Journal of Zhejiang University SCIENCE A
ISSN 1009-3095 (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

A problem on extremal quasiconformal extensions*

CHEN Zhi-guo
(*Department of Mathematics, Zhejiang University, Hangzhou 310027, China*)
E-mail: zgchen@zju.edu.cn
Received Nov. 20, 2005; revision accepted Mar. 21, 2006

Abstract: In this paper we give a short survey on a problem on extremal quasiconformal extensions. It had been a conjecture for a long time that the dilatations $K_0(h)$ and $K_1(h)$ are equal before Anderson and Hinkkanen disproved this by constructing concrete examples of a family of affine mappings of some parallelograms. The problem also engendered many interesting results. At the end of the current paper, we discuss relationships among $K_0(h)$, $H(h)$ and $K_1(h)$ as a concluding remark.

Key words: Quasisymmetric mapping, Extremal quasiconformal mapping, Universal Teichmüller space, Non-Strebel point
doi:10.1631/jzus.2006.AS0198 **Document code:** A **CLC number:** O174.5

INTRODUCTION

Denote the unit disk and the unit circle in the complex plane by Δ and Γ respectively. Let h be a sense-preserving quasisymmetric mapping of Γ onto itself. It is well known that there exist quasiconformal extensions of h onto Δ. We define

$$K_1(h)=\inf\{K: h \text{ has a } K\text{-quasiconformal extension to a self map of } \Delta\}. \quad (1)$$

We call f extremal if f is a quasiconformal extension of h and $K(f)=K_1(h)$. Let z_1, z_2, z_3 and z_4 be four points on Γ in the positive direction. Then they determine a unique topological quadrilateral with domain Δ and vertices z_1, z_2, z_3 and z_4, which we denote by $Q=\Delta(z_1, z_2, z_3, z_4)$. Denote the conformal modulus of Q by $M(Q)$. Similarly, we denote

$$h(Q)=\Delta(h(z_1), h(z_2), h(z_3), h(z_4))$$

and its conformal modulus by $M(h(Q))$. For definition of conformal modulus of a quadrilateral, we refer to the classical book of Ahlfors (1966). Define

$$K_0(h)=\sup\{M(h(Q))/M(Q): Q \text{ is a topological quadrilateral with domain } \Delta\}. \quad (2)$$

By definition, it is obvious that $K_0(h)\le K_1(h)$.

For a point $\zeta\in\Gamma$,

$$H_\zeta(h)=\inf\{K: h \text{ has a } K\text{-quasiconformal extension } f \text{ to } U_\zeta\cap\Delta\}, \quad (3)$$

where the infimum is taken over all neighborhood U_ζ and all quasiconformal extensions f of h to $U_\zeta\cap\Delta$. Obviously, $H_\zeta(h)\le K_1(h)$. We call the point ζ a substantial boundary point if $H_\zeta(h)=K_1(h)$. Define

$$H(h)=\inf\{K: h \text{ has a } K\text{-quasiconformal extension } f \text{ to } \Delta_r\}, \quad (4)$$

where $\Delta_r=\{z:r<|z|<1\}$. By definition, we have

$$H_\zeta(h)\le H(h)\le K_1(h).$$

Fehlmann (1982) proved that

$$H(h)=\max_{\zeta\in\Gamma} H_\zeta(h). \quad (5)$$

Therefore, h has substantial boundary points if and only if $H(h)=K_1(h)$.

It is interesting to study the relationships among

* Project (No. 10101023) supported by the National Natural Science Foundation of China

$K_0(h)$, $H(h)$ and $K_1(h)$. In fact, It had been an open problem for a long time to determine whether or not the equality $K_0(h)=K_1(h)$ always holds before Anderson and Hinkkanen (1995) disproved this by constructing concrete examples of a family of affine mappings of some parallelograms. From then on, many interesting results have been obtained. In 1997, Wu proved that the strict inequality $K_0(h)<K_1(h)$ holds for most of the quasisymmetric mappings which do not have substantial boundary points. Similar result was independently obtained by Yang (1997). For an interesting remark on the related topic on this problem, see (Kühnau, 2000).

PREVIOUS RESULTS

Let us recall the universal Teichmüller space. Let $QS(\Gamma)$ be the full set of quasisymmetric mappings of Γ and let $M\ddot{o}b(\Gamma)$ be the group of Möbius transformations mapping Γ onto itself. Then the right coset space $QS(\Gamma)/M\ddot{o}b(\Gamma)$ is the universal Teichmüller space T. For any $h\in QS(\Gamma)$, let $[h]\in T$ be the Teichmüller class of h. Note that if $h\in QS(\Gamma)$, $g\in M\ddot{o}b(\Gamma)$, then $K_0(g\circ h)=K_0(h)$, $H(g\circ h)=H(h)$ and $K_1(g\circ h)=K_1(h)$. Therefore, we can define $K_0([h])=K_0(h)$, $H([h])=H(h)$ and $K_1([h])=K_1(h)$. We call a point $[h]\in T$ a Strebel point if $H([h])<K_1([h])$ and a non-Strebel point if $H([h])=K_1([h])$. Let T_S be the set of all Strebel points in the Teichmüller space T. For any two points $[h_j]\in T$ (j=1,2) the Teichmüller distance is defined as

$$\mathrm{d}([h_1],[h_2]) = 0.5\log[K_1(h_1\circ h_2^{-1})].$$

Earle and Li (1999) showed that, in the topology induced by the Teichmüller metric, T_S is open in T. Lakic (1995) showed that T_S is dense in T.

We distinguish two cases for $K_0(h)$. If there exists a non-degenerated quadrilateral Q such that $K_0(h)=M(h(Q))/M(Q)$, we adopt the notation $K_0^q(h)$ instead of $K_0(h)$. Otherwise, if there exists no non-degenerated quadrilateral such that $K_0(h)=M(h(Q))/M(Q)$, we use $K_0^d(h)$ instead of $K_0(h)$. Let $U=\{[h]\in T: K_0^q([h])=K_1([h])\}$. It was shown by Wu (1997) that U depends only on two real parameters and that $U\in T_S$. Furthermore, he proved the following main result:

Theorem 1 For every point $[h]\in T_S-U$, $[h]$ has the property that $K_0(h)<K_1(h)$.

As mentioned above, T_S is dense and open in the universal Teichmüller space T, whose dimension is infinite. Therefore, the importance of Theorem 1 lies in the fact that it shows almost all quasisymmetric mappings have the property that $K_0(h)<K_1(h)$.

In the same year, Yang (1997) proved the following result independently.

Theorem 2 If $K_0(h)=K_1(h)$, then either h is induced by an affine map or h has a substantial boundary point.

In their papers, Wu (1997) and Yang (1997) asked the following.

Problem 1 Let h be a quasisymmetric mapping of Γ onto itself. Is it true that $H(h)=K_1(h)$ always implies $K_0(h)=K_1(h)$?

For the above problem, Li *et al.*(1999) gave an affirmative answer under some additional conditions and proved the following result:

Theorem 3 Let f be a Teichmüller mapping of Δ onto itself with its complex dilatation

$$v(z) = k\bar{\varphi}(z)/|\varphi(z)|,$$

where $\varphi(z)\mathrm{d}z^2$ is a holomorphic quadratic differential in Δ. Suppose that $\varphi(z)\mathrm{d}z^2$ is real on $\Gamma\cap U$ and has a second order pole $z_0\in\Gamma$, where U is some deleted neighborhood of z_0. Then z_0 is a substantial boundary point of $h=f|\Gamma$ and

$$K_0(h)=H(h)=K_1(h). \tag{6}$$

Applying Theorem 1, Chen *et al.*(2002) gave other sufficient conditions such that there holds Eq.(6).

Theorem 4 Let f be a quasiconformal mapping of the upper half plane H onto itself with its complex dilatation

$$v(z) = k\bar{\varphi}(z)/|\varphi(z)|,$$

where $\varphi(z)=\log^a(z)/z^2$, $a\geq 0$. Then for the boundary function $h=f|\partial H$, Eq.(6) holds.

Liang and Zhu (2001) discussed a special case on hyperbolic region and obtained the following:

Theorem 5 Let $D=\{z=x+\mathrm{i}y: x^2/a^2-y^2/b^2>1, x>0\}$ and $h=A_K|\partial D$, where $A_K(x+\mathrm{i}y)=Kx+\mathrm{i}y$. Then Eq.(6) holds.

These results are affirmative to Problem 1.

However, Shen (2000) gave a negative answer to Problem 1 by giving the following.

Counterexample 1 For any $K>1$, Define $h=h_K$: $\Gamma\to\Gamma$ as $h(x)=x$ for $x\le 0$ and $h(x)=Kx$ for $x>0$. Then $K_0(h)<H(h)=K_1(h)$ for large K.

The papers mentioned above are closely related with the work of Wu (1997)'s. Another important result on the relationship between $K_0(h)$ and $K_1(h)$ was given by Reich (1997). In his paper, Reich established a necessary condition for $K_0(h)=K_1(h)$, where h is induced by a Teichmüller mapping.

Theorem 6 Suppose that h is the boundary correspondence of a Teichmüller mapping $f(z)$ of Δ onto itself with complex dilatation

$$\mu(z)=t\overline{\varphi}(z)/|\varphi(z)|,$$

where $0<t<1$, and φ is holomorphic in Δ and in class $L^1(\Delta)$. Then a necessary condition for $K_0(h)=K_1(h)$ is

$$\sup_{\Phi}|\iint_{\Delta}\frac{\overline{\varphi}(z)}{|\varphi(z)|}\Phi'^2(z)\mathrm{d}x\mathrm{d}y=1. \tag{7}$$

The sup is taken over all functions Φ holomorphic in Δ for which

$$\|\Phi'^2(z)\|=\iint_{\Delta}|\Phi'(z)|^2\mathrm{d}x\mathrm{d}y=1.$$

Using the necessary theorem, Reich gave an example such that $K_0(h)<K_1(h)$ for some h. Following the work of Reich (1997)'s, Chen and Chen (1997) established a necessary and sufficient condition for $K_0(h)=K_1(h)$ in general cases.

Theorem 7 Suppose $f(z)$ is an extremal quasiconformal mapping of Δ onto itself with complex dilatation $\mu(z)$. Then for its boundary function h, the necessary and sufficient condition for $K_0(h)=K_1(h)$ is

$$\sup_{Q}\mathrm{Re}\iint_{\Delta}\mu(z)|\Phi_Q'(z)|^2\mathrm{d}x\mathrm{d}y=k_1, \tag{8}$$

where $\Phi_Q(z)$ maps $Q=Q(z_1,z_2,z_3,z_4)$ conformally onto a rectangle

$$R=\{\zeta=\xi+\mathrm{i}\eta: 0\le\xi\le a, 0\le\eta\le b, ab=1\}.$$

Later, Qi (1998) generalized the above theorem to the case of topological polygons. In his paper, he introduces a constant $K_0^{(m)}(h)$ instead of $K_0(h)$ for any m ($m>4$) polygons. Let z_j ($1\le j\le m$) be point wise points on Γ. Let

$$K_0(h)(z_1, z_2, \ldots, z_m)=\inf\{K(f): f \text{ is a qc mapping of } \Delta \text{ onto itself } f(z_j)=h(z_j),\ 1\le j\le m\}.$$

Define

$$K_0^{(m)}(h)=\sup\{K_0(h)(z_1, z_2, \ldots, z_m): z_1, z_2, \ldots, z_m \text{ are different points on } \Gamma\}.$$

Then he has the following theorem, which follows the Hamilton-Krushkal-Reich-Strebel theorem characterizing the extremal Beltrami differentials.

Theorem 8 Let f be a quasiconformal mapping of Δ onto itself with complex dilatation $\mu(z)$ and $h=f|\Gamma$. Then the necessary and sufficient condition for $K_0^{(m)}(h)=K_1(h)$ is

$$\sup_{\varphi\in Q_m(h)}\mathrm{Re}\iint_{\Delta}\mu(z)\varphi(z)\mathrm{d}x\mathrm{d}y=\|\mu\|_\infty,$$

where $Q_m(h)$ is the set of m-polygon differentials.

It should be pointed out that a complete answer for arbitrary n-gons was given by Strebel (1999). Let f_0 be an extremal qc mapping of Δ_z onto Δ_w with $f_0|\Gamma_z=h$. Let κ_0 with $\|\kappa\|_\infty=k_0$ be its complex dilatation and $K_0=(1+k_0)/(1-k_0)$ its maximal dilatation. Mark n points z_j, $j=1, 2, \ldots, n$ on Γ_z, $4\le n\le N$. The disk Δ_z with the marked boundary points z_j is called a polygon P_n. The image of P_n by f_0 is the polygon P_n', inscribed in Δ_w, with vertices $w_j=f_0(z_j)$. Strebel (1999) proved the following theorem by using Polygon Inequality (Reich and Strebel, 1974).

Theorem 9 Let $f_0:\Delta_z\to\Delta_w$ with complex dilatation κ_0, $\|\kappa\|_\infty=k_0$, be extremal for its boundary values h. Assume that for a fixed number N the polygon mappings $f_N: P_n\to P_n'=f_0(P_n)$ with complex dilatation $k_N(\overline{\varphi}_N/|\varphi_N|)$ satisfying

$$\sup k_N=k_0. \tag{9}$$

Then, there is a sequence of polygon mappings $f_N^{(i)}$ the quadratic differentials $\varphi_N^{(i)}$ of which, $\|\varphi_N^{(i)}\|=1$, form a Hamilton sequence for κ_0, i.e.

$$\mathrm{Re}\iint \kappa_0(z)\varphi_N^{(i)}\mathrm{d}x\mathrm{d}y \to k_0,\ i\to\infty. \qquad (10)$$

Furthermore, he proved that if the initial extremal map f_0 has no essential boundary points, then, for each $n\geq 4$, Eq.(9) is attained on n-gons only when this f_0 is itself a polygon map for some n vertices. More precisely, he obtained:

Theorem 10 Let $f_0:\Delta_z\to\Delta_w$ be qc mapping which is extremal for its boundary values, and assume that it does not have an essential boundary point. For fixed $N\geq 4$ denote the polygons with $4\leq n\leq N$ vertices inscribed in Δ_z generically by P_n. To every P_n the mapping f_0 determines a polygon P_n' inscribed in Δ_w, simply by mapping the vertices of P_n onto those of P_n'. Assume that the mappings $f_N: P_n \to P_n'$ satisfy Eq.(9). Then, there is a convergent sequence $f_N^{(i)}$ of polygon mappings with $\varphi_N^{(i)}\to\varphi_0$ in norm, where $\kappa_0 = k_0(\overline{\varphi}_0/\varphi_0)$ is the complex dilatation of f_0. f_0 itself is the extremal qc mapping of a polygon with $n\leq N$ vertices, and every maximizing sequence $f_N^{(i)}$, $k_N^{(i)}\to k_0$, tends to f_0 uniformly, $\varphi_N^{(i)}\to\varphi_0$ in norm.

Theorem 10 shows that the answer whether the equation always holds is negative for any $n\geq 4$. Another approach is due to Krushkal (2003), whose proof is based on the strengthened Grunsky inequalities for univalent holomorphic functions.

Theorem 11 For each $n\geq 4$ and every $k\in(0,1)$, there exist quasisymmetric maps h with

$$k(h)=k>\sup k(f_n), \qquad (11)$$

where the supremum is taken over the extremal polygonal maps of all possible polygons.

By now, we see that the problems related with Eq.(9) have been completely disapproved by Theorem 11. The theorem has applications also to the Teichmüller space theory.

Recently, the author and Yao give a necessary and sufficient condition such that Eq.(6) holds. We call $\{Q_n\}$ are a sequence of degenerating quadrilaterals, if z_j^n tend to z_j respectively for j=1, 2, 3, 4, and at least two points of z_j ($1\leq j\leq 4$) coincide.

Theorem 12 Suppose $f(z)$ is an extremal quasiconformal mapping (but not conformal) of Δ onto itself with complex dilatation $\mu(z)$ ($\|\mu\|_\infty=k_1<1$). Let h be its boundary function. Then the following conditions are equivalent:

(a) $K_0(h)<H(h)=K_1(h)$;

(b) $K_0(h)=K_0^d(h)=K_1(h)$;

(c) there exist a family of degenerating topological quadrilaterals Q_n such that

$$\lim_{n\to\infty}\mathrm{Re}\iint_\Delta \mu(z)\Phi_{Q_n}'^2(z)\mathrm{d}x\mathrm{d}y = k_1, \qquad (12)$$

where $\Phi_{Q_n}(z)$ map Q_n conformally onto a rectangle

$$R_n=\{\zeta=\xi+\mathrm{i}\eta:\ 0\leq\xi\leq a_n,\ 0\leq\eta\leq b_n,\ a_nb_n=1\}.$$

in such a manner that the vertices $(z_1^n, z_2^n, z_3^n, z_4^n)$ are mapped onto those of R_n.

RELATIONSHIPS AMONG $K_0(h)$, $H(h)$ AND $K_1(h)$

Proposition 1 If $K_0^q(h)=K_0(h)$, then there exists a non-degenerated quadrilateral Q so that

$$K_0(h)=M(h(Q))/M(Q).$$

Vice versa, if $K_0^q(h)=K_0(h)$, then there exist a sequence of degenerating quadrilaterals $\{Q_n\}$ so that

$$K_0(h)=\lim_{n\to\infty}\frac{M(h(Q_n))}{M(Q_n)}.$$

Proof It is obvious for the first part of the proposition. We only prove the second part.

By the definition of $K_0(h)$ in Eq.(2), there exist a sequence of quadrilaterals $\{Q_n\}$ such that

$$K_0(h)=\lim_{n\to\infty}\frac{M(h(Q_n))}{M(Q_n)}.$$

By passing to subsequences, if necessary, we may assume that the vertices z_j^n ($1\leq j\leq 4$) of Q_n tend to limit points $z_j\in\Gamma$ for $1\leq j\leq 4$ as n tends to ∞. If no points of z_j ($1\leq j\leq 4$) coincide, then $Q(z_1,z_2,z_3,z_4)$ is a non-degenerated quadrilateral. According to Theorem on the convergence of the module (Lehto and Virtanen, 1973), we have

$$\lim_{n\to\infty} M(Q_n) = M(Q) \text{ and } \lim_{n\to\infty} M(h(Q_n)) = M(h(Q)).$$

By definition, $K_0^q(h) = K_0(h)$. This is a contradiction with the assumption of the proposition. Therefore, at least two points of z_j coincide. This ends the proof.

If $K_0^q(h) = K_1(h)$, then there exists a Teichmüller extremal quasiconformal extension of h. In fact, we know from Proposition 1 that there exists a non-degenerated quadrilateral $Q=\Delta(z_1, z_2, z_3, z_4)$ so that Φ conformally maps Q onto a rectangle R and Ψ conformally maps $h(Q)=\Delta(h(z_1), h(z_2), h(z_3), h(z_4))$ onto another rectangle $\check{R}$. Then, $f(z) = \Psi^{-1} \circ A_{K_1} \circ \Phi(z)$ is the extremal quasiconformal extension of h, with $A_{K1}(\xi+i\eta)=K_1\xi+i\eta$ mapping R onto $\check{R}$. Since Φ and Ψ are conformal, $H(h)=H(A_{K_1} \mid \partial R)$. If $\xi\in\partial R$ is not a vertex of ∂R, then $H_\zeta(A_{K_1})=1$. If $\xi\in\partial R$ is one of the four vertexes, then the local dilatations of A_{K_1} at the four vertexes are the same (Strebel, 1976). Let this number be denoted by K^*, which is a constant depending only on K_1. It can be actually computed explicitly that $K^*<K_1$. Therefore, in case $K_0^q(h) = K_1(h)$, we have

$$H(h)=K^*<K_1. \tag{13}$$

Proposition 2 If $1\le H(h)<K^*$ or $K^*<H(h)<K_1(h)$, then $K_0(h)<K_1(h)$.

Proof We prove it by contradictions. Suppose that $K_0(h)=K_1(h)$. Then it follows into two cases.

(1) $K_0^q(h) = K_1(h)$. By Eq.(13), $H(h)=K^*$. This is a contradiction.

(2) $K_0^d(h) = K_1(h)$. By Theorem 12, $H(h)=K_1(h)$. This is also a contradiction with the assumptions. Thus, we have completed the proof.

Proposition 3 Suppose that $K_0(h)=K_1(h)$. Then $H(h)$ is not necessarily equal to $K_1(h)$.

Proof There are two cases for $K_0(h)=K_1(h)$.

(1) $K_0^d(h) = K_1(h)$. By Theorem 12, $H(h)=K_1(h)$.

(2) $K_0^q(h) = K_1(h)$. By Eq.(13), $H(h)<K_1(h)$.

This completes the proof.

Liang and Zhu (2001) gave a concrete example such that $K_0(h)=H(h)=K_1(h)$. While Shen (2000) constructed a counterexample such that $K_0(h)<K_1(h)$ when $H(h)=K_1(h)$. So, We have the following property.

Proposition 4 Suppose that $H(h)=K_1(h)$. Then $K_0(h)$ is not necessarily be equal to K_1.

References

Ahlfors, L.V., 1966. Lectures on Quasiconformal Mappings. Van Nostrand, New York.
Anderson, J.M., Hinkkanen, A., 1995. Quadrilaterals and extremal quasiconformal extensions. *Comment. Math. Helv.*, **70**:455-474.
Chen, J., Chen, Z., 1997. A Remark on "An approximation condition and extremal quasiconformal extensions". *Chinese Science Bull.*, **42**:1765-1767.
Chen, J., Zhu, H., Liang, X., 2002. Quadrilateral and substantial boundary points. *Chinese Annals of Math.*, **23A**(2):197-204 (in Chinese).
Earle, C., Li, Z., 1999. Isometrically embedded polydisks in infinite dimensional Teichmüller spaces. *J. Geom. Anal.*, **9**(1):51-71.
Fehlmann, R., 1982. Quasiconformal mappings with free boundary components. *Annal. Acad. Sci. Fenn., A.I. Math.*, **7**:337-347.
Krushkal, S.L., 2003. Polygon quasiconformal maps and Grunsky inequalities. *J. Anal. Math.*, **90**:175-196.
Kühnau, R.K., 2000. Drei Funktionale eines Quasikreises. *Annal. Acad. Sci. Fenn., A.I. Math.*, **25**:413-415.
Lakic, N., 1995. Strebel Points, Lipa's Legacy. Proceedings of the Bers Colloquium, New York, p.417-431.
Lehto, O., Virtanen, K.I., 1973. Quasiconformal Mappings in the Plane. Springer, Berlin.
Li, Z., Wu, S., Qi, Y., 1999. A class of quasisymmetric mappings and substantial boundary points. *Chinese Science Bull.*, **44**:926-930 (in Chinese).
Liang, X., Zhu, H., 2001. An extremal problem of quasi-conformal mappings in hyperbolic regions. *Journal of Fudan University*, **40**(6):645-648.
Qi, Y., 1998. A problem in extremal quasiconformal extensions. *Sci. in China, Series A*, **41**(11):1135-1141.
Reich, E., 1997. An approximation condition and extremal quasiconformal extensions. *Proc. Amer. Math. Soc.*, **125**(5):1479-1481. [doi:10.1090/S0002-9939-97-03863-X]
Reich, E., Strebel, K., 1974. Extremal Quasiconformal Mappings with Given Boundary Values. *In*: Contributions to Analysis. Academic Press, p.375-391.
Shen, Y., 2000. A counterexample theorem in extremal quasiconformal mapping theory. *Sci. in China, Series A*, **43**(9):929-936.
Strebel, K., 1976. On the existence of extremal Teichmüller mappings. *J. D'Analyse Math.*, **30**:464-480.
Strebel, K., 1999. On the dilatation of extremal quasiconformal mappings of polygons. *Comment. Math. Helv.*, **74**(1):143-149. [doi:10.1007/s000140050080]
Wu, S., 1997. Moduli of quadrilaterals and extremal quasi-conformal extensions of quasisymmetric functions. *Comment. Math. Helv.*, **72**(4):593-604. [doi:10.1007/s000140050036]
Yang, S., 1997. On dilatations and substantial boundary points of homeomorphims of Jordan curves. *Results in Math.*, **31**:180-188.