Journal of Zhejiang University SCIENCE A
ISSN 1009-3095 (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Two-parameters quasi-filled function algorithm for nonlinear integer programming*

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Received Dec. 28, 2005; revision accepted July 3, 2006

Abstract: A quasi-filled function for nonlinear integer programming problem is given in this paper. This function contains two parameters which are easily to be chosen. Theoretical properties of the proposed quasi-filled function are investigated. Moreover, we also propose a new solution algorithm using this quasi-filled function to solve nonlinear integer programming problem in this paper. The examples with 2 to 6 variables are tested and computational results indicated the efficiency and reliability of the proposed quasi-filled function algorithm.

Key words: Integer programming, Local minimizer, Global minimizer, Filled function, Global optimization

doi:10.1631/jzus.2006.A2083 **Document code:** A **CLC number:** O221.2; O224

INTRODUCTION

Considering the following nonlinear integer programming problem:

$$(\mathrm{P_I}) \quad \min f(x), \quad \text{s.t. } x\in X_I, \tag{1}$$

where $X_I\subset I^n$ is a bounded and closed box set containing more than one point, I^n is the set of integer points in $\mathbb{R}^n$.

If we suppose that $f(x)$ satisfies the following conditions: if $x\in X_I$, then $f(x)=f(x)$, otherwise $f(x)=+\infty$, then Problem $\mathrm{P_I}$ is equal to the following nonlinear integer programming problem

$$(\mathrm{UP_I}) \quad \min f(x), \quad \text{s.t. } x\in I^n. \tag{2}$$

The formulation in $\mathrm{P_I}$ allows the set X_I to be defined by equality constraints as well as inequality constraints. Furthermore, when $f(x)$ is coercive, i.e., $f(x)\to+\infty$, as $\|x\|\to\infty$, there exists a box containing all discrete global minimizers of $f(x)$, therefore $\mathrm{UP_I}$ can be reduced into an equivalent problem formulation in $\mathrm{P_I}$. In other words, both unconstrained and constrained nonlinear integer programming problems can be considered in $\mathrm{P_I}$.

PRELIMINARIES

To introduce the concept of the quasi-filled function method, let us recall some definitions involved in nonlinear integer programming problem.

Definition 1 (Shang and Han, 2005) For any $x\in I^n$, the neighborhood of x is defined by $N(x)=\{x, x\pm \boldsymbol{e}_i: i=1,2,\ldots,n\}$, where $\boldsymbol{e}_i$ is the n-dimensional vector with the ith component equal to one and other components equal to zero. Let $N^0(x)=N(x)\backslash\{x\}$.

Definition 2 (Shang and Zhang, 2005) An integer point $x_0\in X_I$ is called a local minimizer of $f(x)$ over X_I if there exists a neighborhood $N(x_0)$ such that for any

* Project (Nos. 10571137 and 10271073) supported by the National Natural Science Foundation of China

$x\in N(x_0)\cap X_I$, $f(x)\geq f(x_0)$ holds. An integer point $x_0\in X_I$ is called a global minimizer of $f(x)$ over X_I if for any $x\in X_I$, $f(x)\geq f(x_0)$ holds. In addition, if $f(x)>f(x_0)$ for all $x\in N^0(x_0)\cap X_I$ ($x\in X_I\backslash\{x_0\}$), then x_0 is called a strictly local (global) minimizer of $f(x)$ over X_I.

It is obvious that a global minimizer of $f(x)$ over X_I must be a local minimizer of $f(x)$ over X_I.

The local minimizer of $f(x)$ over X_I is obtained by using the following Algorithm 1 (Shang and Han, 2005).

Algorithm 1:

Step 1: Choose any integer $x_0\in X_I$.

Step 2: If x_0 is a local minimizer of $f(x)$ over X_I, then stop; otherwise, we can obtain a $x\in N(x_0)\cap X_I$, such that $f(x)<f(x_0)$.

Step 3: Let $x_0:=x$, go to Step 2.

A QUASI-FILLED FUNCTION AND ITS PROPERTIES

A two-parameters quasi-filled function of $f(x)$ and its properties are given in this section. Let x^* be the current local minimizer of $f(x)$. Let

$$S_1=\{x\in X_I : f(x)\geq f(x^*)\}\subset X_I,$$
$$S_2=\{x\in X_I : f(x)<f(x^*)\}\subset X_I.$$

In the following, we will give a definition of a quasi-filled function of $f(x)$ at a local minimizer x^* for nonlinear integer programming problem which is just an extension of the definition of the filled function proposed for the continuous case in (Ge, 1990; Ge and Qin, 1990; Lucidi and Piccialli, 2002).

Definition 3 (Shang and Han, 2005) $P(x,x^*,q,r)$ is called a quasi-filled function of $f(x)$ at a local minimizer x^* for nonlinear integer programming problem if $P(x,x^*,q,r)$ has the following properties:

(i) $P(x,x^*,q,r)$ has no local minimizer in the set $S_1\backslash\{x_0\}$, where $x_0\in S_1$ is a prefixed point.

(ii) If x^* is not a global minimizer of $f(x)$, then there exists a local minimizer x_1 of $P(x,x^*,q,r)$, such that $f(x_1)<f(x^*)$, that is, $x_1\in S_2$.

Similar to (Shang and Han, 2005), we present a two-parameters quasi-filled function of $f(x)$ at local minimizer x^* as follows:

$$P(x,x^*,q,r)=||x-x_0||+q\min[0,f(x)-f(x^*)+r], \quad (3)$$

where $q>0$ and $0<r<\min(|f(x_1)-f(x_2)|)$, $x_1, x_2\in X_I$.

In the following we will prove that the function $P(x,x^*,q,r)$ satisfies the conditions (i) and (ii) of Definition 3, i.e., it is a quasi-filled function of $f(x)$ at a local minimizer x^* satisfying Definition 3. First, we give Lemma 1 as follows:

Lemma 1 (Zhu, 2000; Zhang *et al*., 1999) For any integer point $x\in X_I$, if $x\neq x_0$, then there exists a $d\in D=\{\pm \boldsymbol{e}_i: i=1,2,\ldots,n\}$ such that

$$||x+d-x_0||<||x-x_0||. \quad (4)$$

Proof Since $x\neq x_0$, there exists an $i\in\{1,2,\ldots,n\}$ such that $x_i\neq x_{0i}$. If $x_i>x_{0i}$, then $d=-\boldsymbol{e}_i$. On the other hand, if $x_i<x_{0i}$, then $d=\boldsymbol{e}_i$.

Theorem 1 $P(x,x^*,q,r)$ has no local minimizer in the integer set $S_1\backslash\{x_0\}$ for any $q>0$.

Proof For any $x\in S_1$ and $x\neq x_0$, by using Lemma 1 we know there exists a $d\in D$, such that

$$||x+d-x_0||<||x-x_0||.$$

Considering the following two cases:

(1) If $f(x)\geq f(x^*)>f(x+d)$, then $f(x+d)-f(x^*)+r<0$, $P(x+d,x^*,q,r)=||x-x_0+d||+q[f(x+d)-f(x^*)+r]<||x+d-x_0|| <||x-x_0||=P(x,x^*,q,r)$. Therefore, x is not a local minimizer of $P(x,x^*,q,r)$.

(2) If $f(x+d)\geq f(x^*)$ and $f(x)\geq f(x^*)$ then $P(x+d,x^*,q,r)=||x+d-x_0||<||x-x_0||=P(x,x^*,q,r)$. Therefore, it is shown that x is not a local minimizer of $P(x,x^*,q,r)$.

By Theorem 1, we know that the function $P(x,x^*,q,r)$ satisfies the first property of Definition 3 without any assumption on the parameter $q>0$.

Theorem 2 Given that x^* is a local minimizer but not a global minimizer of $f(x)$, then $P(x,x^*,q,r)$ has local minimizer in the integer set S_2 if $q>0$ satisfies the following condition:

$$q>\frac{||x^{**}-x_0||}{f(x^*)-f(x^{**})-r}, \quad (5)$$

where x^{**} is a global minimizer of $f(x)$.

Proof By the conditions, we have $f(x^{**})<f(x^*)$ and $f(x^*)-f(x^{**})-r>0$.

Therefore, if

$$q>\frac{||x^{**}-x_0||}{f(x^*)-f(x^{**})-r},$$

then $P(x^{**},x^*,q,r)=||x^{**}-x_0||+q[f(x^{**})+r-f(x^*)]<0$.

On the other hand, for any $y\in S_1$, we have $P(y,x^*,q,r)=||y-x_0||\geq 0$.

Therefore, the global minimizer of $P(x,x^*,q,r)$ must belong to the set S_2. Since a global minimizer must be a local minimizer, $P(x,x^*,q,r)$ has a local minimizer in the S_2.

In summary, by Theorems 1 and 2, if parameter q is large enough and r is chosen appropriately small, the function $P(x,x^*,q,r)$ does satisfy all the conditions of Definition 3, i.e., function $P(x,x^*,q,r)$ is a quasi-filled function.

However, we know the value of $f(x^*)$, and generally we do not know the global minimal value or global minimizer of $f(x)$, so it is difficult to find the lower bound of parameter q in Theorem 2.

But for practical consideration, Problem P_I might be solved if we can find an $x\in X_I$ such that $f(x)<f(x^{**})+\varepsilon$, where $f(x^{**})$ is the global minimal value of Problem P_I, and ε is a given desired optimality tolerance. So we consider the case when the current local minimizer x^* satisfies $f(x^*)\geq f(x^{**})+\varepsilon$. In the following Theorem 3 we develop a lower bound of parameter q which depends only on the given optimality tolerance ε.

Theorem 3 Suppose that ε is a small positive constant, $q>D/(\varepsilon-r)$, where $D=\max||x_1-x_2||$, $r<\min\{\varepsilon, \min[|f(x_1)-f(x_2)|]\}$, $x_1,x_2\in X_I$, Then for any current local minimizer x^* of $f(x)$ such that $f(x^*)\geq f(x^{**})+\varepsilon$, quasi-filled function $P(x,x^*,q,r)$ has a local minimizer in the set S_2, where x^{**} is a global minimizer of $f(x)$.

Proof By the conditions, we have

$$q>\frac{D}{\varepsilon-r}\geq\frac{||x_0-x^{**}||}{f(x^*)-f(x^{**})-r}$$

by Theorem 2; the conclusion of Theorem 3 holds.

We construct the following auxiliary nonlinear integer programming Problem AP_I related to Problem P_I:

$$(AP_I)\quad \min P(x,x^*,q,r),\ \text{s.t. } x\in X_I. \qquad (6)$$

According to the above discussions, given any desired tolerance $\varepsilon>0$, if $q>D/(\varepsilon-r)$, where

$$D=\max||x_1-x_2||,$$

$$0<r<\min\{\varepsilon, \min[|f(x_1)-f(x_2)|]\},\ x_1, x_2\in X_I. \qquad (7)$$

then $P(x,x^*,q,r)$ is a quasi-filled function of $f(x)$ at its current local minimizer x^* which satisfies that $f(x^*)\geq f(x^{**})+\varepsilon$. Thus if we use a local minimization method to solve Problem AP_I from any initial point on X_I, then by the properties of quasi-filled function, it is obvious that the minimization sequences converges either to x_0 or to a point $x'\in X_I$ such that $f(x')<f(x^*)$. If we find such an x', then using a local minimization method to minimize $f(x)$ on X_I from initial point x', we can find a point $x''\in X_I$ such that $f(x'')<f(x')$ which is better than x^*. This is the main idea of the algorithm presented in the next section to find an approximate global minimal solution of Problem P_I.

ALGORITHM AND NUMERICAL RESULTS

In optimization studies, modelling, algorithmization and numerical realization are three relevant yet independent aspects. It has been well known that the numerical realization of an established formulation for nonlinear programming usually involves more difficulties. Based on the theoretical results in the previous section and similar to (Shang and Han, 2005), a global optimization quasi-filled function algorithm over X_I is proposed as follows.

Algorithm 2 (The quasi-filled function method):

Step 1: Given a constant $N_L>0$ as the tolerance parameter for terminating the minimization process of Problem P_I, a small constant $\varepsilon>0$ as a desired optimality tolerance, and a small $0<r<\varepsilon$. Choose a fixed integer $x_0\in X_I$.

Step 2: Obtain a local minimizer x^* of $f(x)$ by implementing Algorithm 1 starting from x_0.

Step 3: Construct the quasi-filled function $P(x,x^*,q,r)$ as follows:

$$P(x,x^*,q,r)=||x-x_0||+q\min[0,f(x)-f(x^*)+r],$$

where $q>0$ and satisfies the condition Eq.(5) or Eq.(7). Let $N=0$.

Step 4: If $N>N_L$, then go to Step 7.

Step 5: Set $N=N+1$. Draw an initial point on the boundary of the set X_I, and start from it to minimize $P(x,x^*,q,r)$ on X_I using any local minimization method. Suppose that x' is an obtained local minimizer of $P(x,x^*,q,r)$. If $x'=x_0$, then go to Step 4, otherwise go to Step 6.

Step 6: Minimize $f(x)$ on the set X_I from the initial point x', and obtain a local minimizer x_2^* of $f(x)$. Let $x^*=x_2^*$ and go to Step 3.

Step 7: Stop the algorithm, output x^* and $f(x^*)$ as an approximate global minimal solution and global minimal value of Problem P_I respectively.

Although the focus of this paper is more theoretical than computational, we still test our algorithm on several global minimization problems to have an initial feeling of the practical value of the quasi-filled function algorithm.

Example

$$\min f(x)=\sum_{i=1}^{n-1}[100(x_{i+1}-x_i^2)^2+(1-x_i)^2],$$
$$\text{s.t. } |x_i|\le 5,\ x_i \text{ integer},\ i=1,2,\ldots,n.$$

This problem is a box constrained nonlinear integer programming problem. It has 11^n feasible points and many local minimizers (5, 6, 7, 9 and 11 local minimizers for n=2, 3, 4, 5 and 6, respectively), but only one global minimizer: x^*_{global}=(1, 1, …, 1) with $f(x^*_{\text{global}})$=0 for all n. We considered three cases of the problem: n=2, 3 and 5. There are about 1.21×10^2, 1.331×10^3, 1.611×10^5 feasible points, for n=2, 3, 5, respectively.

In the following, the proposed solution algorithm is programmed in MATLAB 6.5.1 Release for working on the WINDOWS XP system with 900 MHz CPU. The MATLAB 6.5.1 subroutine is used as the local neighborhood search scheme to obtain local minimizers of $f(x)$ in Step 2 and the local minimizers of $P(x,x^*,q,r)$ in Step 5 in Algorithm 2. The function $P(x,x^*,q,r)$ is as follows:

$$P(x,x^*,q,r)=\|x-x_0\|+q\min[0, f(x)-f(x^*)+r].$$

Let ε=0.002, r=0.001 and $q>D/(\varepsilon-r)+1$, $D=\max\|x_1-x_2\|$, the tolerance parameter $N_L=10^n+1$, n is the variable number of $f(x)$.

The partial main of the computational process for the numerical example is summarized in Table 1. The symbols used are as follows: T_S the number of initial

Table 1 Results of numerical example

Parameters	T_S	k	x^k_{ini}	$x^k_{f-l_0}$	$f(x^k_{f-l_0})$	$x^k_{p-l_0}$	$f(x^k_{p-l_0})$	QIN
n=2, ε=0.002, r=0.001,	1	1	(5,5)	(2,4)	1	(1,1)	0	2
$q=10000\sqrt{2}+1$,		2	(1,1)	(1,1)	0			≥10^2+1
$D=10\sqrt{2}+1$,	2	1	(−5,−3)	(0,0)	1	(1,1)	0	0
$N_L=10^2+1$		2	(1,1)	(1,1)	0			≥10^2+1
	3	1	(−4,3)	(−2,4)	9	(1,1)	0	1
		2	(1,1)	(1,1)	0			≥10^2+1
	4	1	(−1,5)	(−2,4)	9	(1,1)	0	1
		2	(1,1)	(1,1)	0			≥10^2+1
n=3, ε=0.002, r=0.001,	1	1	(3,3,3)	(1,2,4)	101	(1,1,1)	0	0
$q=10000\sqrt{3}+1$,		2	(1,1,1)	(1,1,1)	0			≥10^3+1
$D=10\sqrt{3}+1$,	2	1	(−4,0,4)	(−1,2,4)	105	(1,1,1)	0	4
$N_L=10^3+1$		2	(1,1,1)	(1,1,1)	0			≥10^3+1
	3	1	(0,4,4)	(1,2,4)	101	(1,1,1)	0	1
		2	(1,1,1)	(1,1,1)	0			≥10^3+1
	4	1	(−2,−1,5)	(0,−2,4)	410	(1,1,1)	0	0
		2	(1,1,1)	(1,1,1)	0			≥10^3+1
n=5, ε=0.002, r=0.001,	1	1	(0,0,2,0,2)	(1,1,1,1,1)	0	(1,1,1,1,1)	0	1
$q=10000\sqrt{5}+1$,		2	(1,1,1,1,1)	(1,1,1,1,1)	0			≥10^5+1
$D=10\sqrt{5}+1$,	2	1	(−2,2,0,1,1)	(−1,1,1,1,1)	4	(1,1,1,1,1)	0	8
$N_L=10^5+1$		2	(1,1,1,1,1)	(1,1,1,1,1)	0			≥10^5+1
	3	1	(−4, −1,−2,−3,5)	(0,0,0,−2,4)	412	(1,1,1,1,1)	0	0
		2	(1,1,1,1,1)	(1,1,1,1,1)	0			≥10^5+1
	4	1	(0,0,2,0,2)	(1,1,1,1,1)	0	(1,1,1,1,1)	0	1
		2	(1,1,1,1,1)	(1,1,1,1,1)	0			≥10^5+1

points to be chosen; k the times for the local minimization process of Problem P_I; x_{ini}^k the initial point for the kth local minimization process of Problem P_I; $x_{f-l_0}^k$ the minimizer for the kth local minimization process of Problem P_I; $f(x_{f-l_0}^k)$ the minimum of the $x_{f-l_0}^k$; $x_{p-l_0}^k$ the minimizer for the kth local minimization process of Problem AP_I; $f(x_{p-l_0}^k)$ the minimum of the $x_{p-l_0}^k$; QIN the iteration number for the kth local minimization process of Problem AP_I.

CONCLUSION

This paper gives a two-parameters quasi-filled function and discusses its properties. A quasi-filled function algorithm based on the given quasi-filled function was designed. Numerical results on testing functions indicated the efficiency and reliability of the proposed quasi-filled function algorithm.

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