i + i\Delta\boldsymbol{p}_{i-1})$$
$$= \sum_{i=0}^{n} B_i^n(u)[2(\boldsymbol{q}_i - \boldsymbol{p}_i)] + \sum_{i=0}^{n} B_i^n(u)(i\Delta(\boldsymbol{q}_{i-1} - \boldsymbol{p}_{i-1})),$$
$$\Delta\boldsymbol{p}_{-1} = \Delta\boldsymbol{p}_n = \Delta\boldsymbol{p}_{-1}\boldsymbol{q}_{-1} = \boldsymbol{0},$$

and further according to the linear independence for Bernstein basis, we obtain

$$(n-i)\Delta\boldsymbol{p}_i + i\Delta\boldsymbol{p}_{i-1} = 2(\boldsymbol{q}_i - \boldsymbol{p}_i) + i\Delta(\boldsymbol{q}_{i-1} - \boldsymbol{p}_{i-1}),$$
$$i = 0,\ 1,\ \ldots,\ n;$$
$$\Delta\boldsymbol{p}_{-1} = \Delta\boldsymbol{p}_n = \Delta(\boldsymbol{q}_{-1} - \boldsymbol{p}_{-1}) = \boldsymbol{0}.$$

Therefore,

$$\begin{cases} \boldsymbol{q}_0 = \boldsymbol{p}_0 + \dfrac{n\Delta\boldsymbol{p}_0}{2}; \\ \boldsymbol{q}_i = \boldsymbol{p}_i + \dfrac{(n-i)\Delta\boldsymbol{p}_i + i\Delta\boldsymbol{p}_{i-1} + i(\boldsymbol{q}_{i-1} - \boldsymbol{p}_{i-1})}{i+2}, \\ \qquad i = 1,\ 2,\ \ldots,\ n-1; \\ \boldsymbol{q}_n = \boldsymbol{p}_n + \dfrac{n\Delta\boldsymbol{p}_{n-1} + n(\boldsymbol{q}_{n-1} - \boldsymbol{p}_{n-1})}{n+2}. \end{cases}$$

Again, using the above recursion formula, it is easy to know that

$$\begin{cases} \boldsymbol{q}_0 = \boldsymbol{p}_0 + n\boldsymbol{A}_0; \\ \boldsymbol{q}_i = \boldsymbol{p}_i + (n-i)\boldsymbol{A}_i + i\boldsymbol{A}_i,\quad i = 1,\ 2,\ \ldots,\ n-1; \\ \boldsymbol{q}_n = \boldsymbol{p}_n + n\boldsymbol{A}_{n-1}; \\ \boldsymbol{A}_i = \dfrac{1}{(i+1)(i+2)} \sum_{j=1}^{i+1} j\Delta\boldsymbol{p}_{j-1},\quad i = 0,\ 1,\ \ldots,\ n. \end{cases} \tag{16}$$

Fig.4 shows a 3×1 Bézier tangent surface where $\alpha_1(u)$ is the edge of regression.

(2) a=1, b=0.

In this case, $\alpha'=\tau$. From the degree elevation formula for a Bézier curve, we get

$$\sum_{i=0}^{n} B_i^n(u)((n-i)\Delta\boldsymbol{p}_i + i\Delta\boldsymbol{p}_{i-1}) = \sum_{i=0}^{n} B_i^n(u)(\boldsymbol{q}_i - \boldsymbol{p}_i),$$
$$\Delta\boldsymbol{p}_{-1} = \Delta\boldsymbol{p}_n = \boldsymbol{0}.$$

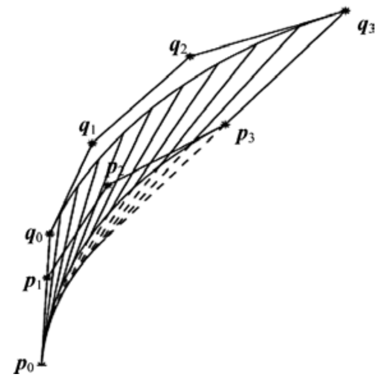

Fig.4 A 3×1 Bézier tangent surface with *a*=2, *b*=*u*

Then according to the linear independence for Bernstein basis, there exist

$$(n-i)\Delta\boldsymbol{p}_i + i\Delta\boldsymbol{p}_{i-1} = (\boldsymbol{q}_i - \boldsymbol{p}_i),\ i = 0,\ 1,\ \ldots,\ n;$$
$$\Delta\boldsymbol{p}_{-1} = \Delta\boldsymbol{p}_n = \boldsymbol{0}.$$

Hence

$$\begin{cases} \boldsymbol{q}_0 = \boldsymbol{p}_0 + n\Delta\boldsymbol{p}_0; \\ \boldsymbol{q}_i = \boldsymbol{p}_i + (n-i)\Delta\boldsymbol{p}_i + i\Delta\boldsymbol{p}_{i-1}, \\ \qquad i = 1,\ 2,\ \ldots,\ n-1; \\ \boldsymbol{q}_n = \boldsymbol{p}_n + n\Delta\boldsymbol{p}_{n-1}. \end{cases} \tag{17}$$

Fig.5 shows a 3×1 Bézier tangent surface where $\alpha_1(u)$ is the edge of regression.

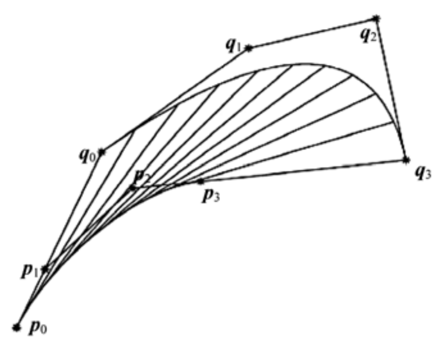

Fig.5 A 3×1 Bézier tangent surface with *a*=1.0, *b*=0.0

(3) a=1/u, b=1.

From $\alpha'=a\tau+b\tau'$, we have

$$n\sum_{i=0}^{n-1} B_i^{n-1}(u)\Delta\boldsymbol{p}_i$$
$$= \frac{1}{u}\sum_{i=0}^{n} B_i^n(u)(\boldsymbol{q}_i - \boldsymbol{p}_i) + n\sum_{i=0}^{n-1} B_i^{n-1}(u)\Delta(\boldsymbol{q}_i - \boldsymbol{p}_i),$$

or

$$nu\sum_{i=0}^{n-1}B_i^{n-1}(u)\Delta\boldsymbol{p}_i$$
$$=\sum_{i=0}^{n}B_i^n(u)(\boldsymbol{q}_i-\boldsymbol{p}_i)+nu\sum_{i=0}^{n-1}B_i^{n-1}(u)\Delta(\boldsymbol{q}_i-\boldsymbol{p}_i),$$

and elevate the degree of Bézier curves on the two sides of the above equation, then

$$\sum_{i=1}^{n}B_i^n(u)(i\Delta\boldsymbol{p}_{i-1})$$
$$=\sum_{i=0}^{n}B_i^n(u)(\boldsymbol{q}_i-\boldsymbol{p}_i)+\sum_{i=1}^{n}B_i^n(u)(i\Delta(\boldsymbol{q}_{i-1}-\boldsymbol{p}_{i-1}))$$

can be obtained. According to the linear independence of the Bernstein basis, it follows that

$$\begin{cases}\boldsymbol{q}_0=\boldsymbol{p}_0;\\ \boldsymbol{q}_i=\boldsymbol{p}_i+\dfrac{i}{i+1}(\boldsymbol{p}_i+\boldsymbol{q}_{i-1}-2\boldsymbol{p}_{i-1}),\ i=1,\ 2,\ \ldots,\ n.\end{cases}\tag{18}$$

Fig.6 shows a 3×1 Bézier tangent surface where $\alpha_1(u)=\alpha(u)-\tau(u)$ is the edge of regression.

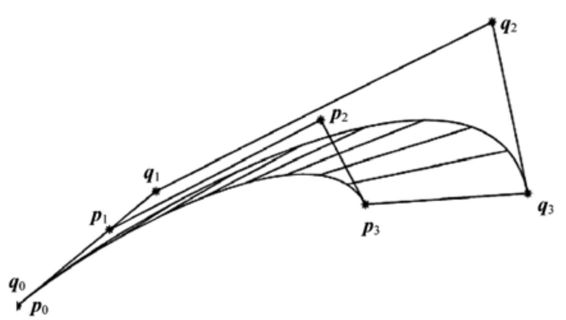

Fig.6 A 3×1 Bézier tangent surface with *a*=1/*u*, *b*=1.0

(4) $a=\dfrac{n(1-\sigma)}{\rho-1},\ b=\dfrac{1-u+\sigma u}{\rho-1},\ \sigma\neq1,\ \rho\neq1$.

From $\alpha'=a\tau+b\tau'$, we get

$$n\sum_{i=0}^{n-1}B_i^{n-1}(u)\Delta\boldsymbol{p}_i=\frac{n(1-\sigma)}{\rho-1}\sum_{i=0}^{n}B_i^n(u)(\boldsymbol{q}_i-\boldsymbol{p}_i)$$
$$+\frac{n(1-u+\sigma u)}{\rho-1}\sum_{i=0}^{n-1}B_i^{n-1}(u)\Delta(\boldsymbol{q}_i-\boldsymbol{p}_i).$$

From the degree elevation formula for a Bézier curve, the following equation

$$(\rho-1)\sum_{i=0}^{n}B_i^n(u)((n-i)\Delta\boldsymbol{p}_i+i\Delta\boldsymbol{p}_{i-1})$$
$$=n(1-\sigma)\sum_{i=0}^{n}B_i^n(u)(\boldsymbol{q}_i-\boldsymbol{p}_i)$$
$$+\sum_{i=0}^{n}B_i^n(u)((n-i)\Delta(\boldsymbol{q}_i-\boldsymbol{p}_i)+\sigma i\Delta(\boldsymbol{q}_{i-1}-\boldsymbol{p}_{i-1})),$$
$$\Delta\boldsymbol{p}_{-1}=\Delta\boldsymbol{p}_n=\Delta(\boldsymbol{q}_{-1}-\boldsymbol{p}_{-1})=\Delta(\boldsymbol{q}_n-\boldsymbol{p}_n)=\boldsymbol{0},$$

can be obtained. Similarly, then we have

$$(\rho-1)((n-i)\Delta\boldsymbol{p}_i+i\Delta\boldsymbol{p}_{i-1})=n(1-\sigma)(\boldsymbol{q}_i-\boldsymbol{p}_i)$$
$$+(n-i)\Delta(\boldsymbol{q}_i-\boldsymbol{p}_i)+\sigma i\Delta(\boldsymbol{q}_{i-1}-\boldsymbol{p}_{i-1}),\ i=0,\ \ldots,\ n;$$
$$\Delta\boldsymbol{p}_{-1}=\Delta\boldsymbol{p}_n=\Delta(\boldsymbol{q}_{-1}-\boldsymbol{p}_{-1})=\Delta(\boldsymbol{q}_n-\boldsymbol{p}_n)=\boldsymbol{0}.$$

After rearranging it, we obtain

$$(\rho-1)((n-i)\Delta\boldsymbol{p}_i+i\Delta\boldsymbol{p}_{i-1})=(n-i)((\boldsymbol{q}_{i+1}-\boldsymbol{p}_{i+1})-\sigma(\boldsymbol{q}_i$$
$$-\boldsymbol{p}_i))+i((\boldsymbol{q}_i-\boldsymbol{p}_i)-\sigma(\boldsymbol{q}_{i-1}-\boldsymbol{p}_{i-1})),\ i=0,\ 1,\ \ldots,\ n.$$

After simplifying it, we obtain

$$(n-i)((\rho-1)\Delta\boldsymbol{p}_i-((\boldsymbol{q}_{i+1}-\boldsymbol{p}_{i+1})-\sigma(\boldsymbol{q}_i-\boldsymbol{p}_i)))$$
$$=i((\rho-1)\Delta\boldsymbol{p}_{i-1}-((\boldsymbol{q}_i-\boldsymbol{p}_i)-\sigma(\boldsymbol{q}_{i-1}-\boldsymbol{p}_{i-1}))),\tag{19}$$
$$i=0,1,\ldots,n.$$

In particular, taking the integer i as zero in the above equation, and letting

$$n(\rho-1)\Delta\boldsymbol{p}_0-n((\boldsymbol{q}_1-\boldsymbol{p}_1)-\sigma(\boldsymbol{q}_0-\boldsymbol{p}_0))=\boldsymbol{0},$$

we get

$$(\rho-1)\Delta\boldsymbol{p}_i-((\boldsymbol{q}_{i+1}-\boldsymbol{p}_{i+1})-\sigma(\boldsymbol{q}_i-\boldsymbol{p}_i))=\boldsymbol{0},$$
$$i=0,\ 1,\ \ldots,\ n-1;$$

or

$$\boldsymbol{q}_{i+1}=\boldsymbol{p}_i+\rho(\boldsymbol{p}_{i+1}-\boldsymbol{p}_i)+\sigma(\boldsymbol{q}_i-\boldsymbol{p}_i),\ i=0,\ 1,\ \ldots,\ n-1,\tag{20}$$

which is the expression of developable Bézier tangent surfaces in (Aumann, 2003). Fig.7 shows two different 3×1 Bézier tangent surfaces where the parameters ρ and σ are distinct.

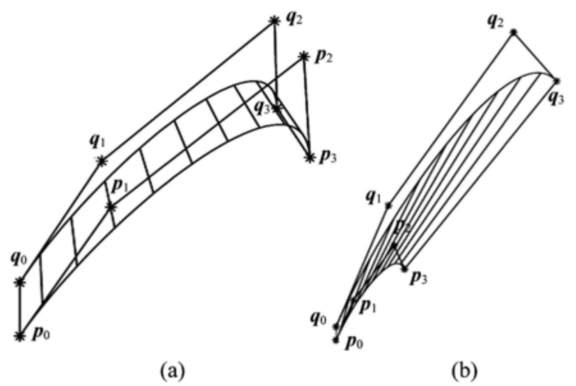

Fig.7 3×1 Bézier tangent surfaces. (a) ρ=0.9, σ=1.1; (b) ρ=3, σ=1.1

CONCLUSION

This paper investigates the geometric design of developable Bézier surfaces through a space Bézier curve $\alpha(u)$ as its directrix. The straight line $\tau(u)$ known as a ruling is easily determined by the differential geometry theory on the necessary and sufficient conditions for a developable surface, the degree elevation formula for parameter curves and the linear independence for Bernstein basis. This method generalizes Aumann's design method. It can further generate more general and more widely-used developable Bézier surfaces than Aumann's method. Moreover, no nonlinear characteristic equations have to be solved and the vertex of a cone and the edge of regression of a tangent surface are directly determined. Therefore, it provides practical and effective tools for designing developable Bézier surfaces and is useful in practical applications in engineering shape design.

References

Aumann, G., 1991. Interpolation with developable Bézier patches. *Computer Aided Geometric Design*, **8**(5):409-420. [doi:10.1016/0167-8396(91)90014-3]

Aumann, G., 2003. A simple algorithm for designing developable Bézier surfaces. *Computer Aided Geometric Design*, **20**(8-9):601-619. [doi:10.1016/j.cagd.2003.07.001]

Bodduluri, R.M.C., Ravani, B., 1993. Design of developable surfaces using duality between plane and point geometries. *Computer-Aided Design*, **25**(10):621-632. [doi:10.1016/0010-4485(93)90017-I]

Chalfant, J.S., Maekawa, T., 1998. Design for manufacturing using B-spline developable surfaces. *Journal of Ship Research*, **42**(3):206-214.

Chen, F., Zheng, J., Sederberg, T., 2001. The μ-basis of a rational ruled surface. *Computer Aided Geometric Design*, **18**(1):61-72. [doi:10.1016/S0167-8396(01)00012-7]

Chu, C.H., Séquin, C.H., 2002. Developable Bézier patches properties and design. *Computer-Aided Design*, **34**(7):511-527. [doi:10.1016/S0010-4485(01)00122-1]

Frey, W.H., Bindschadler, D., 1993. Computer Aided Design of a Class of Developable Bézier Surfaces. GM Research Publications, R&D-8057.

Lang, J., Röschel, O., 1992. Developable (1, n)-Bézier surfaces. *Computer Aided Geometric Design*, **9**(4):291-298. [doi:10.1016/0167-8396(92)90036-O]

Ma, W., Kruth, J., 1998. NURBS curve and surface fitting for reverse engineering. *International Journal of Advanced Manufacturing Technology*, **14**(12):918-927. [doi:10.1007/BF01179082]

Maekawa, T., 1998. Design and tessellation of B-spline developable surfaces. *ASME Trans. Journal of Mechanical Design*, **120**(3):453-461.

Mancewicz, M.J., Frey, W.H., 1992. Developable Surfaces Properties, Representations and Methods of Design. GM Research Publications, GMR-7637.

Piegl, L., Tiller, W., 1997. The NRUBS Book (2nd Ed.), Springer-Verlag New York, Inc., New York.

Pottmann, H., Farin, G., 1995. Developable rational Bézier and B-spline surfaces. *Computer Aided Geometric Design*, **12**(5):513-531. [doi:10.1016/0167-8396(94)00031-M]

Pottmann, H., Wallner, J., 2001. Computational Line Geometry. Springer, Berlin.

Spivak, M., 1975. Differential Geometry. Publish or Perish Inc., Boston.

Tang, K., Wang, M., Chen, L., Chou, S., Woo, T., Janardan, R., 1997. Computing planar swept polygons under translation. *Computer-Aided Design*, **29**(12):825-836. [doi:10.1016/S0010-4485(97)00030-4]

Zheng, J., Sederberg, T., 2001. A direct approach to computing the μ-basis of planar rational curves. *Journal of Symbolic Computation*, **31**(5):619-629. [doi:10.1006/jsco.2001.0437]